Engineering Mathematics - I
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Lecture – 09
Partial Derivatives of Functions of two Variables

Hello, welcome to the lectures on Engineering Mathematics I and today's, this is lecture number 9 and we will be talking about Partial Derivatives of Functions of Several variables.

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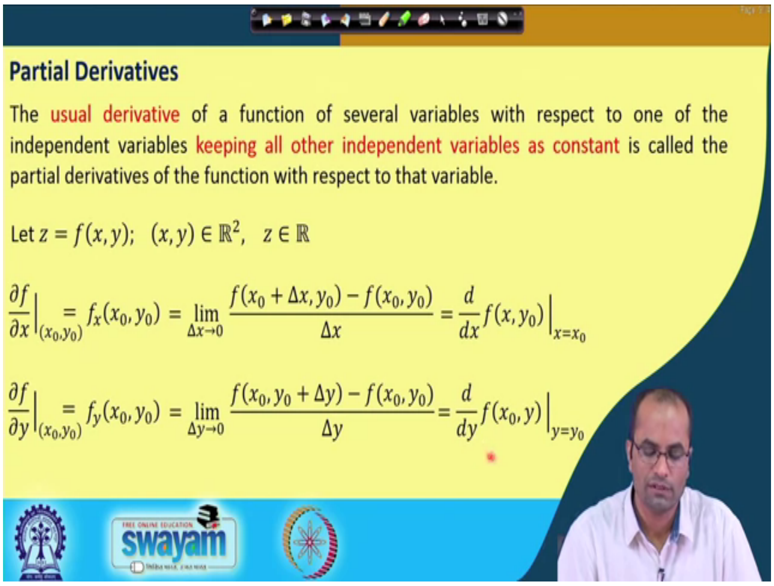

So, what is Partial Derivative? It is a usual derivative of a function of several variables with respect to one of the independent variable while keeping all other independent variables as constant. So, as written here it is a usual derivative, when we have to keep other independent variable as variables as constant and taking the derivative of one particular with respect to one particular variable and this is called the partial derivative with respect to that variable.

So, for example, we have a function z is equal to f x y and x y belongs to some domain in R square and z belong to R the real line. So, this partial derivative of f with respect to x at a point x 0 y 0 which we also denote by f x x 0 y 0. This is defined as we said before it is the usual derivative. So, we are taking here derivative with respect to x. So, the fundamental definition of the derivative that limit delta x goes to 0 and then, we will make an increment here in x 0. So, x 0 plus delta x; this is the increment in x because we are taking partial derivative with respect to x.

So, here x naught plus delta x and then, y naught we are not changing y naught. So, it is at the point y naught. So, y naught will remain as it is and minus the function value at x naught y naught and divided by delta x and if this limit exists, then we call that this is a partial derivative with respect to x at this point x naught y naught and it is denoted by f x at x naught y naught or delta f over delta x at x naught y naught.

We can also understand this as so here in the function we have kept this y 0 as constant and taking this usual derivatives. So, now, this is here the function of x because we have substituted like y 0 in the function and then, this become a function of one variable x and then, we can take the derivative with respect to x and then later on we can substitute x is equal to x 0.

So, as written there it is a usual derivative with respect to x. Like here we have use the usual derivative d over d x of this function which is a function of x because we have substituted y is equal to y naught, the constant value of this y. Similarly, the derivative of f with respect to y at the point x naught y naught and now because we are taking the derivative of with respect to y. So, in that fundamental definition now we will make an increment in y 0 arguments. So, here y 0 plus delta y and minus the function value of f at x naught y naught and here divided by delta y.

And again, if this limit exist we call that the partial derivative exists and this is the notation that f y x naught y naught over del f over del y at x naught y naught. And again, we can also take like fixing this x naught, then we have this function as a function of one variable y and then we can take this usual derivative with respect to y at y is equal to y 0 later on. So, this will be the partial derivative with respect to y at the point x naught y naught.

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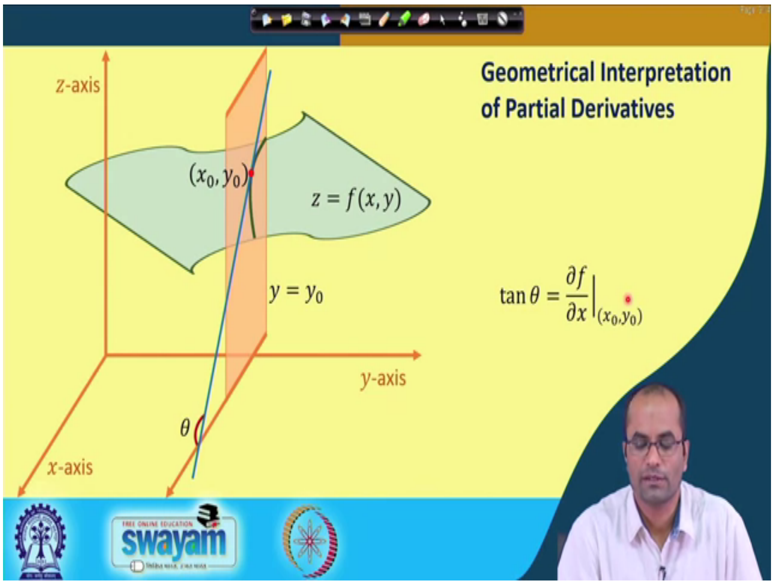

So, coming to Geometrical Interpretation of the Partial Derivative, we consider this plane three dimensional. So, x axis, y axis and then, we have z axis there and we have some surface where the function z is equal to f x y. So, the surface is represented by this function as z is equal to f x y.

