Engineering Mathematics - I

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Lecture - 07
Evaluation of Limit of Functions of Two Variables

So, welcome back to the lectures on Engineering Mathematics I and today, we will be talking about Evaluation of Limit of Functions of two variables.

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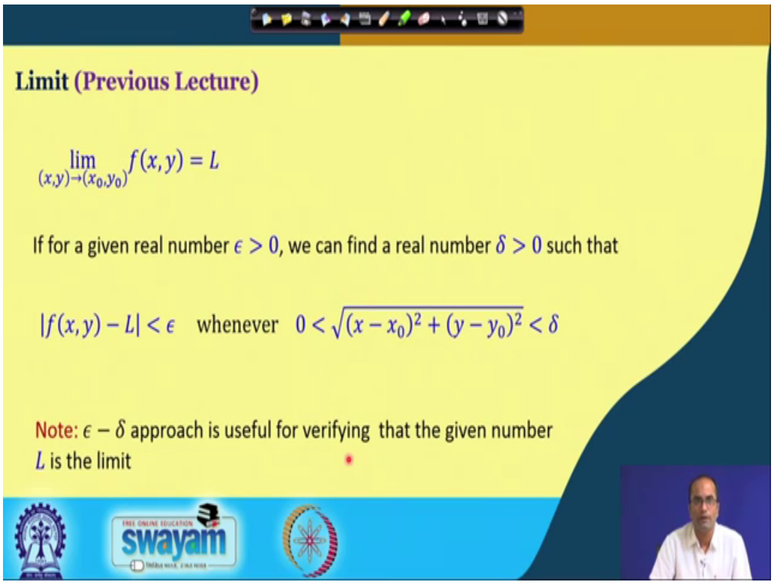

So, in the previous lecture what we have seen that this limit f x y as x y goes to x 0 y 0 is equal to L. It was defined using this epsilon delta approach that means, if for a given real number epsilon positive, we can find a real number delta positive such that f x minus L less than epsilon.

So, the difference between the 2 here f x y and minus its limiting value L is less than epsilon for a given number for an arbitrary epsilon whenever this x y is in the delta neighborhood of x 0 y 0 point and we have seen that this approach is useful in verifying that the given number L is the limit. But we have to guess that what is the limit L. So, in today's lecture we will look for more practical issues that how to get this number L and the epsilon delta approach may be used to verify that this given number is L.

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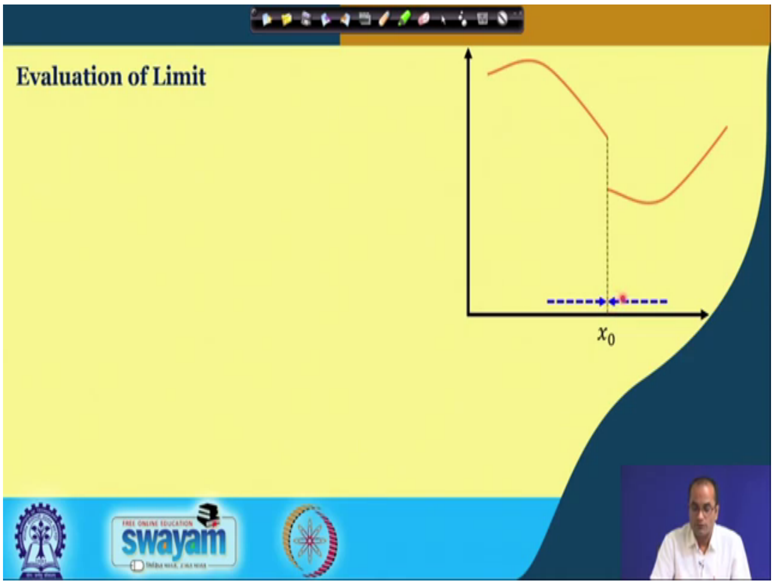

So, while talking the limit for a function of single variable, we had 2 paths to approach a particular point here x 0 on x axis like in this case. So, we can approach from the right side to this point or we can approach from the left side to this point x 0 in case of a functions of single variable.

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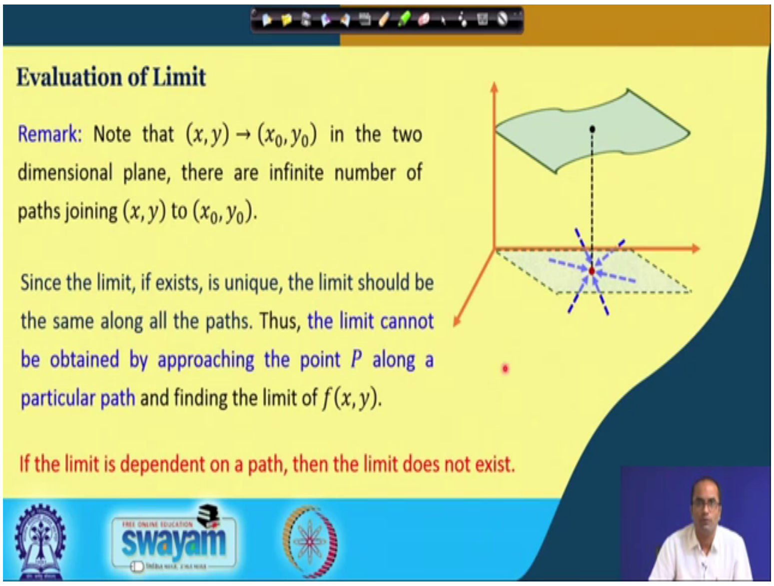

But now, we have functions of 2 or more variables, in this case suppose here this is x axis, y axis and z axis. So, in the x-y plane, we have this domain of this function. Suppose we want to approach to this point P here, then there are several paths to approach this point. We can approach from any direction and using any path to this point to get the limit as x y approaches to this particular point P.

