Engineering Mathematics - I

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Lecture – 08
Continuity of Functions of 2 Variables

Welcome to engineering mathematics 1 and today we will be talking about a Continuity of Functions of 2 Variables.

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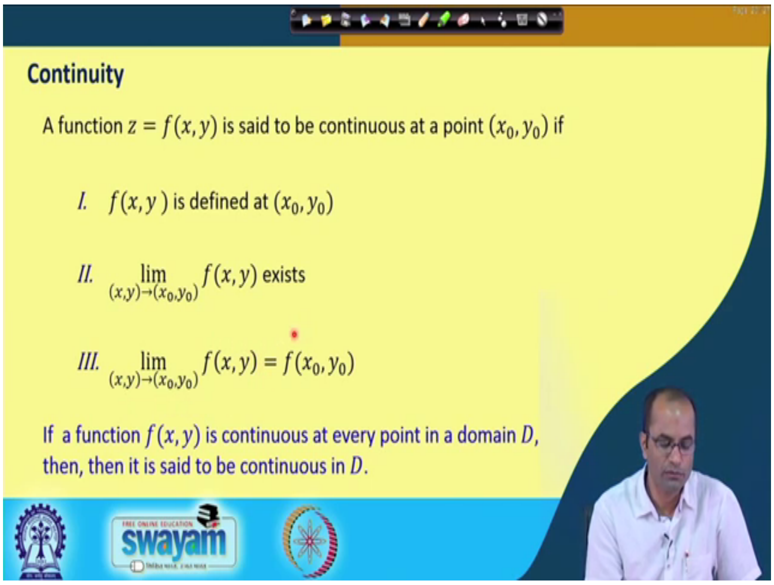

So, we have already seen the definition of a limit indeed the limit delta epsilon definition of limit and this is motivated by that definitions. So, a function z is equal to f x y is said to be a continuous at a point x 0 y 0. If this f x y is defined at x 0 y 0 second, this limit x y goes to x 0 y 0 f x y exists. And the 3rd one that this limit is equal to the function value at that point.

If all these 3 conditions are met; so function is defined this limit exists and the limit is equal to the function value at 0 x 0 y 0, then we call that the function is continuous at the point x 0 y 0. And if a function f x y is continuous at every point in the domain D, then we call that function is continuous in that domain D. So, eventually here the continuity is basically finding the limit as x y goes to x 0 y 0 and then observe that this limit should be equal to the function value at x 0 y 0.

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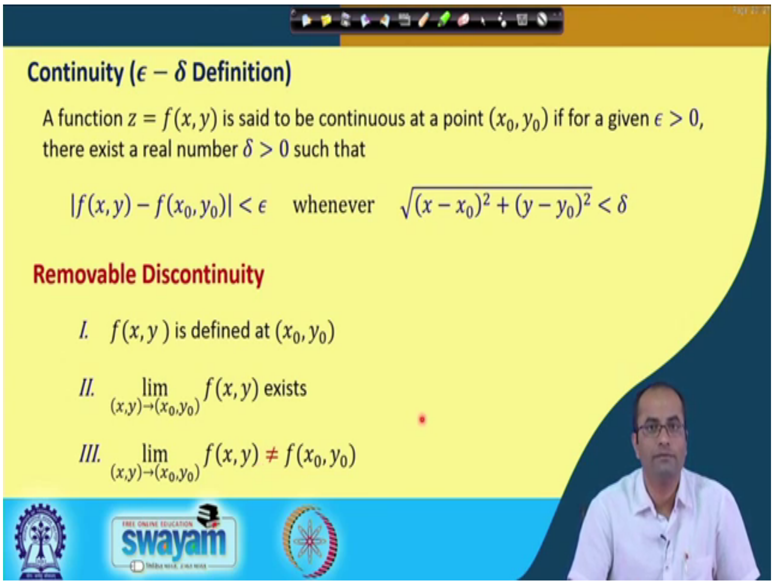

So, we can define delta epsilon definition as for the limit. So, a function z is equal to f x y is said to be a continuous at a point x naught y naught. If for a given epsilon there exist a real number delta positive; such that this difference between f x y and this value of f at x 0 y 0 less than epsilon whenever these x and y's if we take from the delta neighborhood of x naught y naught point.

So, this is the def this was eventually the definition of the limit as well, where we have seen that this function has a limit l. So, this instead of f x 0 y 0 we used l there and now since the function has to be defined at x 0 y 0 and we are looking at whether the limit of f x y as x y goes to 0 is equal to f x 0 y 0 or not. So, this epsilon delta definition is exactly the same as earlier for the limit.

Indeed, here we have not used now that this must be a greater than 0 to avoid this x naught y naught point. Because, now the function is defined at x naught y naught point; earlier in the case of limit function may not be defined at that point is still we can talk about the limit. But, now in this case for the continuity the function has to be defined at x naught y naught and therefore, this inequality at this end is no more required. So, there is another term we used for removable discontinuity. So, all the conditions we have discussed before like the function is defined at x naught y naught, and the limit x y goes to x 0 y 0 f x y exists.

And the 3rd one when this limit is not equal to the function value at x naught y naught, then we call that the function has removable discontinuity. If this is equal, then this is the definition of the continuity we have just discussed.