So, if we cut the surface by a plane here y is equal to y 0. Now, this is the plane y is equal to y 0, this is y axis. So, here we have point on the y axis y is equal to y 0 and this is the plane. So, here all the values on of y as y 0 another are constant. So, here we have this plane y is equal to y 0. If we cut this surface, then we as we can see here there is a curve of intersection here by this plane and the surface. So, now, here on this curve if we take a point for example, x naught y naught and we are talking about the partial derivative with respect to x.

Then, if we draw the tangent on this curve at this point x 0 y 0 and then, we note here the angle from the x axis and the tangent of this angle would be precisely the partial derivative of f with respect to x at x naught y naught. So, this is the same definition same geometrical interpretation as we have for the function of one variable. The only difference is that because we have fixed this y. So, y is equal to y naught if we cut this plane over this surface or by this plane, then we will get a curve and then we have we are again back to in one dimensional case and the geometrical interpretation is same that the tangent of this theta is equal to the partial derivative of f with respect to x at this point x naught y naught.

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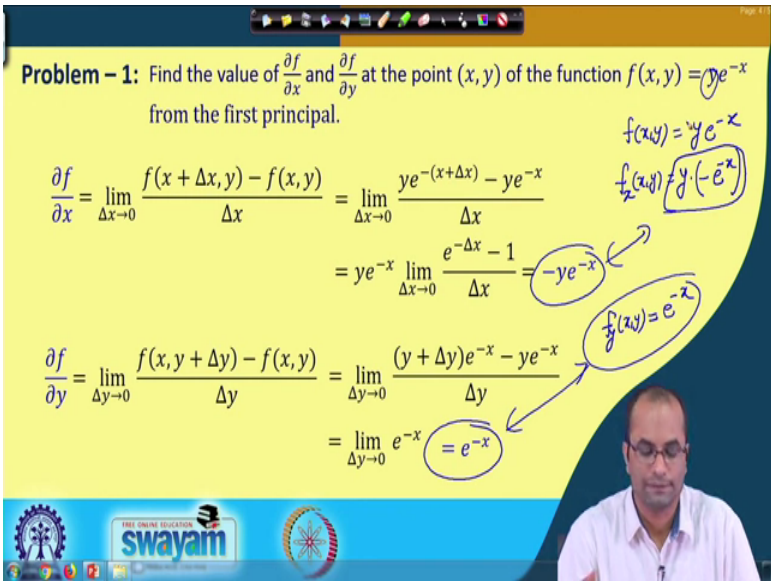

Coming to the problem find the value of del f over del x at the point x naught x y. It is a general point here of the function f x y is equal to y e power minus x from the first principle. So, we will use the definition we which we have discussed in the previous slide to compute this del f over del x and del f over del y at a general point x y.

So, what was the definition? Del f over del x at a point x y. So, we have taken here a general point x y. So, the definition says that we have to make an increment in x because we are talking about the partial derivative with respect to x. So, x plus delta x and y minus the function value at x y; so, f x y and divided by this increment delta x and then, taking the limit delta x goes to 0. So, then we have to take this function which is f x y is equal to y e power minus x.

So, in this case this x plus delta x y. So, it will become y and instead of x we will have x plus delta x. So, e power minus and for x, we have x plus delta x minus the function value at x y which is y e power minus x the function itself and divided by delta x. So, now, if we see here y e power minus x and y e power minus x is common. So, we can take this outside y e power minus x. So, what will remain here? e power minus delta x and then, minus 1 because y e power minus x we have taken common from these 2 terms and then, we have divided by delta x there.

Now, if you compute this limit because as we see delta x goes to 0. This e power 0 it is one minus one 0 over 0 form. So, we can apply the (Refer Time: 08:11) rule which says that this limit will be like minus e power here minus delta x, the derivative of this e power minus delta x and the derivative here will be just 1. So, when delta x goes to 0. So, this limit is going to minus 1. So, the derivative here in this case will become minus 1 and y e power minus x; so, minus y e power minus x.

Now, for the partial derivative with respect to y, so our definition will change now. So, x will remain as it is and they will be incrementing y. So, y plus delta y and minus f x y divided by delta y. So, now, substituting this in the function; so, we have y plus delta y and e power minus x minus the function y e power minus x and divided by this delta y.

So, again here y e power minus x is common term y e power minus x and here also we have y e power minus x. So, what is left then e limit delta y goes to 0 and e power minus x we. So, here we have y e power minus x which cancel out. So, y e power minus x and minus y e power minus x cancel out and then, we have delta y e power minus x over delta y. So, the delta y also cancels out and we get only e power minus x. So, no need to take common because this is y e power minus x and here we have minus y e power minus x is the same term.

So, it basically gets cancelled and then, we have here y e power minus x term divided by delta y and this delta y and delta y will be cancelled. So, we will get e power minus x and there is no delta y term here. So, this is just e power minus x. So, what we can also see? So, this was using the basic definition of or the fundamental definition of partial derivatives. We got the derivatives here with respect to x was minus y e power minus x and with respect to y it was e power minus x.

We can also compute this directly because this function is given as f x y is equal to y e power minus x and the partial derivative with respect to x at the point x y. So, keeping here y constant; treating y constant and taking the usual derivative. So, y will remain as it is and the derivative of e power minus x will be minus e power minus x.

So, we have the partial derivative with respect to x which we have already seen here minus y e power minus x. Same we can do to get the partial derivative with respect to y. So, if we take the partial derivative with respect to y here, then this is now x will be treated as constants. So, e power minus x will be treated as constant and then, when we take the partial derivative with respect to y. So, this will be 1, then we have e power minus x. So, again we get. So, we can directly compute using the usual definition or visual a known reserves of the derivatives keeping other variable constant ok.

