Engineering Mathematics - I
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Lecture – 60
Cauchy – Euler Equations

So, welcome back and this is lecture number 60 we will be talking about a Cauchy - Euler equations.

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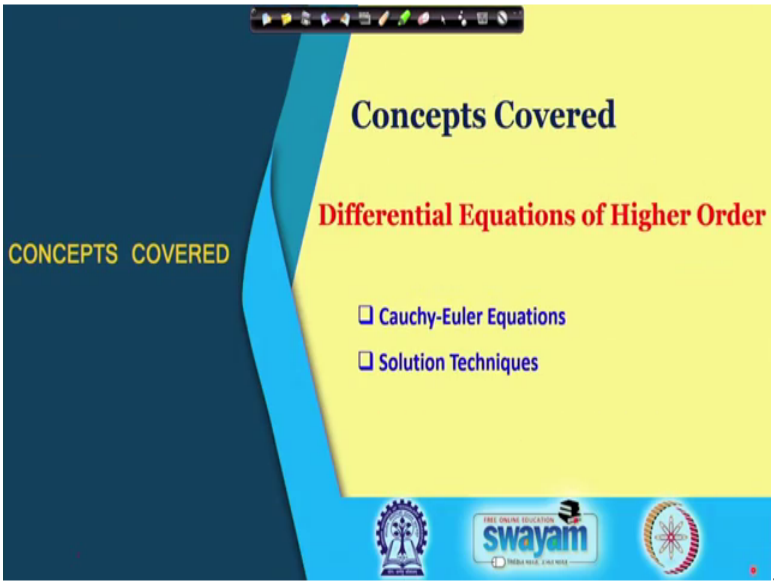

So, we will first introduce what are the Cauchy- Euler questions and then their solution techniques.

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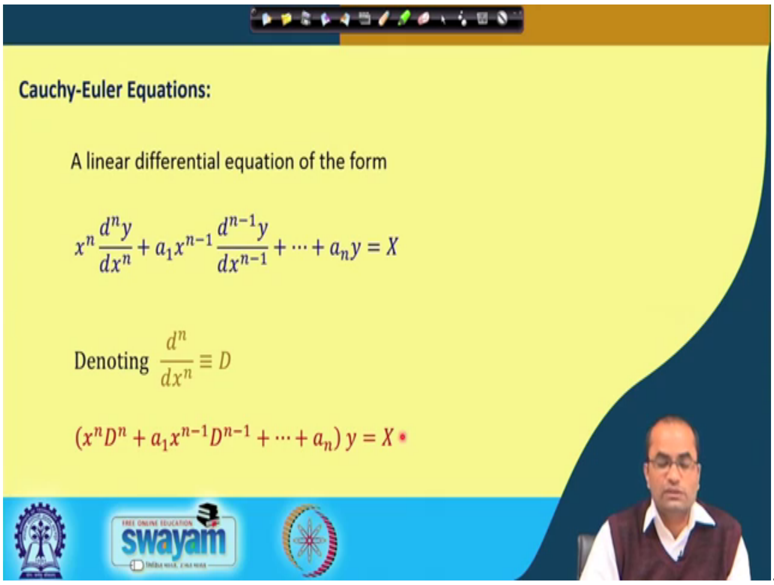

So, here the Cauchy Euler equations are the equations with an on a constant coefficient. So, so far we have learnt linear differential equations with constant coefficients. So, this is one kind of special kind of equation with non constant coefficients. So, like here we see this x power n is coming and then x power n minus 1 is coming with n minus 1 and derivative and so on.

So, here we will consider this equation which has these non constant coefficients. So, this a 1 a 2 an their constant was together with them we have this x power n x power n minus 1 and x power n minus 2 and so on. So, we will today look into that how to solve such equations when we have non constants in as the coefficients of these derivative terms. So, denoting this derivative here the an act derivate as this operator D which usually we do.

So, we can write down this equation in this form x power n D n and then a 1 x power n minus 1 and this operator D n minus 1 I operated on this y is equal to x.

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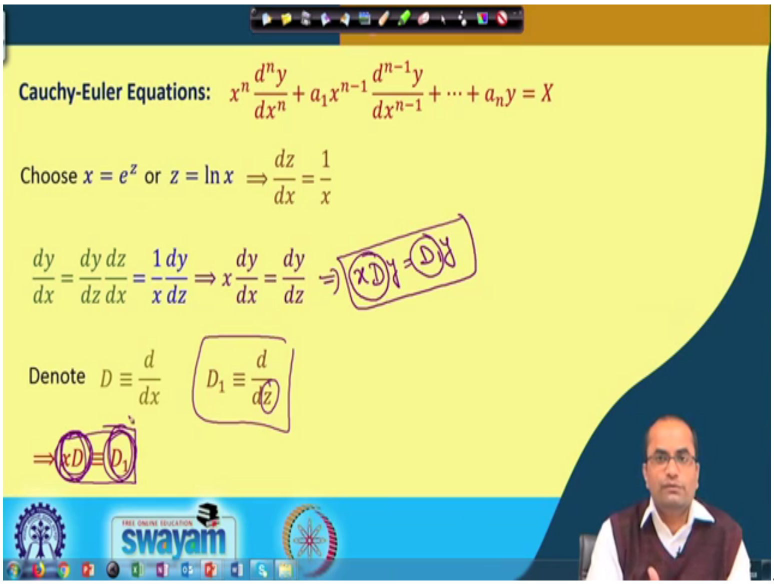

So, now this Cauchy Euler equation now how to solve that what we will do, we will choose this x or we will substitute this x to e power z. So, this independent variable z x in the given equation will be replaced by, will be substituted by, the independent variable z by this relation x is equal to e power z. That is a trick and what this trick will do finally, it will convert the given differential equation into the linear equation with constant coefficient so, that is the idea behind this substitution.