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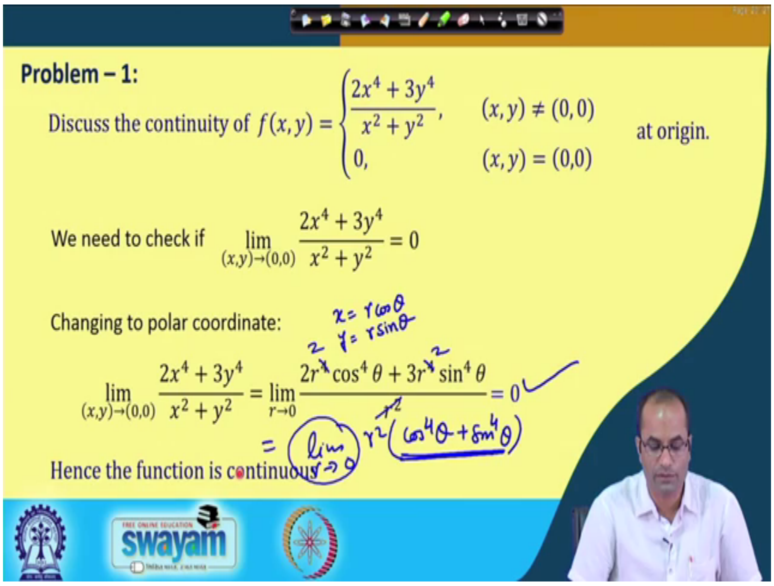

Let us go to some examples now. So, discuss the continuity of this function f x y is equal to 2 x 4 plus 3 y 4 divided by x square plus y square and this is defined when x y is not equal to 0 0 and the value of this function is 0 when x y equal to 0. And we will discuss the continuity of this function at the origin.

So, we need to check basically that this limit here of this function 2 x 4 plus 3 y 4 divided by x square plus y square is equal to 0. If this limit is equal to 0, then the function will be continuous; otherwise, in case this limit does not exists or this is not equal to 0 then we call the function is not continuous. So, as discussed in the last lecture, this changing to polar coordinate is very helpful for finding out the limit.

So, in this case also we will follow those steps. So, we will change now from Cartesian to the polar coordinate, because whenever we see the x square plus y square term in the denominator than this changing to polar coordinate is very, very useful. So, in this case when we change the coordinates from x y to r theta, then x will be a so we basically substitute here. The x is equal to r cos theta and y is equal to r sin theta. So, by doing so we got here 2 times r 4 cos 4 theta plus 3 r 4 sin 4 theta and then your r square cos square theta plus r square a sin square theta which becomes here r square.

So now this r square will get cancel with this, so here also we will get 2. And now we have 2 or rather r square if we take common; so we have here cos 4 theta plus sin 4 theta times r square. So, let us just see and which, so this will become as the limit r goes to 0 and then here r square, and then cos 4 theta plus sin 4 theta and since this is a bounded function cos 4 theta plus sin 4 theta when r goes to 0 this limit will be 0. And the function value at 0 0 is also 0, so the function is continuous; function is continuous.

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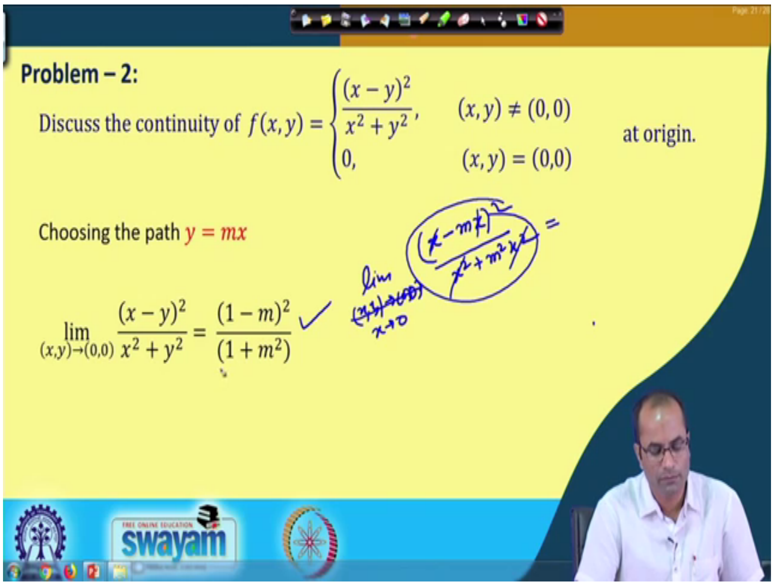

Now, moving further to the next example, we will discuss now the continuity of the function f x y is equal to x minus y minus whole square divided by x square plus y square, when x y not equal to 0 0 otherwise at a origin the function is defined as 0. So, in this case we will choose the path y is equal to mx because here this choosing path is again conclusive, because the order here is 2 in y also it is 2 and there also it is 2. So, choosing y is equal to mx the x will get cancel and this limit will depend on m, in most of the cases this will work.

So, when we do this substitution here for y is equal to mx we will get this limit as x y goes to 0 0 and x and y is substituted as mx divided by x square m plus m square x square. So, from there we take common x so, this x will be removed and we will get this limit 1 minus m whole square divided by 1 plus m square free from x. So, this limit which is x goes to 0 will be nothing but 1 minus m square by 1 plus m square.

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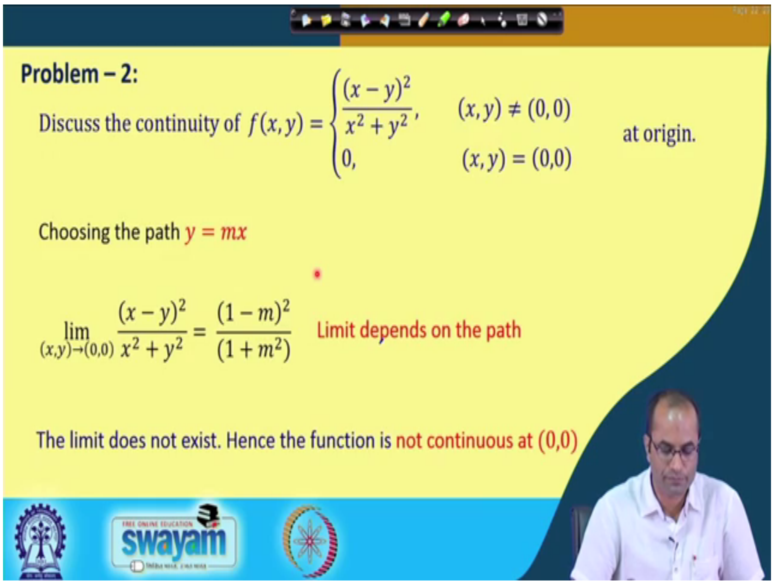

So, this limit depends on path because choosing different value of m we will get a different value of this limit and hence this limit does not exist and then again we will call that this function is not continuous.