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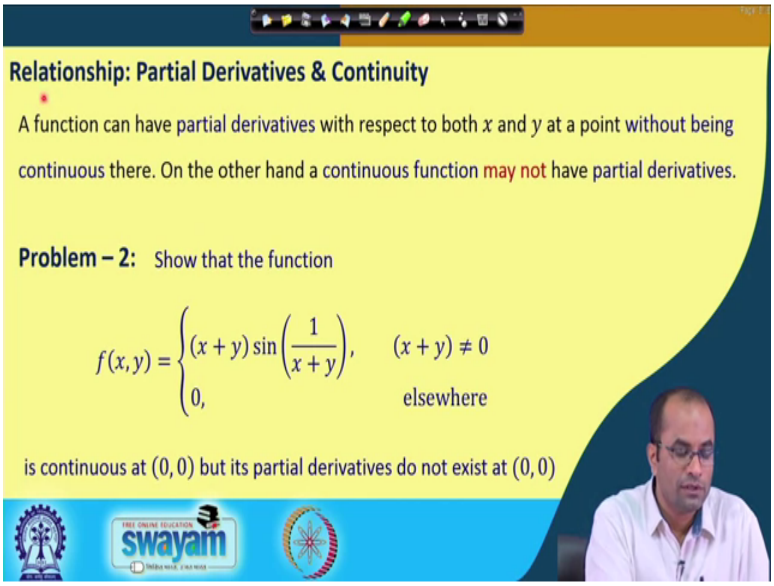

So, the next. So, what is the Relationship between the Partial Derivatives and the Continuity? We have already discussed in the previous lecture Limits and Continuity. So, if a function can have partial derivative that is a interesting result. So, if a function can have partial derivative without or with respect to both x and y at a point without being continuous there. So, we do not need continuity to for the existence of partial derivative in this case. When we talk about the usual derivatives, the continuity is must for the differentiability, but that is so far we called differentiability or getting the derivative there in case of single functions of single variable, we need continuity of the function.

But in this case a function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand, a continuous function may not have partial derivatives. So, other way around, we can have a continuous function, but it may not have partial derivative. So, anything is possible and we will see now in this example which shows that this function we will check now is continuous, but its partial derivatives do not exist. So, that is a one example which shows that the function is continuous, but the partial derivative both the partial derivatives do not exist at 0 0.

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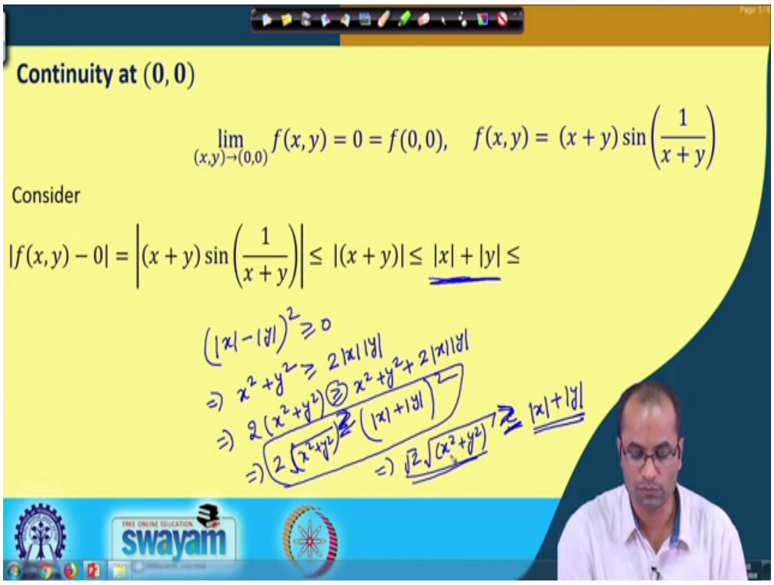

So, let us check the continuity at 0 0. So, we have this function here f x y is equal to x plus y sin 1 over x plus y and we want to show for the continuity that this limit if we take the limit x y goes to 0 0 of this function f x y, then this limit is 0.

If this limit is 0 because the function value we have seen a 0 here at 0 0. So, we will show that this limit here is equal to the function value than the function is continuous. We can use for example, the delta epsilon definition or we can also do directly as we have done in the previous lecture. So, let us just go through the standard definition of epsilon delta.

So, here f x y minus this 0, the function value is equal to this difference. So, here that is in the function itself x plus y sin 1 over x plus y and we know that the sin is bounded by 1. So, we can write down this x plus less than modulus x plus y. So, this is bounded by 1. So, this will be a big quantity than this one and again, we can we know also this result the triangular inequality that the absolute value of x plus y will be less than equal to the absolute value of x plus absolute value of y. And now, we will make a calculation here to substitute this x plus y or to estimate this absolute value of x plus y in terms of the neighborhood.

If you remember we need to set this inequality in terms of the neighborhood so that we can get the relation with delta and epsilon. So, here if we consider this absolute value of x minus absolute value of y whole square, this will be always positive because this is a whole square term and then, we have here x square plus y square and this will be minus 2 times absolute value of x minus absolute value of y which I can bring to the right hand side. So, the right hand side will become 2 absolute value of x and absolute value of y.

Now, if I add both the sides this x square plus y square. So, here it will become 2 times the x square plus y square and the right side, it will be x square plus y square and this plus 2 times x and y. This the right hand side, we can write down as absolute value of x, absolute value of y whole square and the left hand side is a 2 times sorry x square plus y square. And now, we can take the square root here. So, this will become as a square root 2 and a square root of x square plus y square and less than equal to this is less than less than equal to absolute value of x plus absolute value of y.

This is other way round. So, this is greater than equal to because here it was greater than. So, this is greater than, this is greater than equal to this absolute value of x plus absolute value of y. So, now, we can replace this term here the absolute value of x plus y by greater the larger quantity here a square root 2 and a square root x square plus y square.