So, once we do this one or we can rewrite this as z is equal to ln x and that gives us that dz over dx because that will be required when we substitute all these derivatives terms in the form of z so, this dz over dx will be one over x. So, that is the one derivative which is required now to change the dy dx to the dy dz form. So, now, the first change here because we need to get all these derivative terms change into the z form. So, this dy dx by this chain rule so, we have dy over dz and dz over dx.

So, dy over dz we would like to have now in the equation and dz over dx is 1 over x. So, we can replace this dz over dx by 1 over x. So, we have this one here 1 over x dy over dz which we can write down if you multiply this x to the whole equation. So, what we are getting that x times dy over dx is equal to dy over dz and that is precisely the point here because in our equation the order of the derivative here there is n minus oneth order derivative then we have x power n minus 1 term, here when the nth order derivative we have x power n term.

So, here with the first order derivative x and the first order derivative is nothing, but the first order derivative of this y with respect to z. So, if we denote here this D as d over dx this differential operator then what will happen that, this x into the d here x into d will be replaced by the second first let me introduce this D 1 here which is the operator d over dz. So, having these 2 operate or the notations here D 1 will be used for d over dz and d the standard notation which we are using from earlier lectures as well that is d over dx. So, in that case this x into the d over dx because this equation says just now what we have in terms of this x D operated on y is equal to D 1 operated on y.

So, from this equation we realize that this x D operator is equivalent to the D 1 operator which is written here that x D is equivalent to this D 1. So, whether we operate x D or we operate D 1 both are the same here and that will help actually to change this equation into the constant coefficients here because all these x D and x square D square which you will see now that will be also converted into D 1 form and without presence of this new independent variable z. So, we will convert the whole equation to this constant coefficient equation.

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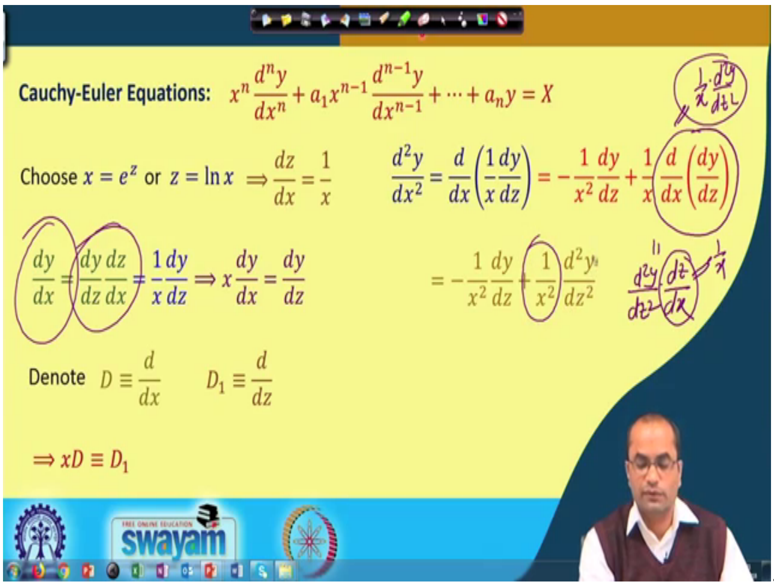

So, now coming back to the second order term how this will look like when converted to the said independent variable.

So, d 2 y over dx means d over dx of this 1 over x dy dz term because this was dy dx. So, d of dx of dy over dx and dy over dx was 1 over x d y over dz. So, we can use this product rule here. So, the 1 over x the derivative will be minus 1 over x square and then this dy over dz will remain as it is plus this 1 over x, now we will differentiate this 1 dy over dz with respect to x. So, that will give the d over dx of dy over dz. So, this operation here will be then minus 1 over x square dy over dz and 1 over x as it is so, the square will come because of this now here.

So, d over dx of this dy over dz will be so, this operation here will be d 2 y over dz 2 and dz over dx as the case we have done before also when we are taking the derivative here dy over d z dz over dx so, here as well. So, the derivative of this dy over dz with respect to z and then dz over dx and dz over dx is nothing, but that is 1 over x. So, here the result of this will be 1 over x into d 2 y over dz square. So, therefore, we call this 1 over x square and d 2 y over dz square.

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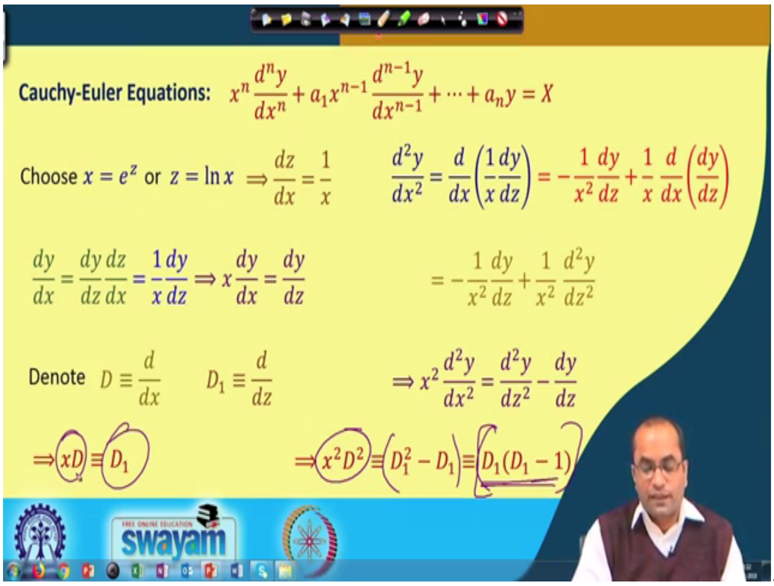

So, what are we getting here this x square when we multiply this x square to each term we are getting x square d 2 y over dx square, the right hand side will be then d 2 y over dz square minus dy over dz and now if we write in this operator form we have x square the d square term there and then here the D 1 square and then minus D 1 x square D square is equal to D 1 square from here and D 1 from here which we can again write down this term as this term is written as D 1 into D 1 minus 1. So, when we have x square D square it is converted to D 1 D 1 minus 1, when we have x D this is converted to simply D 1 and now if we continue this process what we will get.