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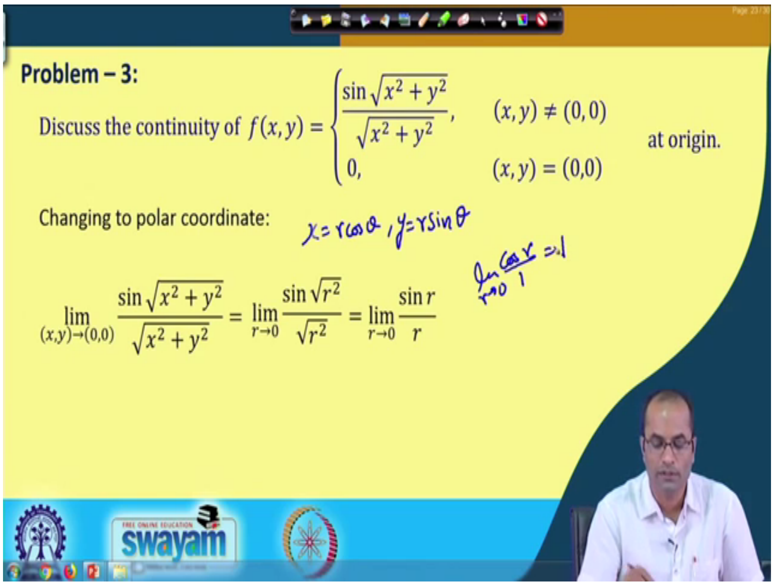

The next example we will discuss the continuity of this function x square plus y square the square root and then we have here a square under a square root x square plus y square otherwise the value is 0. And we will discuss again the continuity of this function at 0 0.

Because, when x y is not equal to 0 0, the function can be easily shown that this is continuous. So, at this point again because x square y square term is there, changing to polar coordinate will be helpful and we will do so the limit x y goes to 0 0 sin this given function will be changed to as limit r to 0. So, here sin this x square plus y square will be r square by that transformation x is equal to r cos theta, y is equal to r sin theta x is equal to r cos theta and y is equal to r sin theta.

So, by this substitution we will get here r square and here also this r square and then this will be like limit r goes to 0, this is sin r and here we have r. So, limit this sin r over r as r goes to 0 we have only one variable limit now. So, this is using the for example, L Hopital's rule we can get the derivative of sin r. So, that will be cos r and this derivative 1 and limit r goes to 0, so this will be 1.

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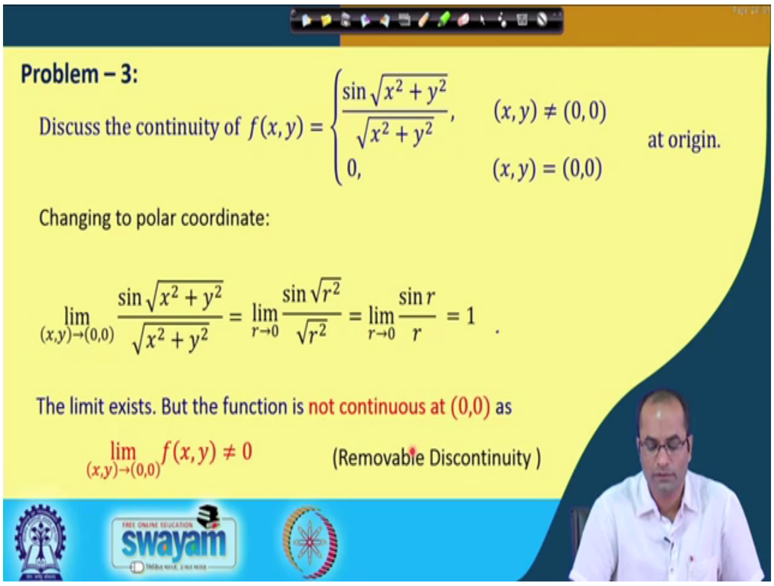

So, this limit here is 1 and the function value defined at 0 0 point is given as 0. So, the limit exist in this case, but the function is not continuous at 0 0, because this limiting value is not equal to the function value which is prescribed at 0 0. So, this is the case of the removable discontinuity where the limiting value.

So, the limit exists the function is defined, but this limiting value is not equal to the functional value at that point. So, this is the example of the removable discontinuity.