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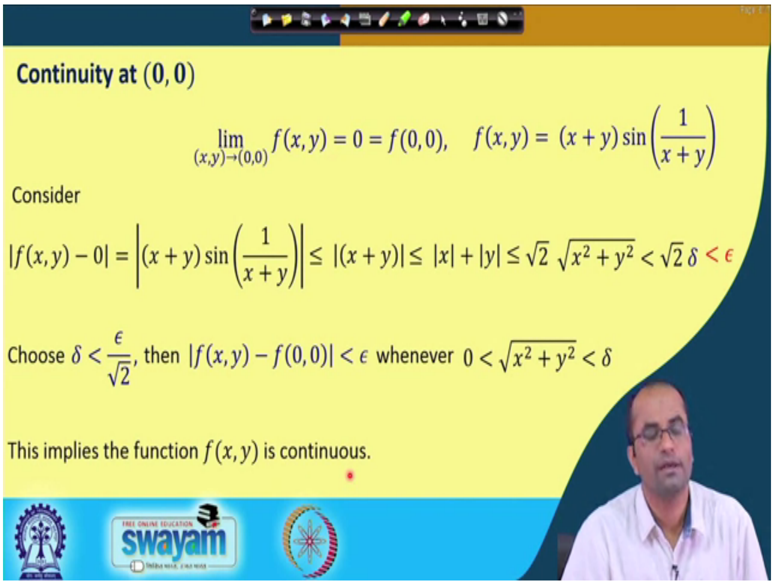

So, let us do that. A square root 2 and a square root x square plus y square and now here exactly we have the neighborhood of this 0 0 point which we can bound by delta. So, here this term can be we can use that inequality for the neighborhood and then, we have a square root 2 into delta and now this difference the aim is to make this difference of f x y minus 0 less than epsilon. So, this is less than epsilon now, if you make then we will get the relation between delta and epsilon.

So, here choosing this delta from here epsilon by a square root 2 because epsilon is given and then we can find this delta by this relation anything is smaller than this epsilon divided by square root 2 and then, we have that this difference is less than epsilon whenever we choose x y from this neighborhood; from this neighborhood delta neighborhood of 0 0 point.

So, this is the epsilon delta approach to prove that the given number is the limit of a function which is here x plus y sin 1 over x plus y. So, what we have seen that this 0 is the limit of this f x y function. So, in that case we have proved the continuity. So, this implies that f x y is continuous.

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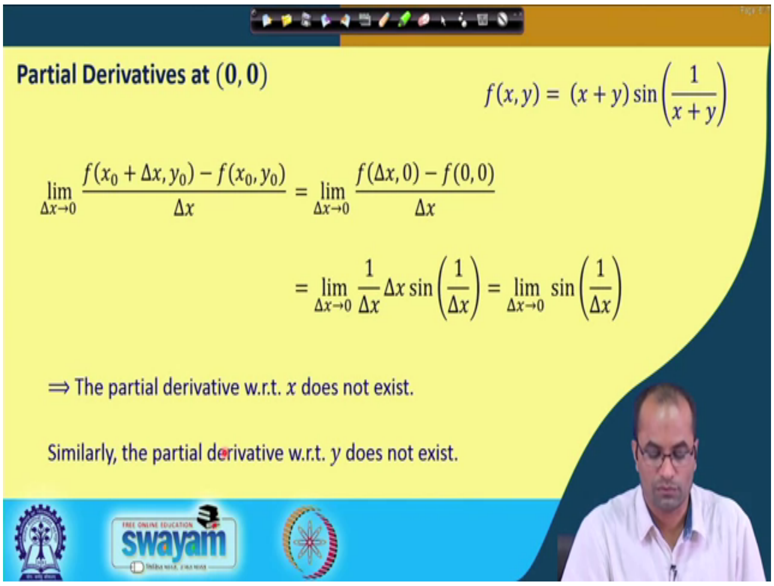

Next moving to the partial derivatives at 0 0 points, so we have this function and we can use now this definition delta x goes to 0 and x 0 plus delta x and minus f x 0 over delta x. So, since x 0 and y 0 0 0, so we are talking about the derivative at 0 0 point. So, we are taking this limit if this limit exist, then we call the partial derivatives with respect to x exist. If it does not exist, we will call the partial derivatives or partial derivative with respect to x does not exist. So, here in this case we have as per the definition now and delta x goes to 0.

So, this f delta x 0. So, here y is 0 and then, x will be replaced by delta x. So, we have delta x and sin 1over delta x and this 1over delta x term is here. So, delta x and delta x gets cancelled and then, we get simply this limit delta x goes to 0 sin 1 over delta x and delta x approaches to 0. So, in this case here we do not know what is this limit because the sin is not definite when this delta x goes to 0. We cannot conclude the value of this one.

So, basically in this case this limit does not exist because we do not know what number is this. So, the limit; that means, the partial derivative with respect to x does not exist in this case and obviously, the similar calculation we can do for y or the partial derivative with respect to y to get this limit and we will find that that the partial derivative with respect to y also does not exist.

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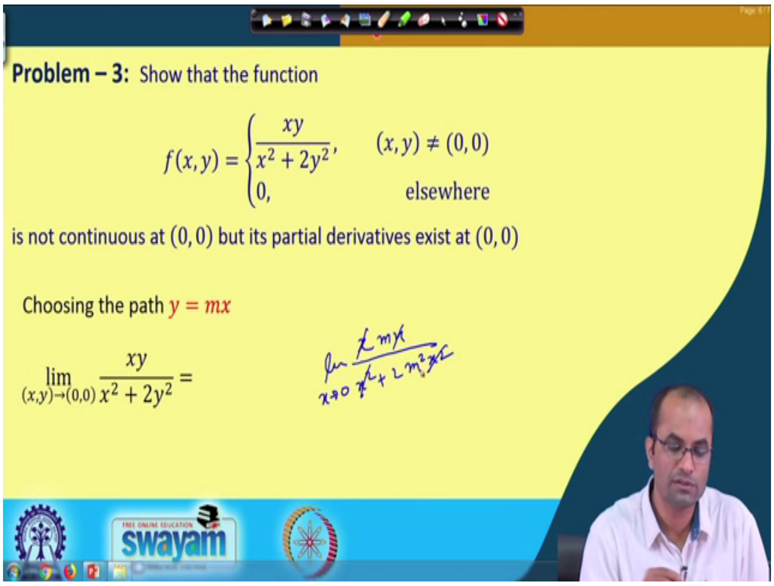