While in the case of single variable, we had only 2 paths. So, we used to check the limit from the right side and limit from the left side and if they are equal, then we say that the limit exist and limit is equal to that value or if those 2 limits are different then, we call that the limit does not exist. In this case we have infinitely many paths a ways to approach to a particular point. So, in that case it is difficult to find a limit along some particular path and saying that this is the limit.

So, as I have remarked here say note that this x y goes to x 0 y 0 point in two dimensional plane, there are infinite number of paths joining x y to this particular point x 0 y 0. Since, the limit if exist is unique the limit should be same along all the paths. That means, the limit cannot be obtained by approaching the point P along a particular path and finding the limit of the function f x y as x y goes to x 0 y 0. However, if the limit is dependent on the path, so if we find along 2 different paths that the limit is different; then, at least we can conclude that limit does not exist.

But getting the limit along 2 or many different paths though that limit may be the same, but we cannot conclude that that particular number is the limit. Because there maybe some other path where the limit may be different. So, concluding that the limit along some particular path is not possible, but what we can conclude that if we find 2 different paths and along those paths, the limit is different then we can say that the limit does not exist. So, now, we will see some possibilities finding the limit of a function of f x y as x y approaches to some particular point.

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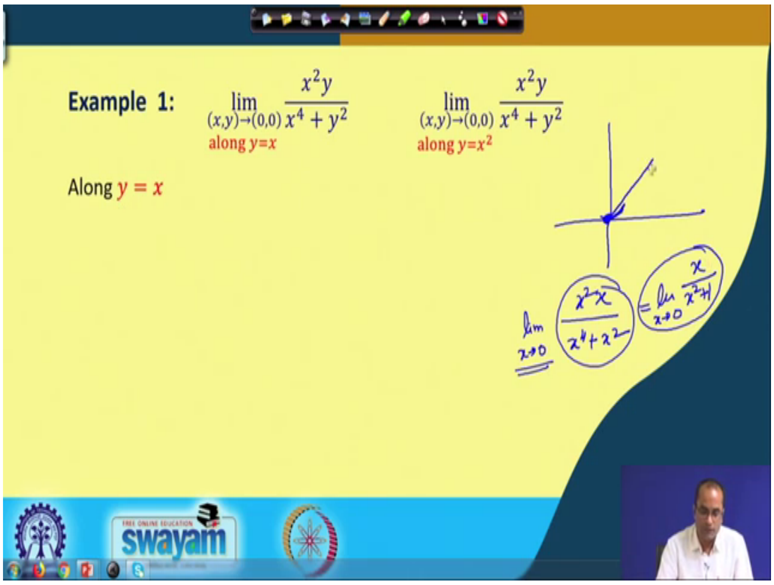

For example, consider this example x square y over x 4 plus y square and x y goes to 0 0 along y is equal to x path and second we will take again the same function and x y goes to 0 0 along this path y is equal to x square. So, we have chosen 2 different paths to approach this origin here. One the straight line y is equal to x and the second, we have taken this y is equal to x square paths.

So, here how what will be the limit of this function as we approach to 0 0 point along this y is equal to x square or y is equal to x 2 different path. So, if we take along y is equal to x here, we are approaching to the origin and in this case so when y is equal to x. So, we will put here x square and then, y will be x again.

So, we have here x power 4 plus x square and now, we can approach as x goes to 0 because along this path now we have restricted this y is equal to x or we have substituted here y is equal to x. So, now, we can take that x goes to 0, what will happen to this limit. So, here the limit x goes to 0 and this is x square we can cancel out. So, you have still x over x square plus 1 and then, this limit will go to 0. So, along this path y is equal to x along this straight line which is approaching to the origin, we are getting this limit as 0.

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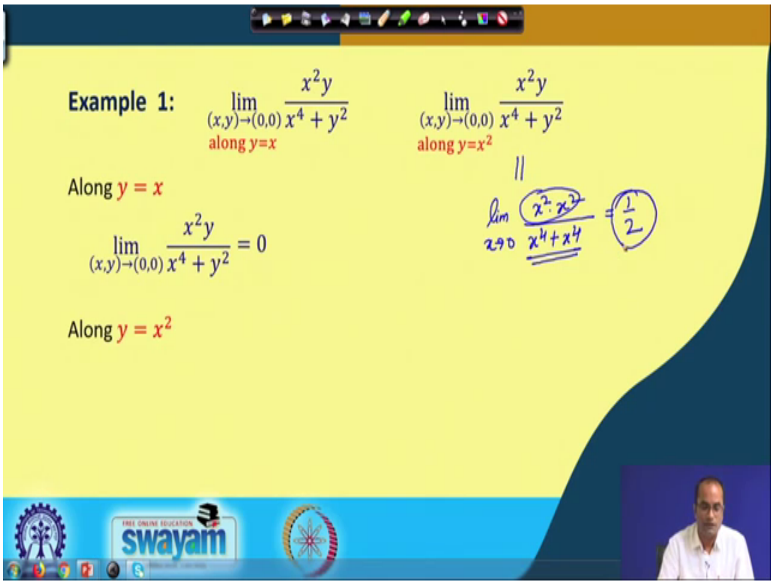

Now, if we take y is equal to x square. So, along this path we have this limit the given limit as the limit x goes to 0. Again, this path y is equal to x square is also approaching to 0, but with this parabolic path. So, here we have x is equal to 0 and x square the y is again x square. So, we have x power 4 and y square is x power 4. So, in this case what we observed because here x power 4 and then, you have 2 x power 4 here. So, this limit will be just half.