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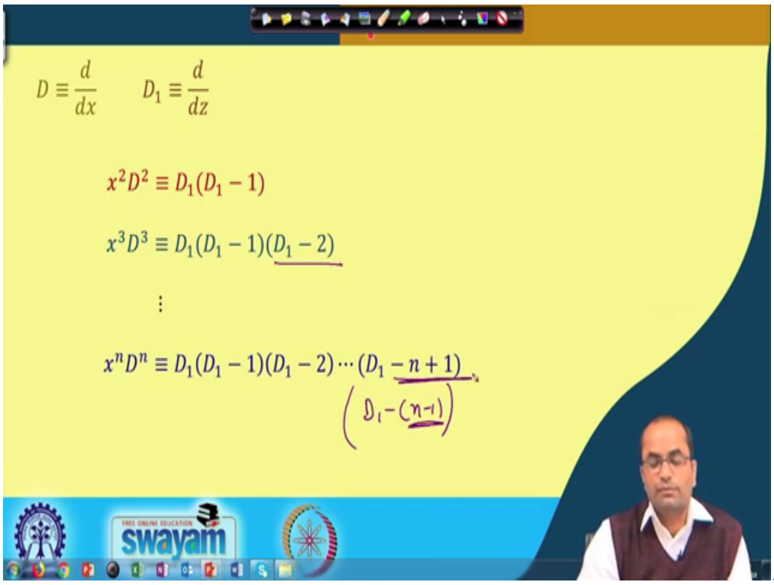

So, in the general forms we can now write introducing these operator here D 1 into implies the d over dx and D implies d over dx and D 1 d over dz. So, this x square D square we have seen that D 1 D 1 minus 1 when we compute x cube D cube what will come D 1 and multiplied by D 1 minus 1 and D 1 minus 2. If we continue this process in general what we are getting x power n D n is D 1 D 1 minus 1 D 1 minus 2 and it will go up to D 1 minus and n minus 1.

So, here when we have power 3 we are going up to 2 here we have power n we will be going up to n minus 1. So, this is D 1 minus n minus 1 which is written here minus n plus 1.

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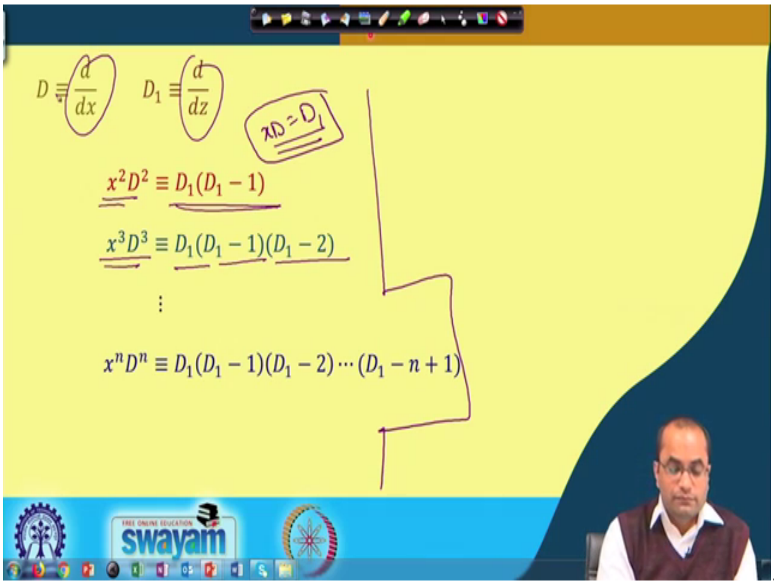

So, these are the general forms and we already remember that this x D was D 1. So, we have x D is equal to D 1 x square D square D 1 D 1 minus 1 x 3 D 3 D 1 D 1 minus 1 and D 1 minus 2. So, once we know that how this D operator will be changed to this D 1 operator the D 1 is with respect to z where D is with respect to x.

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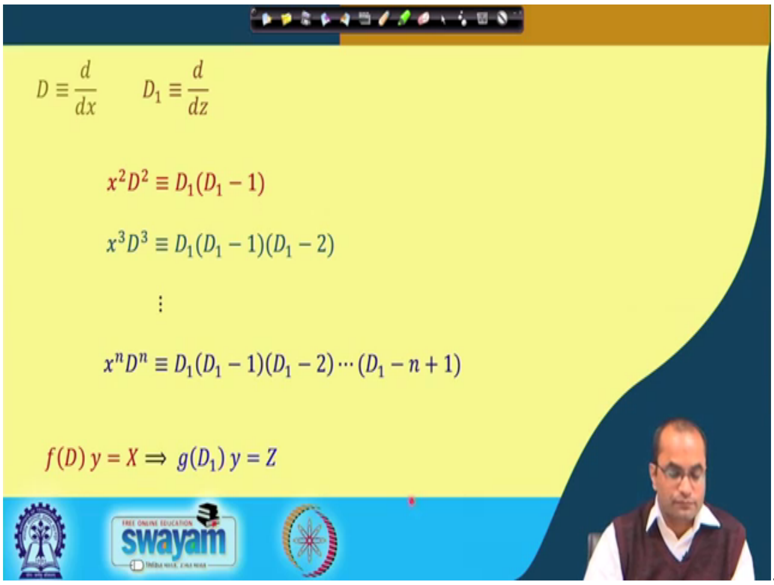

Then what we will realize that our original equation will be converted to this linear equation with constant coefficient, meaning if we have this f D y is equal to X equation with that substitution.