Moving to another problem, we have a now we will show that this function is not continuous, but its partial derivatives exists at 0 0. In the earlier problem, we have seen the function was continuous, but the partial derivatives do not exist. In this case, we will show that the function is not continuous, but its partial derivatives exist; both the partial derivatives exist. So, why this is not continuous? If we choose the path y is equal to mx; then, what we find because y is equal to mx and then here also y is equal to mx.

So, here x square here x square and here also x square will come and what we will get. So, here we have x and then y is mx. So, we have x square and 2 m square x square and then, the limit x goes to 0.

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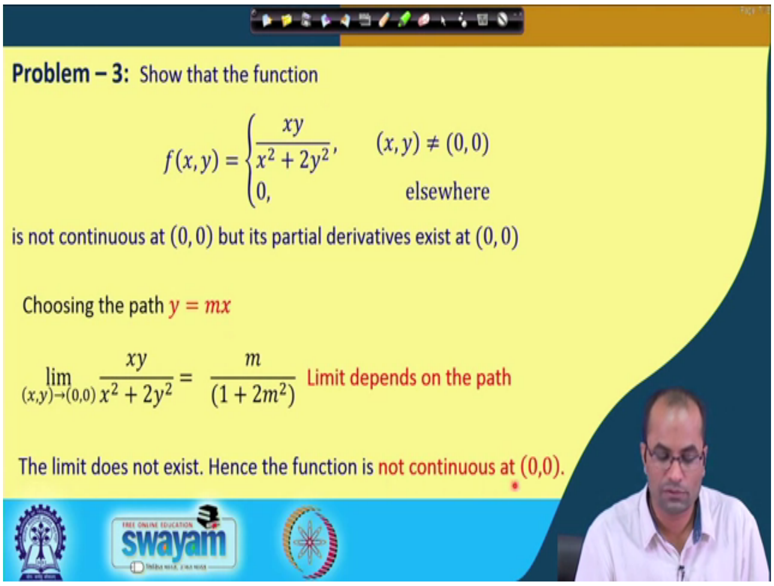

So, here x square here x square here x x square. So, we will get this limit as m over 1 plus 2 m square. m over 1 plus 2 m square and hence, this limit does not exist because it depends on m. For different values of m, we are getting different value of this limit.

Hence, this limit does not exist and the function is not continuous. Now, we will show that though the function is not continuous, but we can compute the partial derivatives with respect to x and with respect to y.

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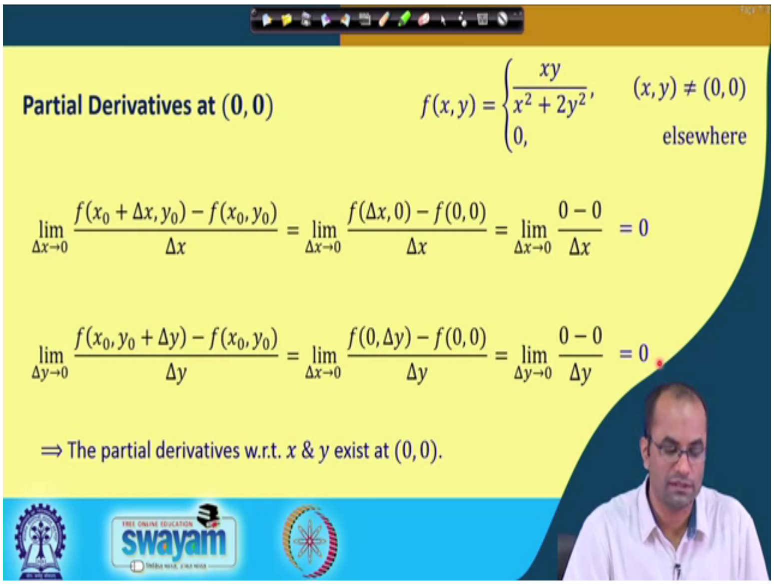

So, the partial derivatives again with the definition we have this fundamental definition which at the 0 0 points become delta x 0 and f 0 0 over delta x. Now, f delta x 0. So, one argument is 0, the y is 0. So, this will become 0 because of this product. So, here it is 0 and then, f 0 0 this is also 0. So, 0 minus 0 over delta x and this is 0 whatever delta x's so, we have here the value of this limit is 0; that means, the partial derivative with respect to x is 0.

Now, the partial derivative with respect to y. So, we have y naught plus delta y in this case and x naught y over delta y. As per the definition, we have because this x naught y naught; you will put 0, we are talking about the partial derivatives at 0 0 point. So, f 0 delta y minus f 0 0 over delta y, again the same argument because here the x is 0 and the product in the numerator is there x y term. So, here this will also become 0. So, this will become 0 and this is 0.

So, 0 minus 0 over delta y and this is 0 and we got again this limit as 0. So, both the partial derivatives with respect to x and with respect to y exist the value of the partial derivatives in both the cases are 0 and the function was are not continuous in this case.