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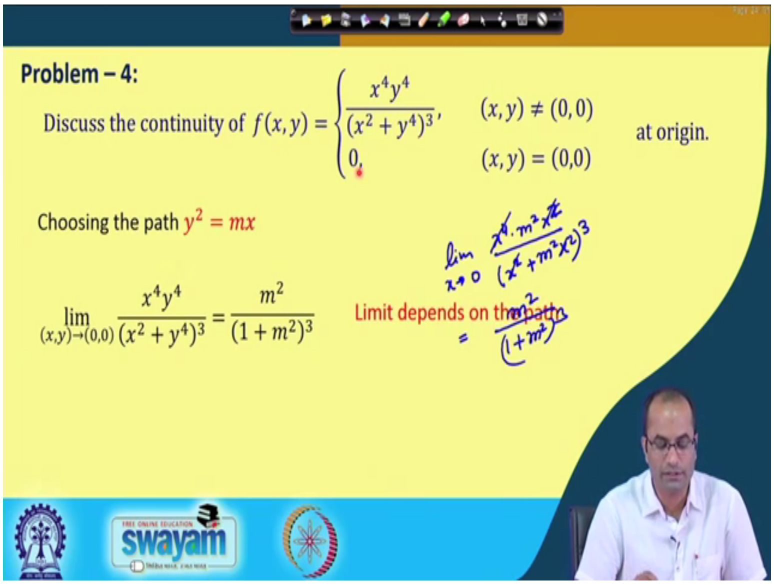

Another example we will discuss the continuity of this function again at origin. So, we have x power 4 y power 4 and then x square plus y power 4 power cube. So now, in this case we will choose the special path y square is equal to mx because, y is equal to mx will not be helpful we have a different order here. So now, choosing this y square is equal to mx due to this y power 4 here. So, if y square will be in then y is m square x square and then we can have this x square common. So, if we take this we substitute this y square there, then in that case this limit x goes to 0 because this path will take to the origin only.

So, now we will take the limit x goes to 0. We have their x 4 and then y square whole square. So, this will become x square and x power 2. So, m square x power 2 and then divided here by x square plus this y square whole square again. So, this is m square x square and power this 3. So, this x power 6, so from here also x square and then power 3. So, x power 6 will be cancelled and then we will get this limit m square over 1 plus m square cube; which is given here.

So, this is m square over 1 plus m square over cube. So, again we see that if we choose this path, then the limit depends on the path again; for different values of m you will get a different value of this limit and therefore, this limit does not exist and hence this function is not continuous.

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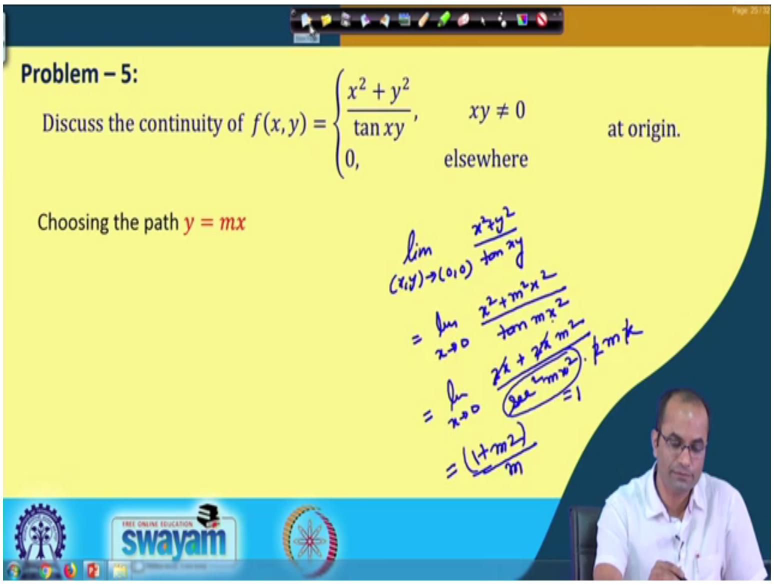

Now, moving next to the next problem. Here we have the x square plus y square over tan x y. So, looking at this function is a difficult to see that what will be the limiting value where the limit will exist on what substitution we should make. So, again let us choose this y is equal to mx and see what is happening in this case. So, this given limit if we have this x y goes to the 0 0 and then this x square plus y square over tan x y. And if we substitute this y is equal to mx, so we will take then the limit x goes to 0.

And here we will have x square plus m square x square and then this tan m x square. So, y is mx, so x square, so now, when a x goes to 0 we are getting like a 0 in the numerator and also tan 0 0 in the denominator. So, 0 by 0 case this is a one variable limit so, we can use basically L Hopital.

So, in that case we will get this limit equal to this limit 2 x plus again 2 x m square and then here we will have the sec square mx square and then this is 2 mx. So, this 2 x 2 x and then also 2 x from here will get cancel and when x goes to 0. So, this is 1 here, this is 1 and in this case it is a there as m square.

So, we have m square over this is 1 and this is; so 1 plus m square this is 1 here plus m square divided by m. So, again this limit is depending on the path.

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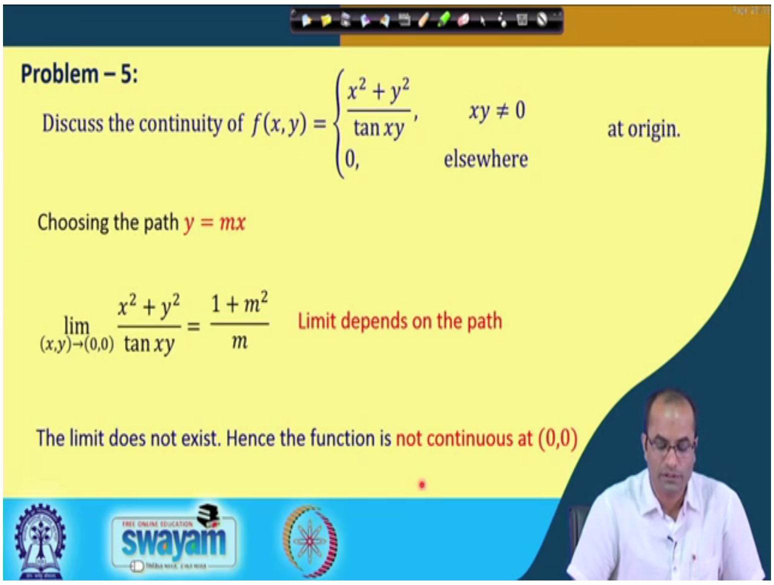

And this is 1 plus m square over m which we have just evaluated there. So, limit depends on path and hence the limit does not exist and the function is not continuous again in this case.