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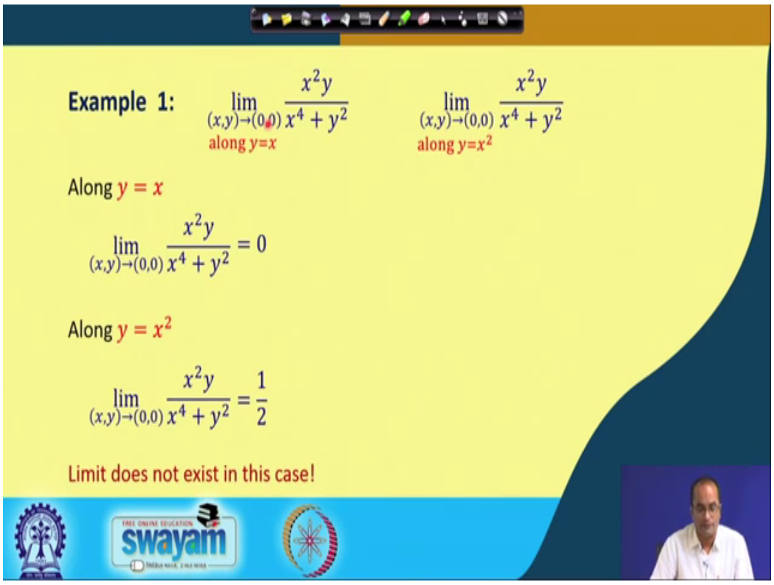

So, along this path we are getting the limit half; along the straight line, y is equal to x, the limit is 0; along y is equal to x square, the limit is half. So, now, we can conclude that this limit as x y goes to 0 0 x square y over x 4 plus y square does not exist.

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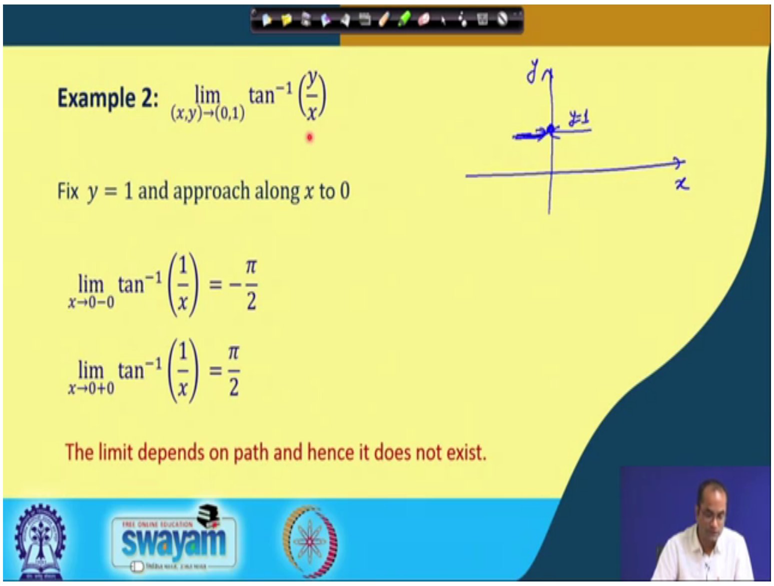

The second example of this kind, we will take limit x y goes to 0 1 tan inverse y over x. So, in this case we have again this x-y plane and we are approaching to the point 0 1. So, x 0 y 1 so, somewhere here, we will take 2 paths again. We can fix this y and we will take these 2 paths as x approaches to this point from the right side or from the left side. So, along this path we will fix y as 1 and then, we will take or we will evaluate this limit along these 2 particular paths. So, in this case if we fix y is equal to 1 and then, approach x to 0 from this 2 directions. So, if we go from the left hand side as x goes to 0 tan inverse 1 over x.

So, you note that if x is negative. So, all these numbers here 1 over x will be negative. So, as x approaches to infinity. This 1 over x will approach to minus infinity. So, so tan inverse minus infinity will get minus pi by 2. Similarly, if we approach this point from the right side of this 0 1 point, then in this case this will become infinity and the tan inverse infinity.

So, we will get limit as pi by 2. So, I will again in this example we have seen that along 2 different paths, we have 2 different limits. So, what we can conclude that again the limit the given limit here x y goes to 0 1 tan inverse y over x does not exist because this limit depends on the path.