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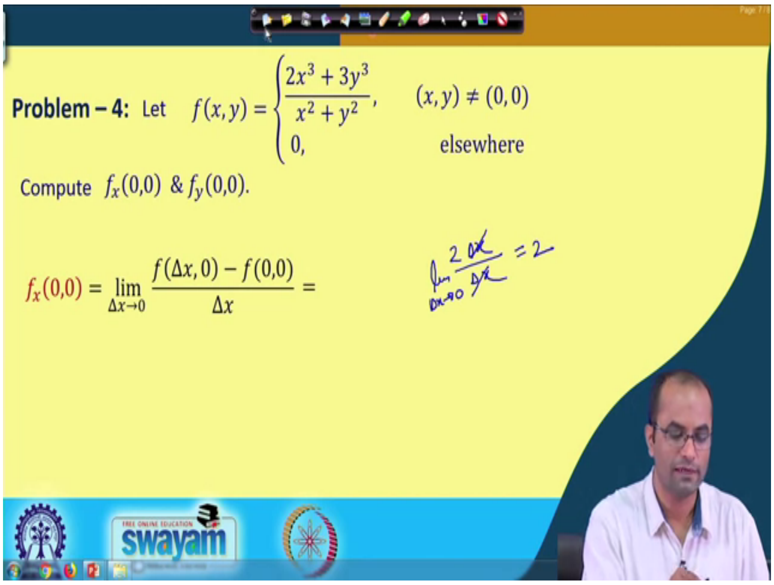

Another problem, so here we have f x y is equal to 2 x cube plus 3 y cube divided by x square plus y square. So, we want to compute f x and f y at 0 0 point.

So, again as per the definition, we have f x 0 0 is equal to f delta x 0 minus f 0 0 divided by delta x and delta x goes to 0. This is the partial derivative with respect to x. So, we will substitute now f delta x 0. So, y will be set here 0; there also 0 and we will get this 2 times. So, this when y will be 0. So, this function will become 2 times x; simply 2 times x. So, we will get this one 2 times delta x and divided by again delta x here and then, limit delta x goes to 0. So, delta x gets cancel and we will get this value as 0.

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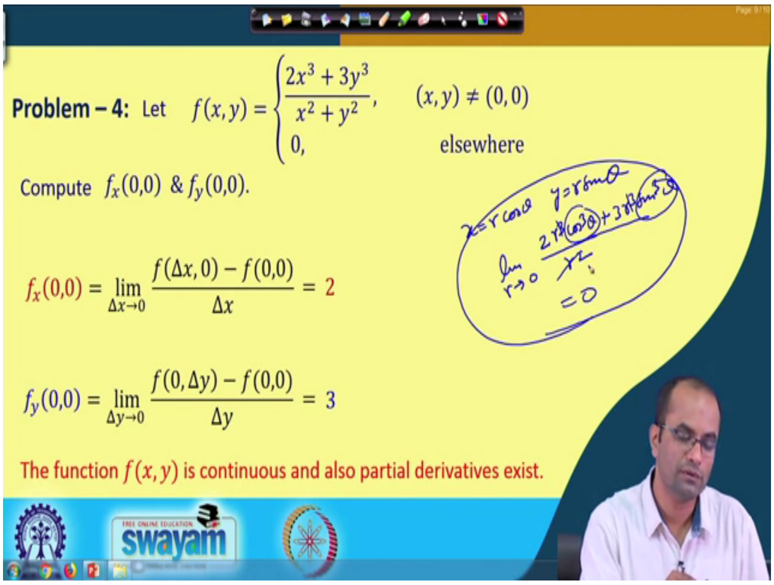

So, we will get here the value 2 ok. Now, the second when we compute f y at 0 0. So, we have a similar situation again; the 0 delta y. So, this time the delta y will remain. So, we will have here 3 times and the delta y minus. This f 0 0 is 0 and over delta y and delta y goes to 0. So, here this gets cancelled and you will get this value as 3. So, this is 3 here. So, we have the partial derivative with respect to x as to partial derivative of f with respect to y here as 3. The function is continuous and also partial derivatives exist.

Because in this case we can easily a prove that the partial derivatives exist and we have already done this in previous lectures. So, here for example, to see the continuity, we can put x is equal to r cos theta and y is equal to r sin theta and then, this will become simply here the r square will come and then, 2 r cube and cos cube theta plus 3 r cube and sin cube theta and limit r goes to 0.

So, in this case the 1 r will survive in the numerator and the rest term here is bounded. So, this will go to 0. So, the function is continuous and the partial derivatives exist. So, anything is possible. We have seen the example when the function was not continuous partial derivatives exist or the function was continuous, partial derivative do not exist. In this case function is also continuous and partial derivatives also exist.

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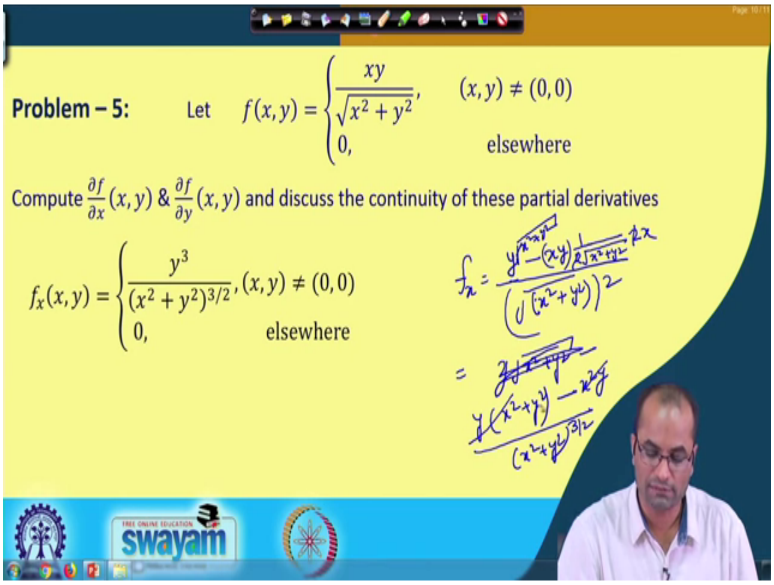

So, this example again to compute the partial derivatives with respect to x and with respect to y and also you will discuss the continuity of these partial derivatives. So, here the f x when x y is not equal to 0 0. So, when x y is not equal to 0 0, we have this nice function. The problem is at 0 where we have to defined separately as 0. So, when x y is not equal to is not equal to 0 0, we have this function and we can take the partial derivative x here treating this y as constant.