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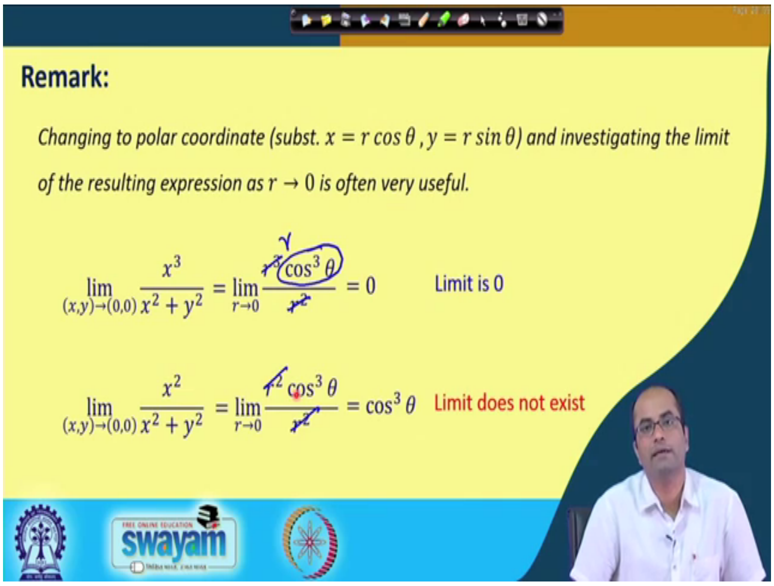

So, some remarks which are very useful for finding out the continuity or basically the limit. So, changing to polar coordinate as we have seen by substituting x is equal to r cos theta and y is equal to r sin theta. And it is investigating the limit of the resulting expression as r goes to 0 is very useful.

As we have seen in many examples ah, like again if we take the simple example x cube over x square plus y square; so we can just change the polar co ordinate. So, our limit will be the r goes to 0 and here we will have r square cos q sorry r cube cos cube theta over this r square. And then this r square will be cancelled with this r here.

So, you will have r and then limit r goes to 0, so this cos square theta is a bounded function. So, here when r goes to 0 this limit will be 0 so limit is 0. In another example what we have seen this changing to polar coordinate is also useful for determining that the limit does not exist. So, this was the case when limit exist and now in this case when we substitute this again x is equal to r cos theta and y is equal to r sin theta. In this so we will get here r square and here r square cos square theta. So, this r square will get cancelled with this r square and in this case then there is no r left so, this is cos squared theta.

So, this limit is cos square theta and it depends on theta. So, when the limit depends on theta; that means, the limit does not exist the limit should be independent of theta. So, here for different values of theta we are getting the different real number. So, this limit is not unique and hence it does not exist. So, we have seen other examples also that this choosing to polar coordinate is very useful for determining the limit as well as for determining that the limit does not exist.

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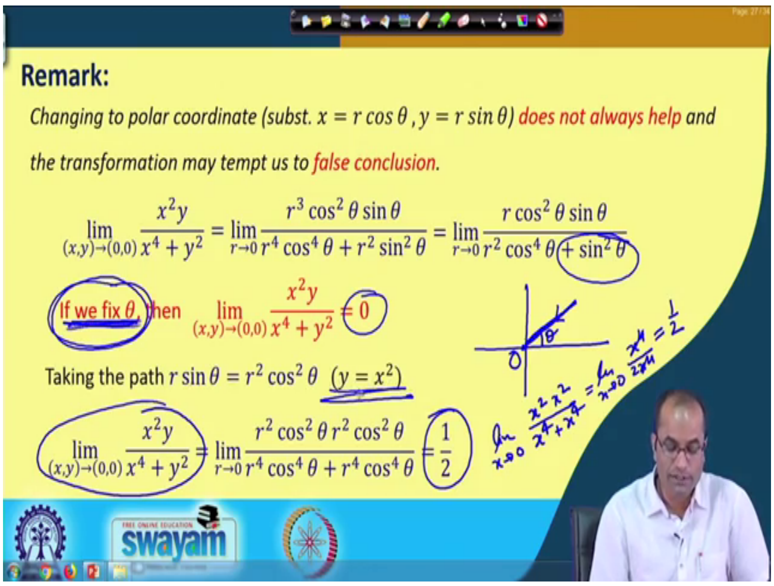

The next remark that changing to this polar coordinate does not always helps, you know this is very important now that it does not always help and the transformation may tempt us to false conclusion we will see some supporting example in this case. So, if we take this x square y over x 4 plus y square, this is example; then if we substitute x is equal to r cos theta and y is equal to r sin theta. So, here we will get because one r square from here r from here. So, we will get r cube, and then cos square theta here sin theta and again here r 4 cos 4 theta and r squared sin a square theta.

So, in this case this r square we can cancel out. So, will get r cos square theta sin theta, and in the denominator we will get r square cos 4 theta plus sin square theta. And now if someone just directly try to put the limit here and think that r goes to 0. So, this term becomes 0 we have something a sin square theta. And then here r cos square theta sin theta when r goes to 0, that this becomes 0; but that is the wrong conclusion.

Indeed, if we fix theta; so if we fix some value of theta, in that case this limit is 0 because, whatever theta including 0 also when theta is 0. So, this will become 0 and then everything will become this 0, so limit will be 0 in that case also. If we take any other theta so, some number will be sitting here and then r goes to 0. So, we can conclude that this is 0, no doubt about it. But the point to be noted here that we have we have fixed theta. So, for fix value of theta, this is 0, but as for the limit we have already discussed that we should not fix theta to conclude that this is the limit.