We will get this some function of D 1 y is equal to Z and this g D 1 will be free from this Z meaning there will be constant coefficients only in this equation and we know how to solve equation with constant coefficient.

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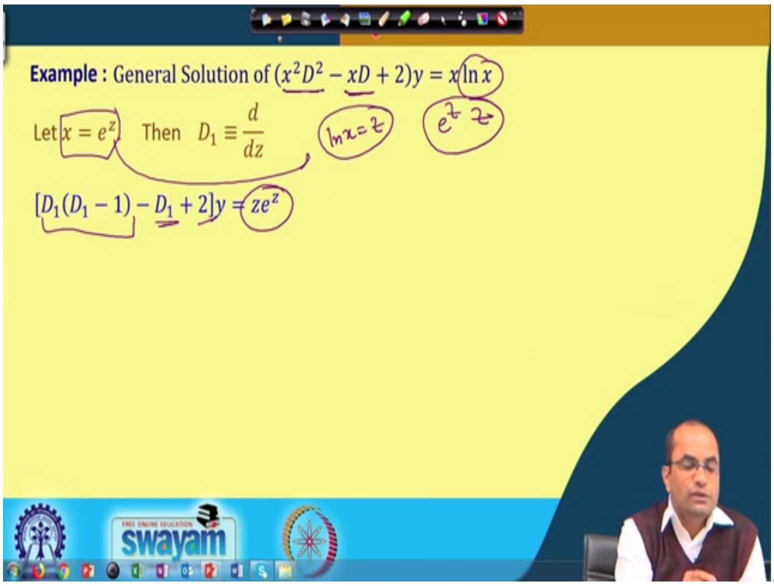

So, let us just go through this example to demonstrate this idea, we have the general solution we want to get the general solution of this x square d square minus x D plus 2 y is equal to this x ln x equation.

So, we will make this substitution x is equal to e power z or that is equal to logarithmic of x. Then we know already that if we have this D 1 as d over dz then this our equation here will be converted because this x square D square term. So, x square D square term will be taken as D 1 D 1 minus 1 the x D term or a will be replaced by D 1 and plus this 2 operated on y is equal to the x here is e power z x here is e power z. So, this is given here and then ln x.

So, from here what we have we have the ln x is equal to z. So, this ln x will be place by z and this x will be replaced by e power z. So, that will be the right hand side here z in to e power z and these operator here with x square D square we have now the new operator D 1 D 1 minus 1 and with this x D we have this D 1 and plus 2. So, our new equation which is in terms of the z now is given here.

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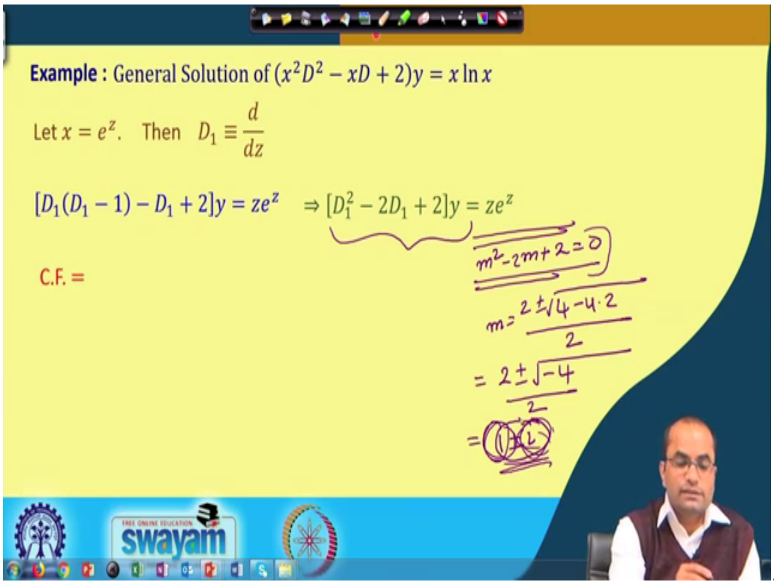

And now we can simplify this so, it is a D 1 square minus D 1 and this minus D 1. So, minus 2 times D 1 and this plus 2 operated on y and this gives us z into e power z.

So, this equation now is with constant coefficients. So, if we look the coefficients here it is 1 here it is minus 2 it is 2. So, all the coefficients here are constant whereas, this in original equation the coefficients were not constant. So, that is the benefit of such transformation that this equation is reduced now to a question with constant coefficients which are easy to which is easy to solve now. So, the complementary function for this equation because we need to write down the auxiliary equation that will be m square minus 2 m and plus 2 is equal to 0.

So, we have this equation here. So, the roots will be minus plus minus in the root b square minus 4 ac and divided by 2 times a. So, we have 2 plus minus this is 4 minus 8 so, minus 4 divided by 2. So, we have 1 plus minus this will be 2 i. So, 2 get cancel then we have 1 plus minus i. So, the roots of the auxiliary equation corresponding to this differential equation we have 1 plus minus i and that we know how to write down the solution. So, this will be e power now not x our independent variable is z.