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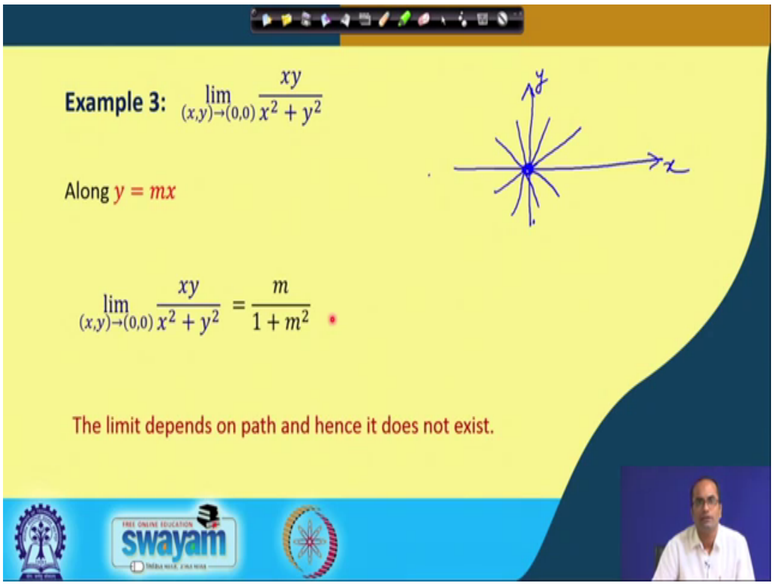

So, if you take this next example x y over x square plus y square and x y goes to 0 0, in this case we are approaching to the origin and the function is x y over x square plus y square. So, we take a general path y is equal to mx; that means, here y is equal to mx with varying m. So, we are approaching to this point along different straight lines. So, the slope of these straight lines are different. So, m can take any value and then, we are approaching to the point here the 0 0 along these straight lines. So, in this case what will be this limit here?

So, y we can substitute as earlier, y is equal to mx. So, here we will have m x square and then x square plus m square x square and x square will get cancelled and we will get m over 1 plus m square. So, what we observe again that for different values of m, we are getting a different limit. Therefore, the limit depends on the path and again, this limit does not exist.

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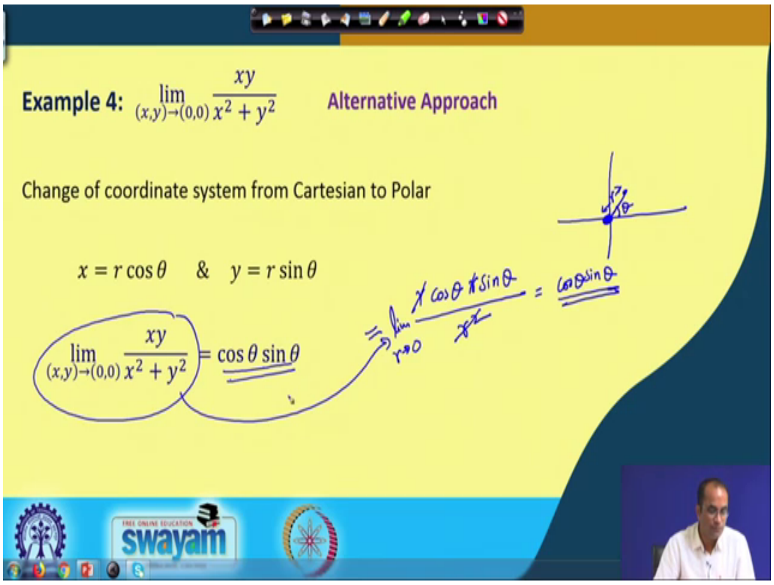

An alternative approach to compute such limit could be using a different coordinate system and we can use the change of coordinate system from Cartesian to Polar coordinates. So, in this case we know the transformation that is x is equal to r cos theta and y is equal to r sin theta.

So, in this case we are will be now working on different coordinate system and in polar coordinate. So, this limit here x y goes to 0 0 x y over x square y square will be given as sin theta cos theta because if we change here x, we substitute as r cos theta and y as r sin theta. And then, x square plus y square will become r square and this limit here x y goes to 0 0 will be the limit as r goes to 0. Because in the polar coordinate, now we have this r and this is theta and this is r. So, independent of theta, we if we approach r to 0; we will approach to infinite to this origin. So, here this limit x y goes to 0 0 will be equal to this limit r goes to 0.

And r cos theta r sin theta over r square and this r gets cancelled. So, we get this cos theta and sin theta. So, what we observe again that this limit is depend on is depending on theta. So, if we choose a different angle to approach to this limit or to this approach to this point here the origin then the value will be different.

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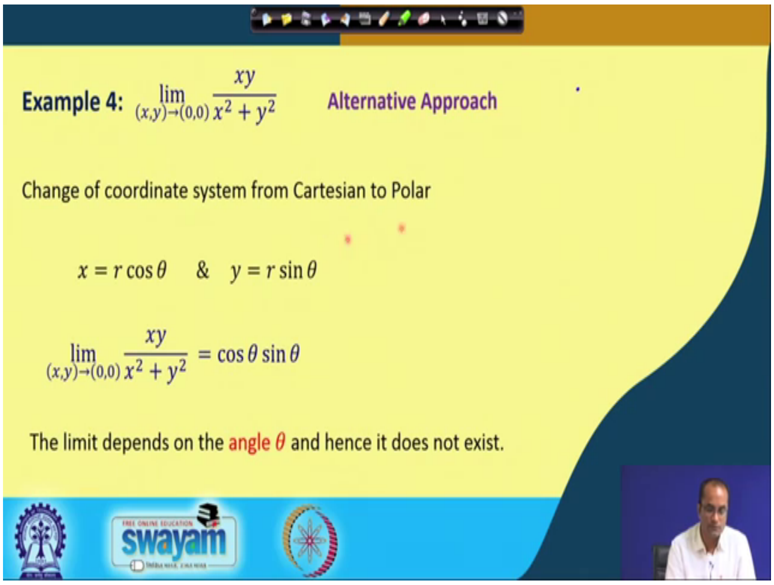

So, again the similar situation that the limit depends on the angle theta or it depends on the path, we can say and hence this does not exist. But what we will realize in today's lecture that this change of coordinate system in many cases is very helpful.