Fixing theta means fixing the path. So, in some particular path this limit is 0; where we are fixing the theta it is like the straight line in the case of the straight line when we are fixing the theta, we are basically approaching towards. So, if we have this is our r theta plane, and then by fixing theta; so if we have fixed theta and then approaching to origin here. In that case we are basically approaching to origin by the straight line. So, again this fixing theta will not conclude that the limit is 0.

So, for example, if we take another path where y is equal to x square path so, this is again going to 0 to origin. And in this case if you choose this particular path y is equal to x square or in polar coordinate r sin theta is equal to r square cos square theta, then this limit will become r goes to 0 and here this x square r square cos square theta and then this r sin theta again we have used this r square cos square theta.

Same thing here r a square cos 4 theta and this r sin theta is replaced with r 4 cos 4 theta. So now, what are we getting here it is a r 4 cos square theta and again cos square theta and cos square. So, we are getting here r power 4 cos 4 theta and here we are getting 2 times r 4 and cos 4 theta. So, this will get cancel and we will get this limit half or directly one can see just substituting y is equal to x square there taking this path.

So, we will get that limit the desired limit as x goes to 0 and then x square. So, y is replaced by x square and then we have x 4 and plus y square again here x 4. So, again this x 4 and so, limit x goes to 0 and we have this x power 4 divided by 2 x 4. So, x 4 get cancel and then we get this limit half. So, this is the limit in this case, along this particular path while by not closely looking at this here we could have done this mistake by putting taking this limit here as r goes to 0 and then this limit is 0.

But this is not the case. As we have seen that this is indeed 0, but in that case we have to fix this theta and then the limit is 0. And we have any way seen here by taking this path the limit is half. So, limit and does not exist in this case.

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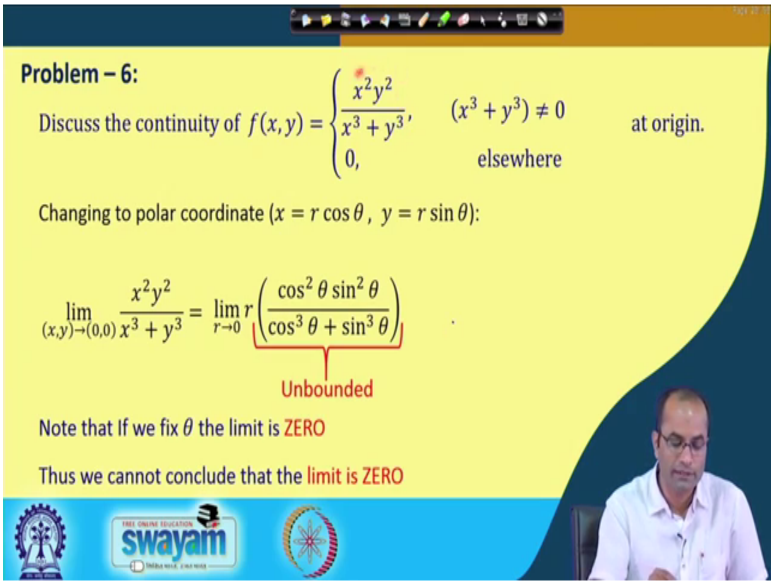

And we will discuss now the continuity following by that motivation the continuity of this x square y square over x cube plus y cube. And in this case if we change the coordinate to this polar coordinate what will happen now? So, this is r squared so r 4 will come and cos square theta sin square theta and from here r cube will come. So, we have one r in the numerator and then cos cube theta and sin cube theta.

So, again when r goes to 0, we should not do that mistake that r goes to 0 and something is here; so it is it becomes 0, no. We have to closely look at this function whether if it is a bounded function of theta sitting here and then we are taking this limit r goes to 0, then this will be 0. But in this case this function here for example, this cos cube theta minus sin cube theta may become unbounded because this may go to very close to 0 and this is actually unbounded function so this is not a bounded function. So, we cannot use that fact that this limit as r goes to 0 is 0.

So, if we fix theta, if we fix theta, then again like in the previous example this limit is 0. Thus, we cannot conclude that the limit is 0. So, we will see in this in the next slide now again that limit in this case does not exist.

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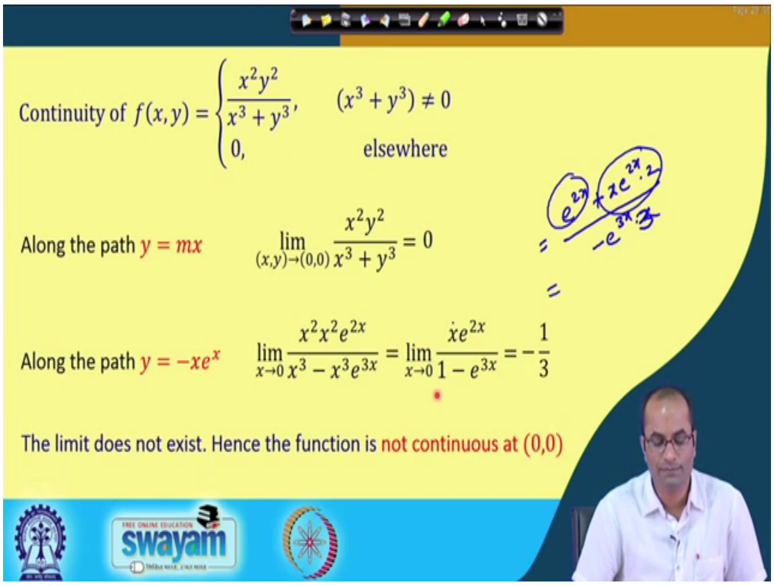

So, if we look at the continuity of this function; so along this path y is equal to mx or fixing theta it is the same thing. So, you will get this limit as 0. So, we take a y is equal to mx or changing to polar co ordinate and fixing theta as we discussed both are equivalent. So, here this limit is 0 along this path, but if we take this special path y is equal to minus x e power x, then we are getting here x square and for y square again x power e power 2 x and then x cube and this y cube will be x cube with minus sign and e power 3 x.