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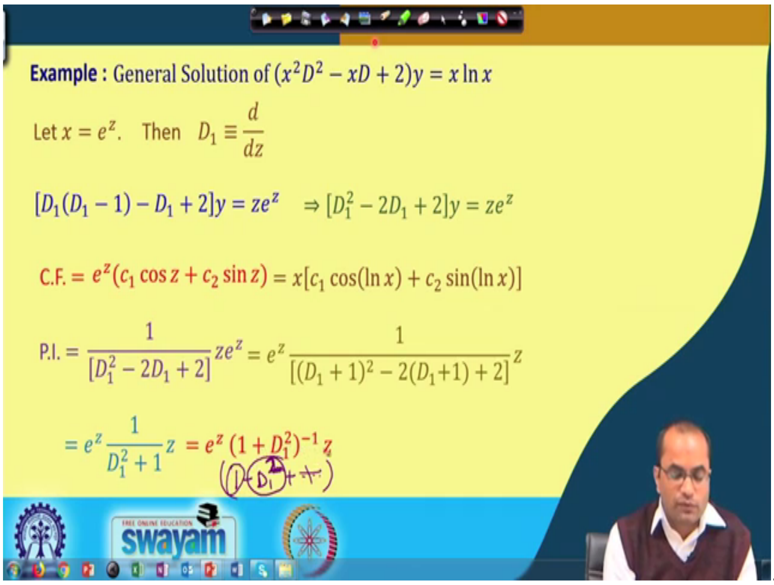

So, this will be e power z and then c 1 cos z plus c 2 sin z e power z, c 1 cos z and plus the c 2 z that is the complementary function of this equation or rather to say the general solution of the homogeneous equation meaning that right hand side is set to 0. So, here this complementary function we have to write down back to the original variables.

So, here e power z will be replaced by x c 1 cos the z was the logarithmic of x. So, cos of ln x plus c 2 sin of ln x and the P I the particular integral will be 1 over this D 1 square minus 2 D 1 plus 2 this particular integral on this z e power z will take us now to this result here, because z e power z the formula which we have already discussed in the previous lecture z into e power z into some function of z. So, this e power z we can take we can bring to the left hand side and then as a result this D 1 will be replaced by this D 1 plus 1. So, that is what we have done here a power z we have taken out of this operator and this D 1 is replaced by D 1 plus 1 D 1 plus 1.

So, that was the direct formula which was discussed already in the previous lecture and now we need to only apply this on this z here. So, let us simplify, but this one first. So, D 1 square and then we will get here 2 D 1 here also we have minus 2 D 1 that will be cancelled out here we have 1 and 2 here so, 3 and minus 2 so, there will be plus 1 there. So, D 1 square and plus 1 operated on this z there. So, e power z and then we can bring into the numerator with this minus 1 which now we can expand it and then we can apply this operator on z.

So, here the power of the z is 1. So, we do not have to write down many terms because D 1 when we expand this 1 here what we will get at 1 m minus this D 1 is square and plus those higher order terms, indeed this D 1 the 2 times when we apply on this z that will set it to 0 and also the higher order terms will make this z to 0. So, you will get basically only this z the result of this operation here.

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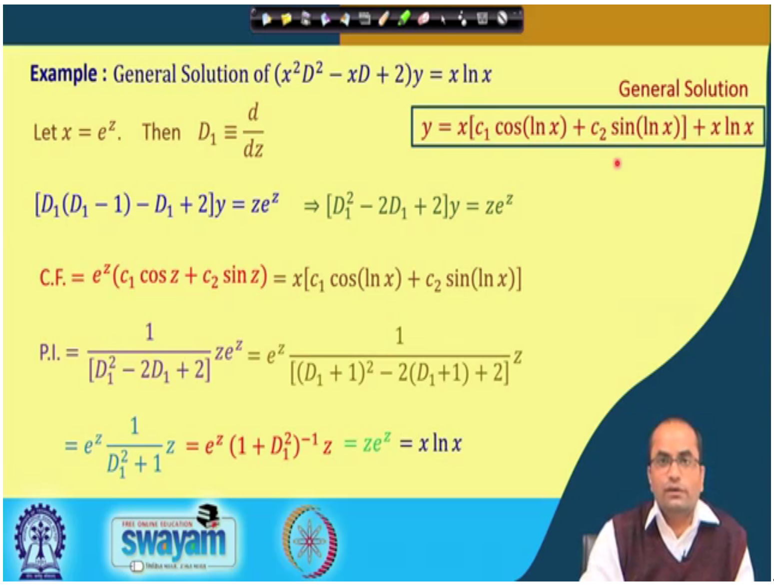

So, this is just the z here e power z so, that is the answer that is a particular integral to this equation to this equation with a constant coefficient. So, what we are getting now this particular integral we have to also write down back to in the x form and this e power z was x and this z was the ln x. So, we our answer is this x ln x the particular integral and this is one particular solution of the given of the given equation and we have the complimentary function we have this particular integral if we add these 2 we will get the general solution of the given differential equation.

So, general solution is given by this y is equal to the complimentary function which is given here x c 1 cos ln x plus c 2 sin ln x and plus this particular integral which is x ln x. So, this will satisfy our given equation it has these 2 constants 2 arbitrary constants. So, this serves as a general solution of the given non homogeneous equation with non constant coefficients or the Cauchy - Euler equation.