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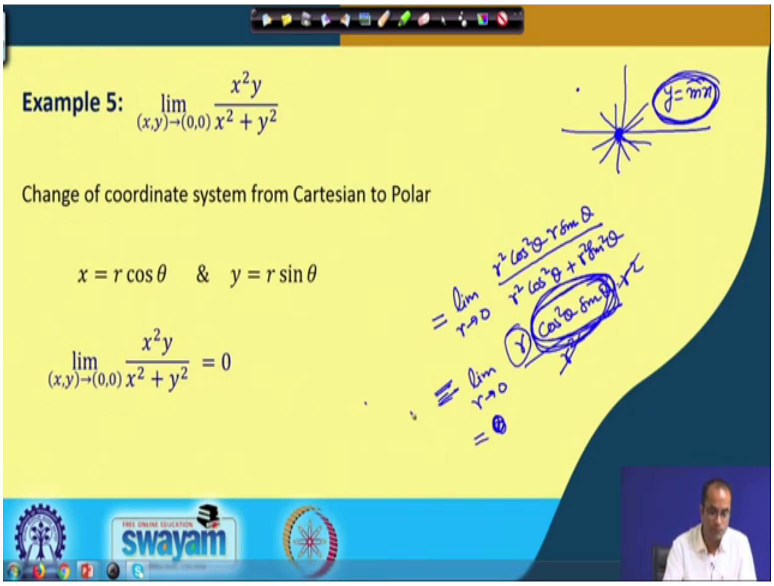

For example, we consider this problem x square y over x square plus y square and if we change the coordinate system to the Polar, then by the substitution as earlier, we can write down this limit x y goes to 0 0; x square y over x square y square is equal to. So, the limit again r goes to 0 and x square as r square cos square theta y is r sin theta and we have here x square again r square cos square theta plus r square sin square theta.

This will become r square. So, we have limit r goes to 0. So, this is r; the r square will get cancelled. And then, we have cos square theta and sin theta and divided by r square which is already cancelled. So, r square and then here r square. So, these cancel out and r goes to 0. So, independent of theta, we are not fixing theta here because whatever the value of theta is we have a finite number here and then, r goes to 0 then this will become 0. So, in this case this limit here is 0.

So, by changing to the coordinate system, the evaluation was easy and we could conclude now that the limit is 0. We have again note that we have not fixed the path because this limit here, we got independently of theta. So, without fixing the value of the theta, we have approach to this 0. So, this is this limit is not along a particular path. This is independent of the path. So, that that is what I said that thus changing the coordinate system is very helpful.

For example, we have seen in this particular case, one might again get confused that approaching to this point along y is equal to mx line will also give limit as 0, but here we have fixed the path. So, y is equal to mx these are the straight lines approaching to 0 0 point. So, by doing so y is equal to mx and letting this x goes to 0, we cannot conclude that the given number or the given limit is the limit of the of the function.

But in this case by changing the coordinate system. So, now, we have a different coordinate system equivalent coordinate system where we have not fixed anything. So, here if it is independent of theta we are approaching to some number then that would be the limit we will explore this in some more example.

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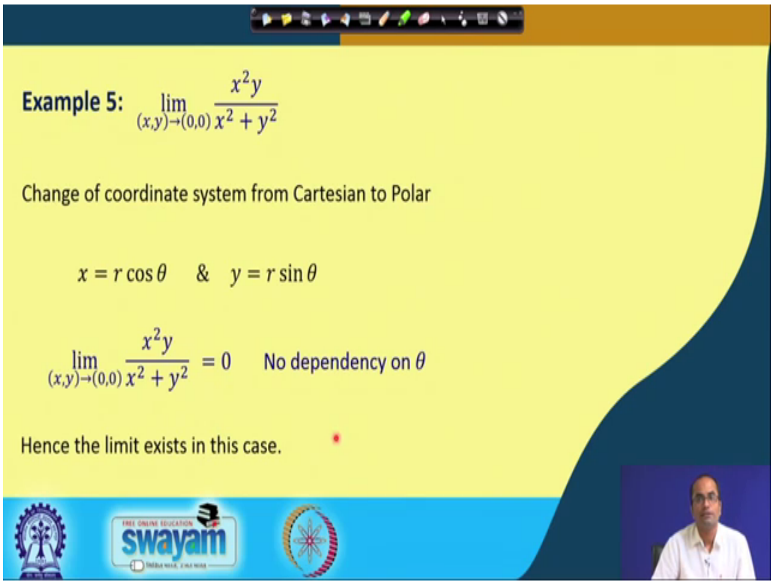

So, no dependency on theta. Therefore, in this case the limit exists and the limit is equal to 0.