So, now in this case this x cube will get cancel. So, we have in the numerator x e power 2 x and denominator 1 minus e power 3 x. So, what is the limit here? X goes to 0 so it is a

0 by 0 case. So, we can use the L Hopital rule. So, this limit will be equal to; so e power 2 x plus x as it is e power 2 x. And then 2; so this is a differentiation there and then minus e power 3 x into 3 into 3.

So, when x goes to 0 this goes to 0 so, we have 1 over 3. So, minus 1 over 3 is the limit of this function along this particular path again. This path is going to 0 so, we have taken a very special path to show this is a very typical example and difficult to realize that a long this path we will get a different limit. But this proves that the limit does not exist and hence that function is not continuous at 0 0. The last example, we will discuss here the continuity of this function at origin.

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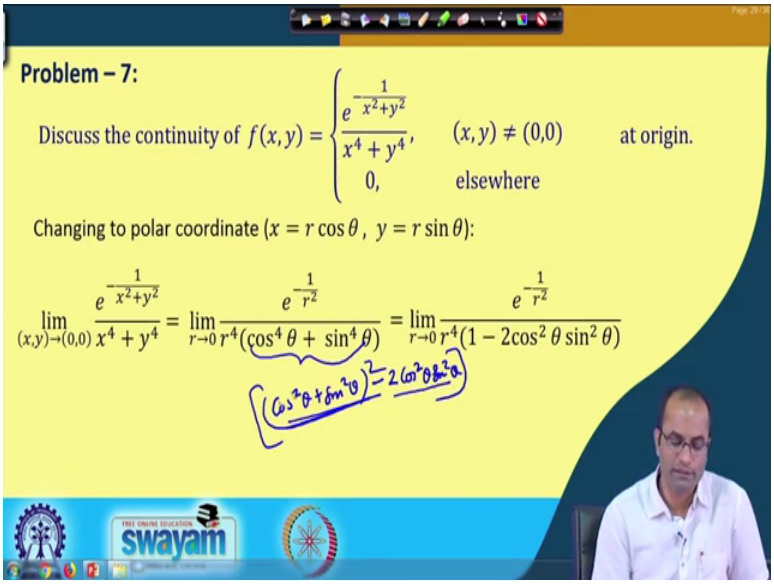

So, again we change the to the polar coordinate, it will simplify this x square plus y square term and we have x y goes to 0 0 e power minus 1 over x square plus y square x 4 plus y 4. So, changing to polar coordinate it will be like e power minus 1 over r square.

And this here this will be r 4 and cos 4 theta plus sin 4 theta because of this term. So now, here we can do little bit manipulations. So, here cos 4 theta plus sin 4 theta we can just a make 4 a square term here so, the cos square theta plus sin square theta whole square. So, the 2 times cos square theta will come and then minus we can subtract that cos square theta plus into sin square theta. So, this term is equal to this term. The whole square minus 2 times cos square theta sin square theta. Because when we open this square we will get cos 4 theta plus sin 4 theta and 2 times the product which is it will be cancelled by this one, so we will get this term. So, that sin square theta plus cos square theta became 1 here, and then we have this 1 minus this term.

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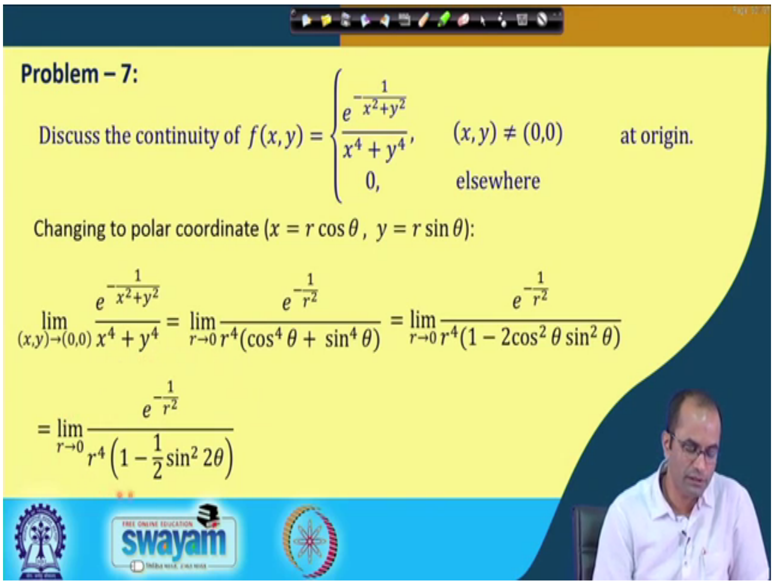

Which again we can write down here 2 times cos theta sin theta as sin 2 theta. So, this is square because of the square this square will appear here, and we have to make this 4. So, we multiplied and divided by 2. We have this term now so we have to see what is this limit here of this function.