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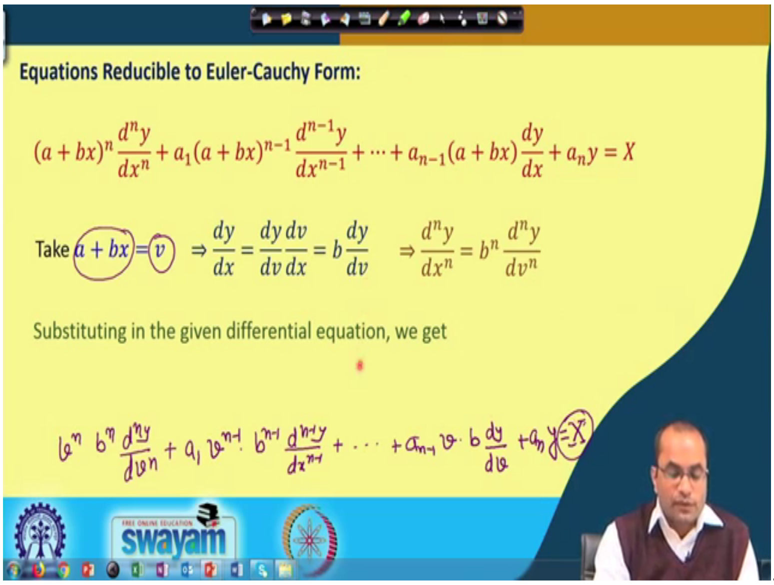

Now, here we will be talking about the idea that there are some equations which can be reduced to this Cauchy - Euler form and then we know how to solve the Cauchy- Euler form.

So, for instance this type of equation when we have a plus bx power n here also a plus bx power n minus 1 and so on. So, these are the coefficients now in the Cauchy equations we have just these x power and n or x power n minus 1 etcetera, but in this case we have a plus bx here also we have a plus bx form. So, we need to substitute this a plus bx as a new variable v here. So, if we do that a plus bx is equal to v with this substitution our equation will be converted to this Cauchy - Euler equation and we know how to solve Cauchy - Euler equation.

So, with this change when we make such a substitution a plus bx is equal to v with is our dy over dx will be written as dy over dv and dv over dx. So, what is dv over dx that is nothing, but b here. So, dy over dx dy over dx in terms of dy over dv with this factor b there and if we continue this if we take d 2 y over dx 2 that will be coming just simply again this one more b so, that would be b square when we compute the third derivative this will be b cube with dy over dv and this can be continued for any order derivative, meaning that the n th order derivative also we can compute and this b power n will come out and then we will have d n y over dv n.

So, now, if we substitute this into this equation so, what will happen here this will be having this v power n and then d n y over dx n will be this b power n and d n y over dv n. The second term will be a 1 and here we will have v power n minus 1 and this will be converted again the nth minus oneth order derivative from here. So, the b n minus 1 will come and d n minus 1 y over dx n minus 1 and this will be continued for this a n minus 1 a plus b x is v here and dy over dx is this b dy over dv and this a n a n y will be equal to this x and x is a function of this x here but that will be converted again this x will be set to do this form of v. So, some function of this we will come instead of the right hand side x.

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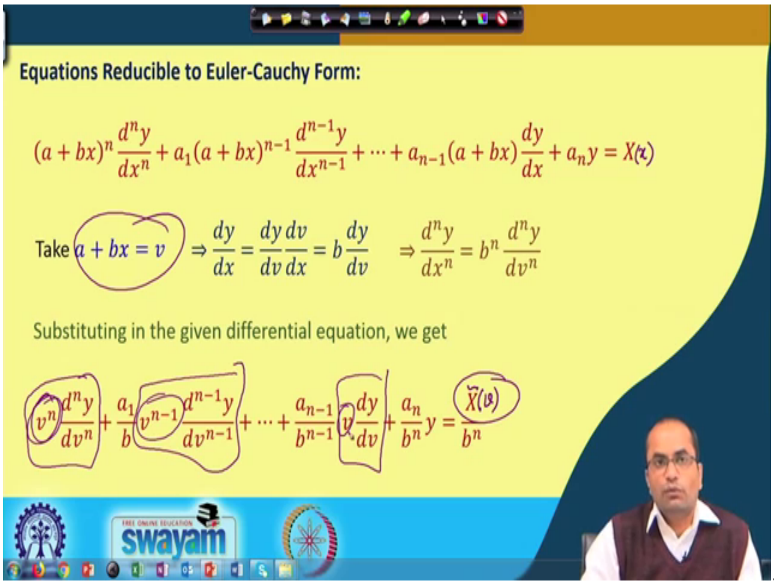

So, when we convert this what are we getting here so, by changing that by dividing this right hand side by this b n by dividing the whole equation by this b n. So, the right hand side become this x which again we have to substitute whatever X is given there in terms of v there and it will be divided by this b n and then the left hand side here v n and d n y over dv n here we will get this a 1 over our b here because there was a term b and minus 1 and when we are dividing by this b n. So, 1 will be survive in the denominator here d n minus oneth order term and so on.

So, this right hand side again this is the maybe a new X here which is a function of v because this was X was a function of x, but now x is replaced in terms of v so, we will get also this function of v the right hand side. But what we realize now that this equation is exactly the same equation which we have discussed before this a Cauchy - Euler equation. So, the coefficient here the nth order term is v power n with n minus 1 other term we have this n minus 1 and here we have v. So, this is the Cauchy - Euler equation which we know that how to solve by this substitution.