So, just doing this calculation for a partial derivative with respect to x, we can just follow the usual calculations. So, here x square plus y square and the square root; so this will be its whole square and then, in the numerator when we take the derivative treating this y as constant.

So, here we will get the partial derivative with respect to x of x y will become y and then, minus this x y as it is and the partial derivative here which will be 1 over 2 and then, here this is square root x square plus y square and we are taking partial derivative with respect to x. So, this 2 x will come and this 2 will get cancel and then, we can multiply here this y term in x square plus this y square and then we have minus. So, here again this term will go there because this quotient rule.

So, x square plus y square. So, this will become y into x square plus y square and minus this x square y and then, divided by this x square plus y square term and this will also come. So, this will become 3 by 2; 1 and this half will make it 3 by 2.

So, this here x square y will get canceled, we will get y cube. Exactly this term here y cube over x square plus y square 3 by 2 and similarly, we can compute this f y with respect to y.

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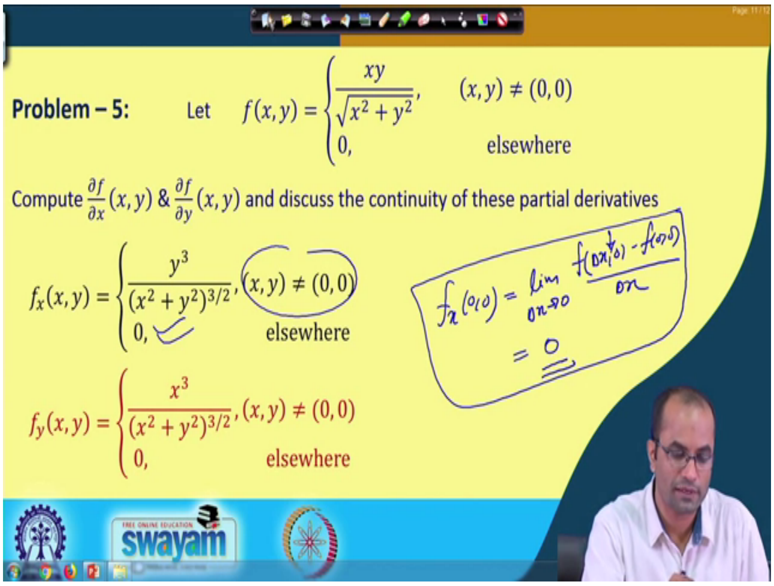

So, we will get just instead of the y will be replaced by x because of the symmetry of the functions. So, we got these 2 and here this partial derivative with respect to x, we have to also compute this with the definition the fundamental definitions.

So, we have to use here what is the f x at 0 0. So, at non zero point that non zero point, we can directly compute from here and at 0 0 we have to use the fundamental definitions. So, the delta x goes to 0 and we have f delta x 0 minus f 0 0 divide by delta x. So, f 0 0 is 0 and here if one of the argument is 0.

So, this function will become 0 and so, we have 0 minus 0 over delta x. So, this will become 0. Therefore, we have used here 0, but we have to use this fundamental definition to get the partial derivative at 0. So, here also we get 0 and now moving next. So, we want to discuss the continuity. So, to discuss the continuity, we have to see what is the limit of this? Whether the limit of this function is 0 or same thing here that the limit of this is 0; then, the function is continuous, otherwise it is not continuous.

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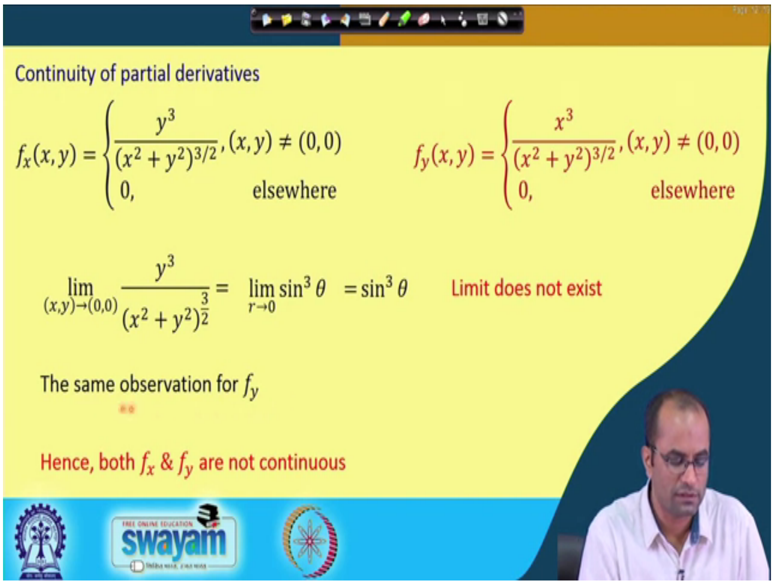

So, we have to check those limits here. In this case, when we substitute x is equal to r cos theta y is equal to r sin theta. So, we will get here r cube and here also r cube sin cube theta. So, r cube will get cancelled and we will get sin cube theta. So, this limit depends on theta. So, therefore, this is not continuous. So, this f x y is not continuous because this limit does not exist the limit depend on theta.