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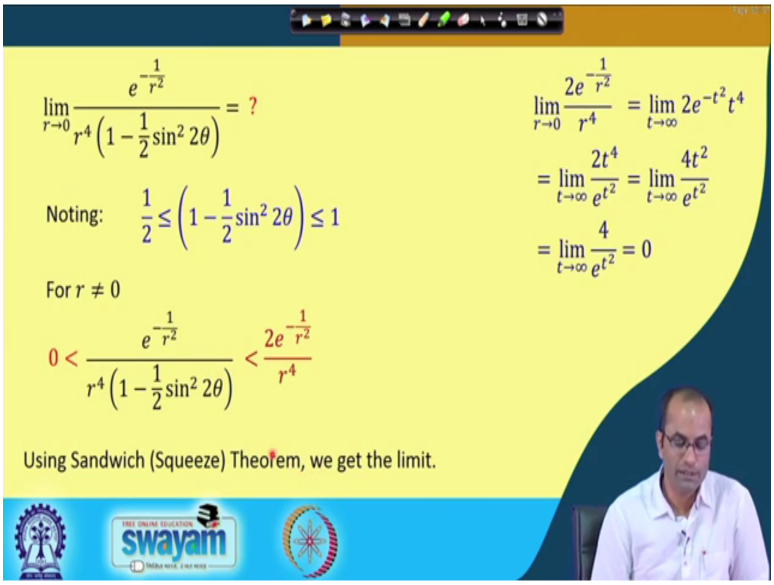

So, what is the limit? So, in this case we have to note now, that this 1 minus half sin square 2 theta this term here lies between half and 1. So, this is like 1 when theta is 0 and then this half and the sin square theta is 1. So, this will become as half, so this lies between half to a 1. So, in this case, then 4 r not equal to 0 we can estimate this term the whole term there 1 over r square and 1 minus this one.

So, this is a positive term, this is a positive term and here also we have this is greater than half. So, certainly this is greater than 0 and also now because we know that this is always greater than half. So, we can replace this half also and get the upper bound for this term. So, replacing this by half we will get this 2 e power minus 1 over r square over r 4. So, this time here lies between 0 and this term. And now for this we can look at what is the limit of this as r goes to 0 of this 2 e power 1 over r square r 4, which we have this used another variable t here, which goes to infinity; because 1 over r we are substituting as t.

So, if r goes to 0 t will goes to infinity. So, here we have 2 e power minus t square and then this 1 over r 4 will become t 4 and this we can now handle easily. So, this is 2 t power 4 and e power t square. So, infinity by infinity case, and then we can use L Hopital rule with to get this limit as 0. So, this limit is 0 so, what we see that this from here which we want to get the limit as r goes to 0. This is between 0 and something here which again goes to 0 as r goes to 0. So, by this squeeze theorem or sandwich theorem, we can get now the limit this as because this is 0 and here also in the limiting scenario this is also going to 0 so, this limit has to be 0.

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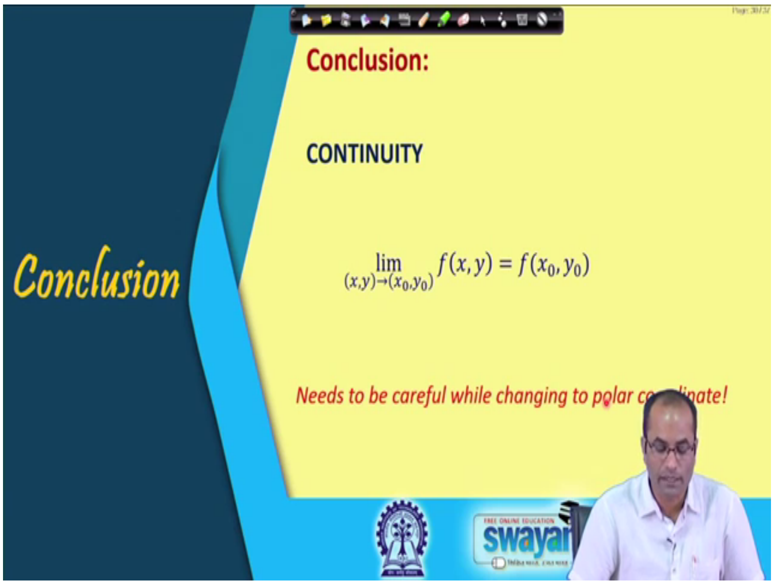

So, in this case now we can conclude this. So, we have discuss the continuity; that f x y is equal to the limiting value here is equal to the function value of the of that function, then we call that the function is continuous. And naturally the function must be defined at this point, then only we can talk about the continuity. And what we have also observed that we need to be very careful while changing to polar coordinate, because that may mislead to some wrong limit in min many cases.

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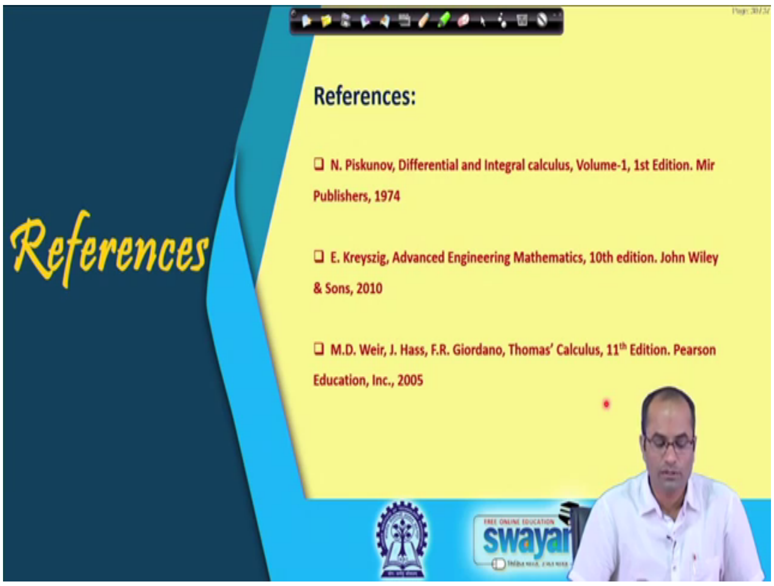

So, these are references we have used to prepare these lectures.

And thank you very much.