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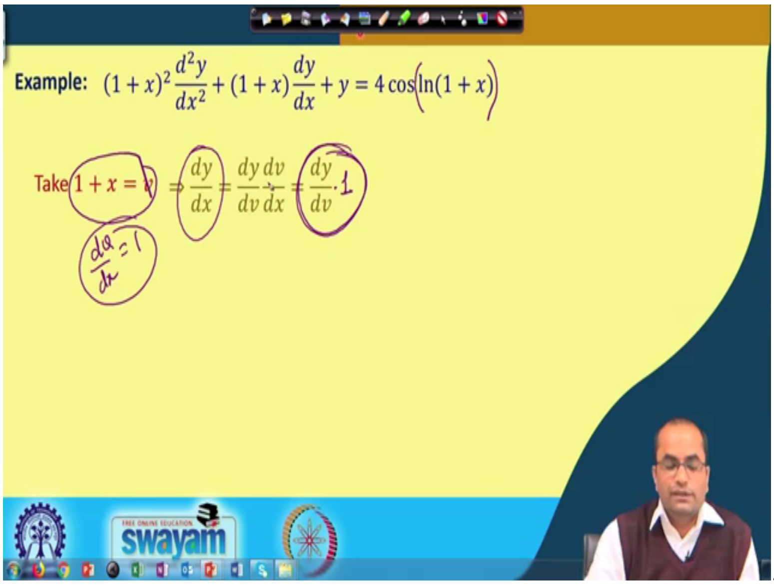

And now to demonstrate this let us go through this example so, we take this 1 plus x whole square and the second order derivative plus this 1 plus x and the first order derivative plus y is equal to 4 the cos here with this cos of the log 1 plus x and now if we substitute this 1 plus x here as v I mean whatever given here 1 plus so, in general a ax plus v term here. So, we have taken this 1 plus x is equal to v this substitution we have made here and that will convert now dy over dx is equal to dy over dv and dv over dx.

So, this dy over dv and dv over dx from this equation is 1 because when we compute this dv over dx that is just 1. So, that is what here the 1 is coming. So, what relation we have that dy over dx will be replaced by dy over dv there is no factor coming because of this 1 plus x.

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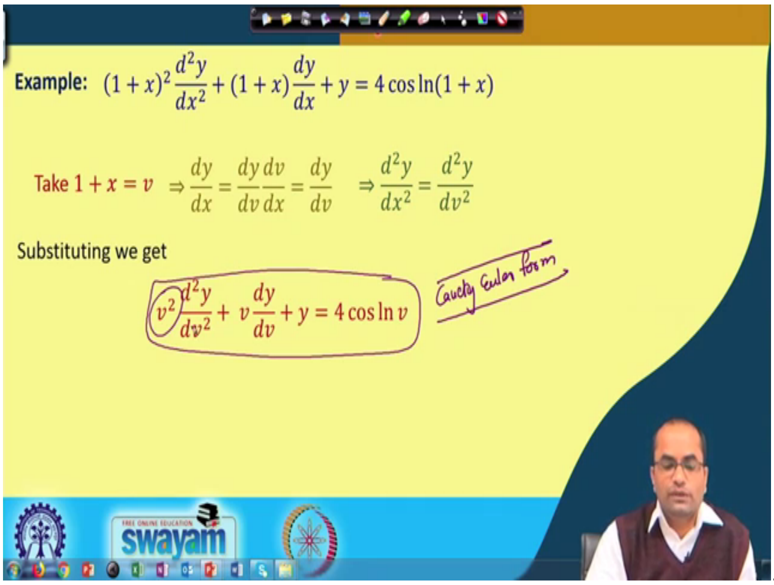

And then the second order derivative when we compute so, d 2 y over dx 2 again because the factor of this x was 1. So, we will get no change, but you will get just the second order derivative with respect to v and if we substitute this in the given equation then we will get v square here and d 2 y over dv square plus the v term and dy over dv plus this y and the right hand side we have 4 times the cos and ln of v.

So, now this equation here is exactly the equation discussed this Cauchy- Euler form. So, this is Cauchy - Euler form of Cauchy - Euler equation which we have solved just before by the appropriate substitution of this independent variable to a new independent variable so, we will do and now exactly the same.

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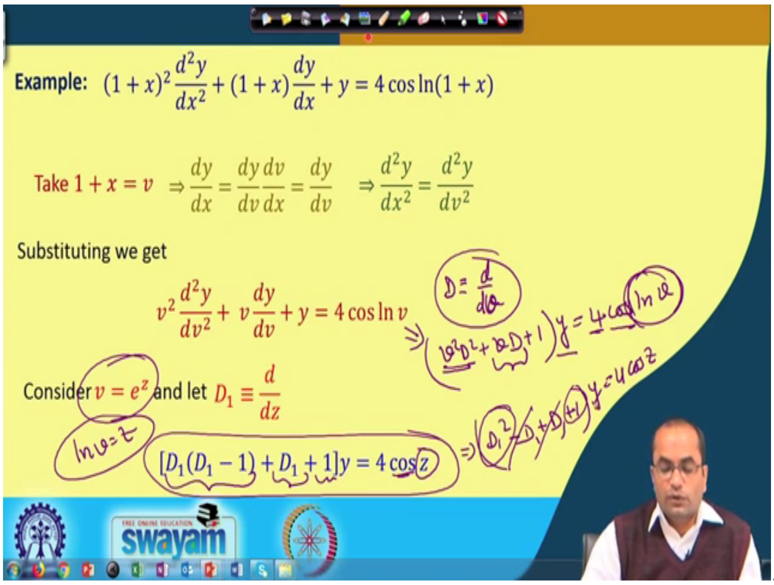

So, we will substitute here this v the independent variable in the form of exponential z and then by doing this substitution and assuming this D 1 now a new operator in terms of this d over dz our operator D here is d over dv the new operator D 1 will be d over dz.