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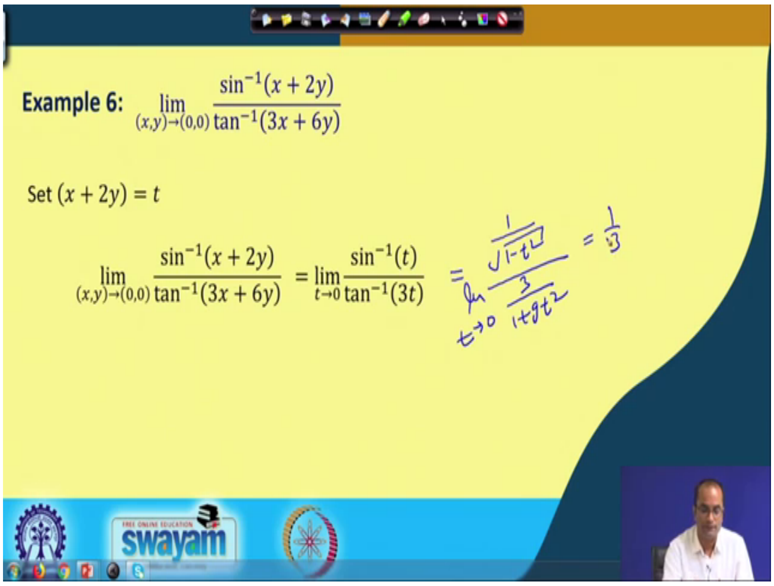

So, here sin inverse x plus 2 y and 3 x plus 6 y. So, this is another approach without changing to the coordinate system. We can observe here that this is x plus 2 y and here also the tan inverse the argument again it is 3 times x plus 2 y. So, we have the same number x plus 2 y and here also 3 times x plus 2 y. So, if we substitute x plus 2 y as a new parameter, as a new variable t. Then, we can convert this limit to the limit sin inverse. So, this is substituted as t and then, tan inverse three times t and the limit t approaches to 0.

So, this is another convenient way without fixing the path. So, we are not fixing again the path, but we have substituted this relation here x plus 2 y which appear everywhere in this function. So, here it was x plus 2 y and again same functionality, but with the 3 times x plus 2 y which we have replaced by t. So, it becomes 3 times t and now, we have changed the problem to the problem of limits of single variable which we can compute. So, in this case it is like 0 by 0.

So, we can apply the L'Hospital's rule which says this will be 1 over square root 1 minus t square and then here this will be 3 and 1 plus 9 t square and then, the limit t goes to 0. So, in this case t goes to 0, it will be 1 and then, we have 3 here. So, 1 by 3.

So, this limit will be 1 by 3.

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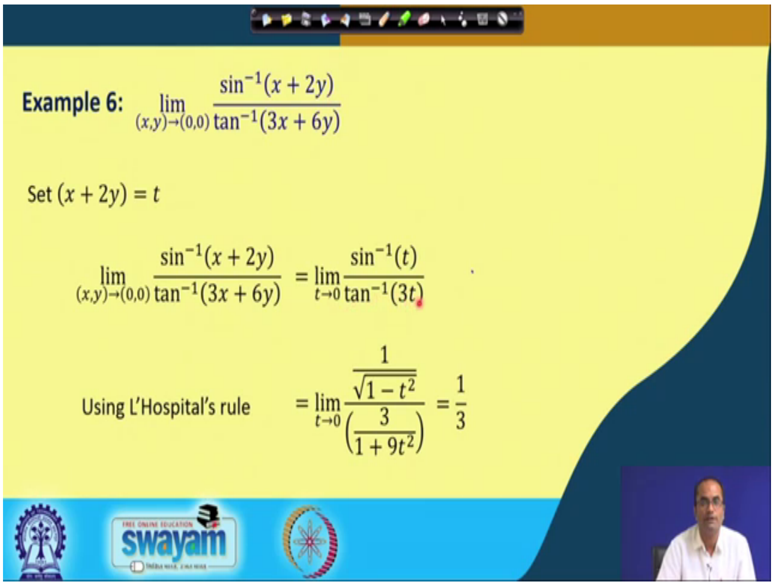

So, again this is another approach which could be helpful in some cases when we can change the functions of two variables to 1 variable.

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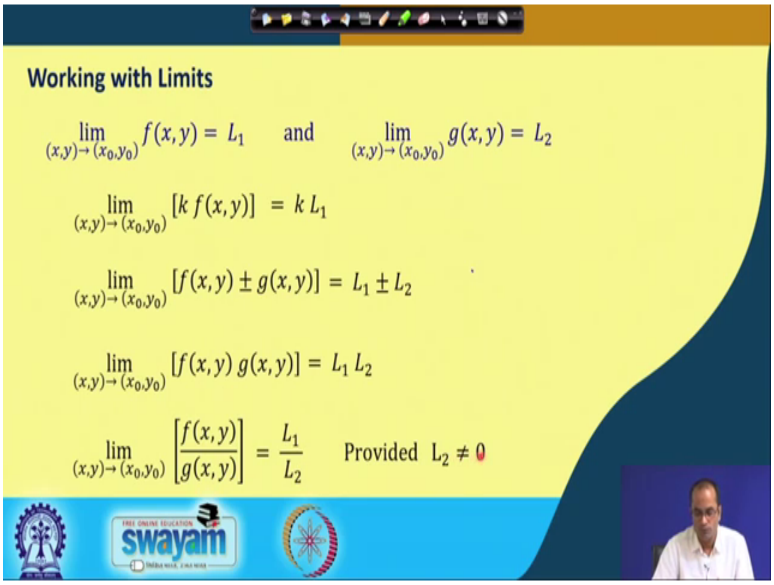

So, now few remarks on working with the limits. So, what we have if the limit f x y goes to L 1 and x y goes to x 0 as L 2; if these 2 limits are given, then we can compute this k times f x y will be the k times the limit of the f x y as x y goes to x 0 y 0. So, this k is a constant. So, this limit will be k times L 1.