For continuity, the limit should exist and it should be equal to the value at 0 0. So, in this case the limit does not exist. Therefore, these partial derivatives are not continuous; (Refer Time: 31:18) did we can do the similar calculation for f y. So, both a partial derivatives f x and f y, they are not continuous.

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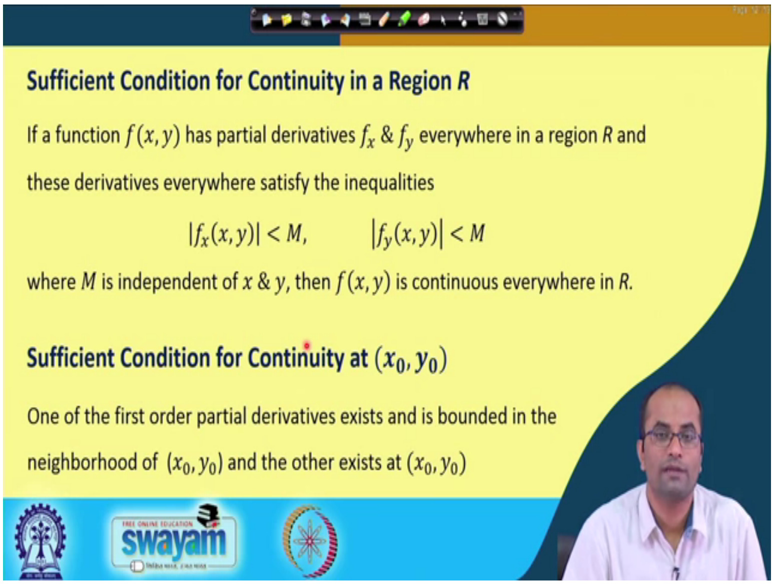

We have one result on the sufficient which is the sufficient condition for continuity in a region r.

So, what it says if I pass if a function f x y has partial derivatives f x and f y everywhere in a region r. So, the existence of the partial derivative here and moreover, these partial derivatives are bounded by some number M here. So, f x x y in that region is bounded and f x also the f y is bounded, where this M is independent of x and y. So, then the f x y is continuous everywhere. So, that is the one sufficient condition for the continuity again.

So, if the partial derivatives exists and they are bounded, then the function will be continuous and we can also have a sufficient condition to discuss this continuity at x naught y naught. So, here if one of the partial derivatives exist and is bounded in the neighborhood; so, here we have to talk about the neighborhood. So, if one of the partial derivatives exist and is bounded in this neighborhood and the other one exist at this x 0 y 0 point. Then, also we can say that the function is continuous at x naught y naught point ok.

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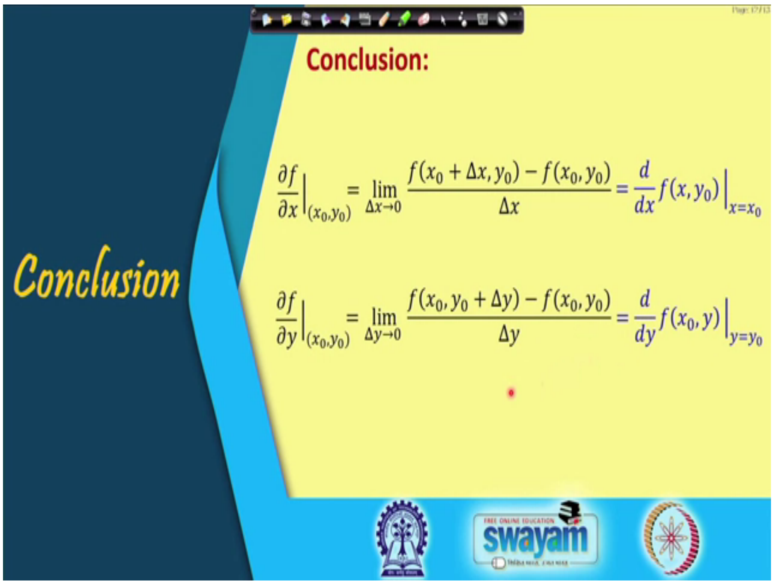

So, what we have learnt today which will be used in following lectures is the Partial Derivatives; so, it with respect to x this definition we are going to use in many other lectures. So, here the increment with respect to x and then, this quotient we have to take the limit if this limit exists, we will call it partial derivative with respect to x and same thing for the partial derivative of with respect to y.

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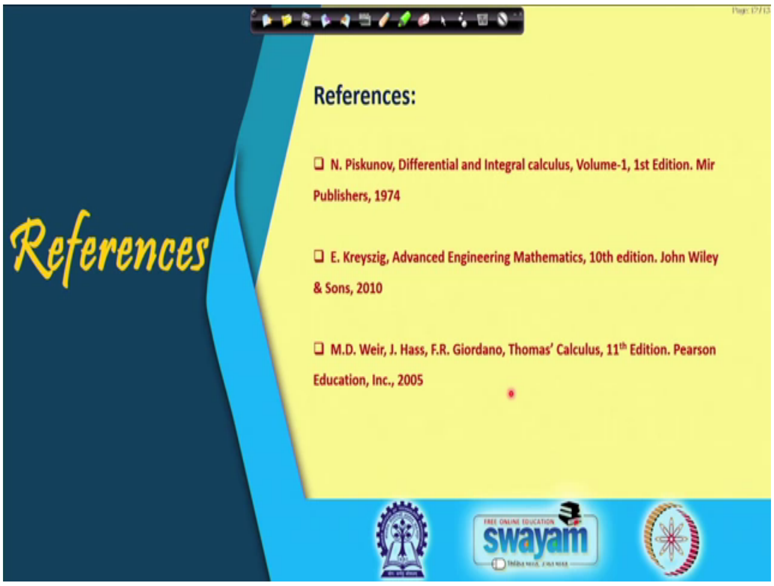

These are the references used for preparing these this lecture and.

Thank you very much.