Again, when we add this f x y and plus or minus g x y, we have finite quantities there L 1 and L 2, then this limit will be also as L 1, L 2. L 1 plus and minus if it is a plus there here plus will appear; if it is a minus here, then minus L 2 will come. Now, we have the product here of the 2 functions f x y into g x y. So, this will be the product of L 1 and L 2. If we have the quotient here f x y over g x y; in this case if this L 2 is not zero because we cannot divide by 0. So, if L 2 is not zero then this limit will be also as L 1 L 2 provided L 2 is not equal to 0.

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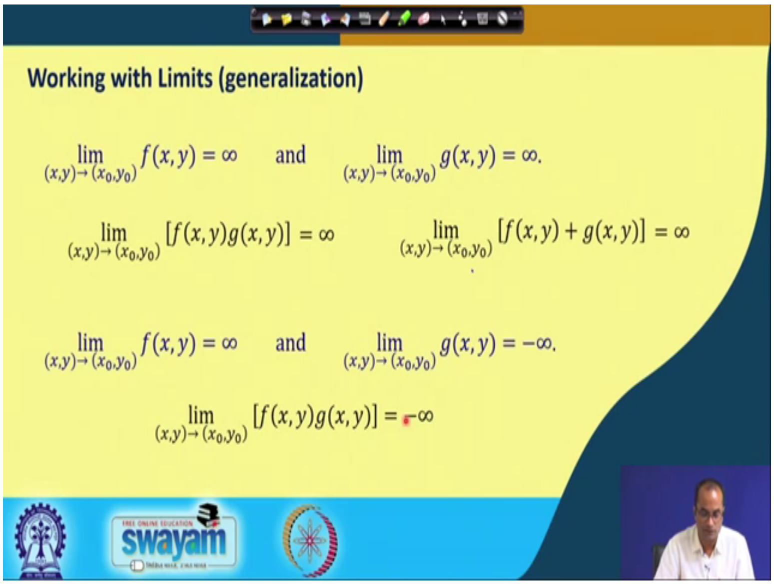

Some more generalization of these working rules, we can also work when we have infinities there. So, if this limit f x y as x y goes to x 0 y 0 is infinity.

And here, the limit of this g x y as x y goes to x 0 y 0 is also infinity. In that case, we can compute this limit of the product of these 2 functions f x y g x y as x y goes to x 0 y 0. So, infinity and here also we have plus infinity plus infinity. So, this limit will be also plus infinity. We can also add the 2 functions and take the limit as x y goes to x 0 y 0 and this will be infinity plus infinity which we know it is the infinity again.

So, here we have the limit plus infinity and the g is approaching to minus infinity as x y goes to x 0 y 0. So, in this case since one is minus infinity another one is plus infinity. So, the product will be infinity again with minus sign because one is negative here, one is plus. So, plus and into minus 1 is minus 1 and then, here we have infinity. So, this will approach to infinity.

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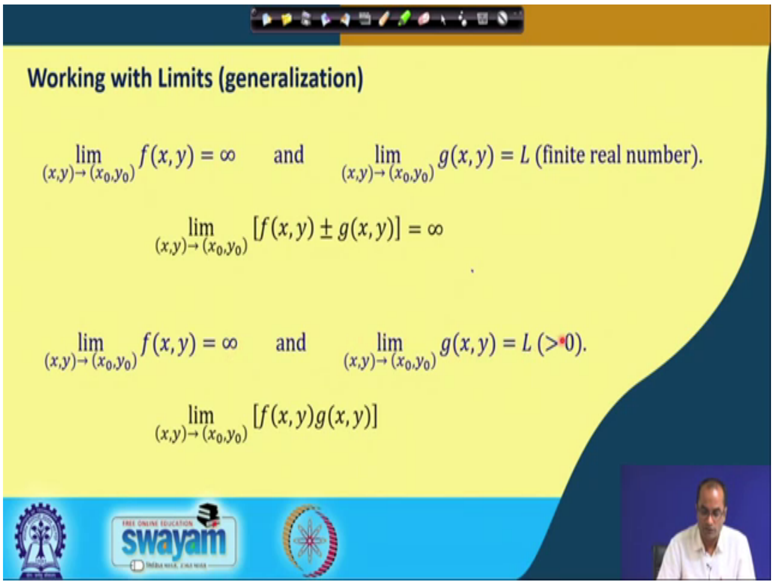

If we have f x y as is going to infinity and g x y is approaching to some finite real number, in this case we can evaluate such limits f x y plus or minus g x y, since this is infinity already and then we have here g xy is equal to L. So, will does not matter what is this number here as long as this is a finite number. Then, this limit will be just infinity.

Second, if we have infinity and then this limit L is positive, in this case we can go for the product rule. So, limit x y goes to x 0 y 0 f x y and g x y and this product will be equal to because this L is positive. So, sign will not change and it will be plus infinity.

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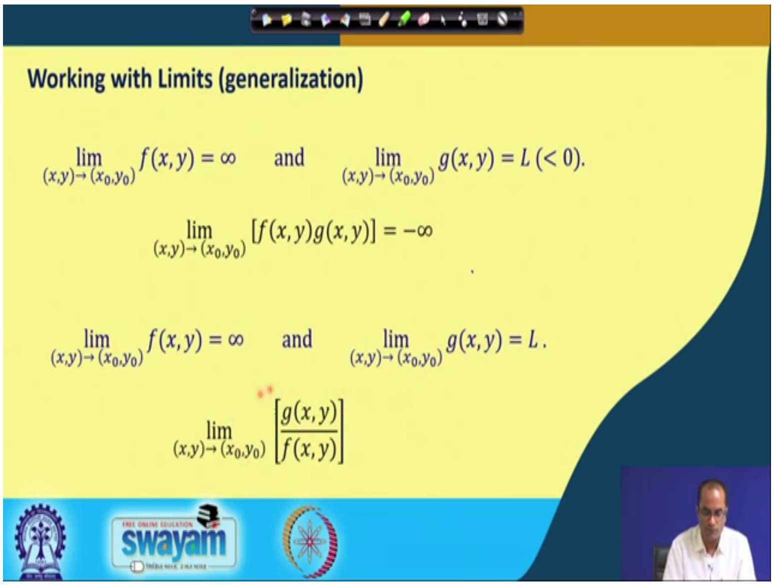

Some more. So, here we have f x y as approaching to infinity and this g x y approaching to L which is a negative number. In that case, this product since f x y is approaching to infinity; the product will go to infinity. But because of this negative L, this will approach to minus infinity.

Now, if f x y approaching to infinity and g x y is a real number, the limit is a real number; in that case the quotient here the g x y over f x y. So, here this f x y is going to infinity and g x y is some number L, in that case this limit will be 0.

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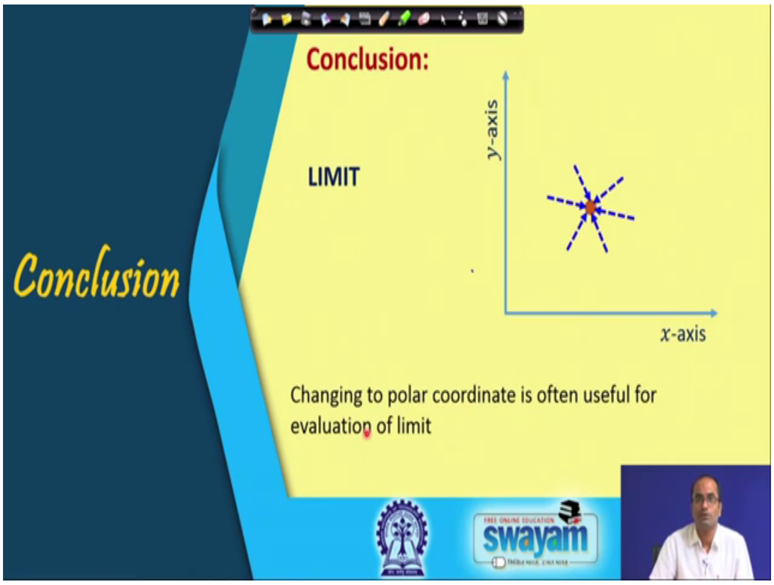

So, what we have seen today that the limit, we can evaluate the limit in 2 in case of 2 variables. But we have to be careful because in the x-y plane now we can approach to a particular point from several directions, while in case of 1 variable we had only 2 directions from the right hand side from the left hand side, but now we have infinitely many directions. So, computing limit along a particular direction will not help, but at least if we find 2 directions where the limits are different, then we can conclude that limit is a limit does not exist.

But we cannot conclude by evaluating the limit along 2 or many paths that this is the limit, even though that limit may be same along several paths. So, computing the limit then what we have seen 2 approaches the one was changing to the polar coordinate would be helpful. In that case if r approaches to 0. For example, if the limit in the question is approaching to the origin. So, in that case we can say if r approaching to 0 independently of theta. So, then whatever number we get that does not depend on the path and that will be the limit.

Another approach we have seen that we can convert in many situation a number into 1 variable problem and then, we can apply L'Hospital' rule for example, to conclude the limit. So, one again we will see more in the next lecture that changing to the Polar coordinate is helpful for evaluating the limit. But we have to be again very careful that while taking the limit as r goes to 0 that should not depend on the angle theta.

So, if the independently of theta we are approaching to some number, we can conclude that that is the limit. But if that limit is depending on theta, then again it is a path dependent limit. And therefore, the limit does not exist. So, we will talk more on these issues that what could be the problem by while changing the coordinate system to polar coordinate and more examples later.

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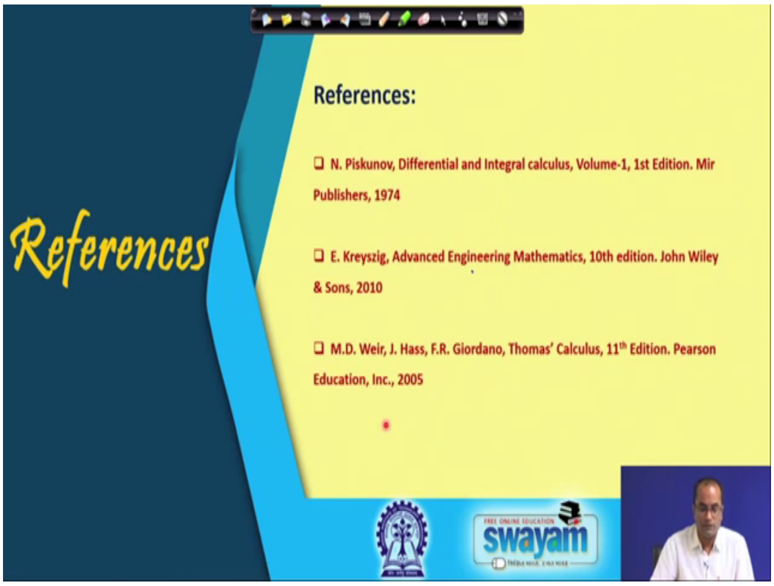

So, these are the references used for preparing these lectures and.

Thank you very much for your attention.