So, with that change of this variable what will happen this v square d 2 y over dv square or v square the d square because this we can write down in terms of the D if we define the operator D as d over dv now because our differential operator is written in terms of v, then this equation will be like v square D square plus this vD and plus this 1 operated on y the right hand side is 4 cos ln v. So, this v square D square will be replaced with the D 1 D 1 minus 1 and this v D with this substitution v is equal to x for e power z this will be replaced with the D 1 and then this is 1 here y is equal to the 4 times the course and the ln v because v is equal to e power z so, here the ln v is z. So, this ln v the logarithmic of e is replaced by the z here.

So, our new equation now becomes D 1 square D 1 square minus D 1 then we have plus this D 1 and we have plus 1 here operated on y is equal to 4 times this cos z and D 1 D 1 gets cancelled and we have D 1 square plus 1 operated on this y is equal to 4 times cos z

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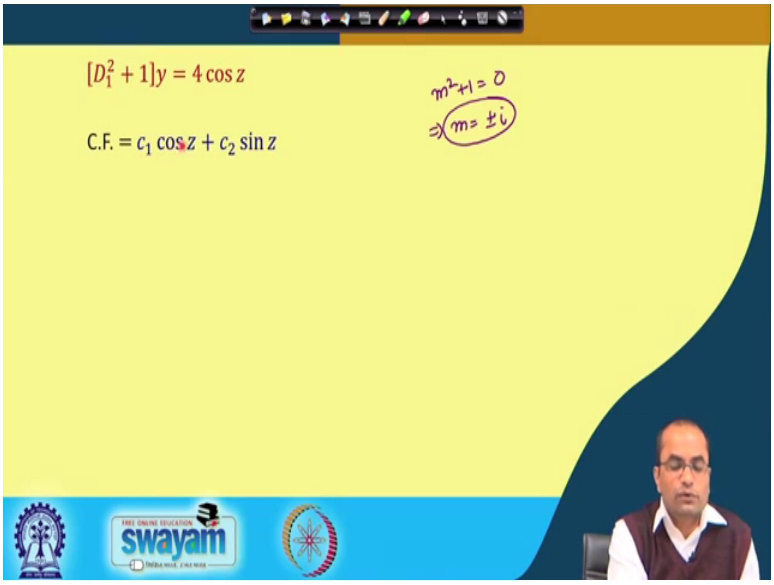

So, that is a new equation now and D 1 square plus y plus 1 y is equal to 4 times the cos z. So, the complementary function for this because the auxiliary equation will be m square plus 1 is equal to 0 that is m is equal to plus minus i. So, with this auxiliary equation we can write down the solution in terms of z that c 1 because e power this is 0 here e power 0 will be 1. So, we have c 1 cos z plus c 2 sin z that is the complementary function.

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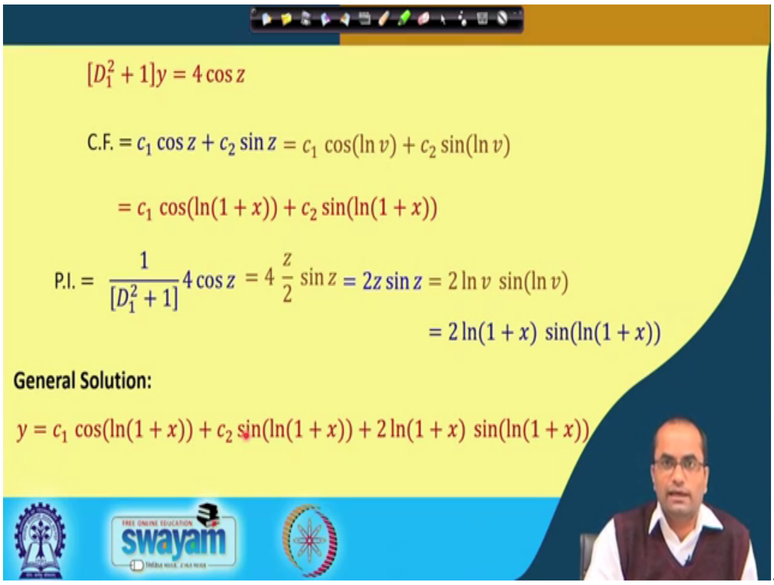

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And then this z here must be replaced back to the original variable so, that was ln v. So, we have the ln v for z and this v was also 1 plus x in the earlier substitution. So, we have to get back to the x so, this will be the complementary function will be c 1 cos this ln of 1 plus x c 2 sin ln 1 plus x and the particular integral which is 1 over this D 1 square plus 1 operated on this 4 times this cos z.

So, this 4 we can bring out here and then this cos z this is the formula which was also derived there when we substitute this D 1 square minus 1. So, this is becoming 0 and then we have derived that will be these dz over 2 and the cos will be just the sign there. So, said over 2 a and 2 a here is just 1. So, z by 2 and sin z that will be the result of this operation 1 over D 1 square plus 1 on cos z that will give z by 2 and sin z.

So, this 2 gets cancelled we have to z and sign z the particular integral of this equation to z into sin z which again we have to get back to v and then x. So, 2 times this ln v and sin ln v and in terms of x we have v was substituted for this 1 plus x. So, we have to replace this v for 1 plus x. So, finally, the particular integral is 2 times ln 1 plus x and sin ln 1 plus x and now the general solution we can write down we can add the complementary function into this particular integral and this will be the general solution of the given differential equation which was first converted to the Cauchy - Euler form.

And then the Cauchy - Euler equation was changed to the linear equation with constant coefficient and that we know how to solve with the help of this complementary function and the particular integral.

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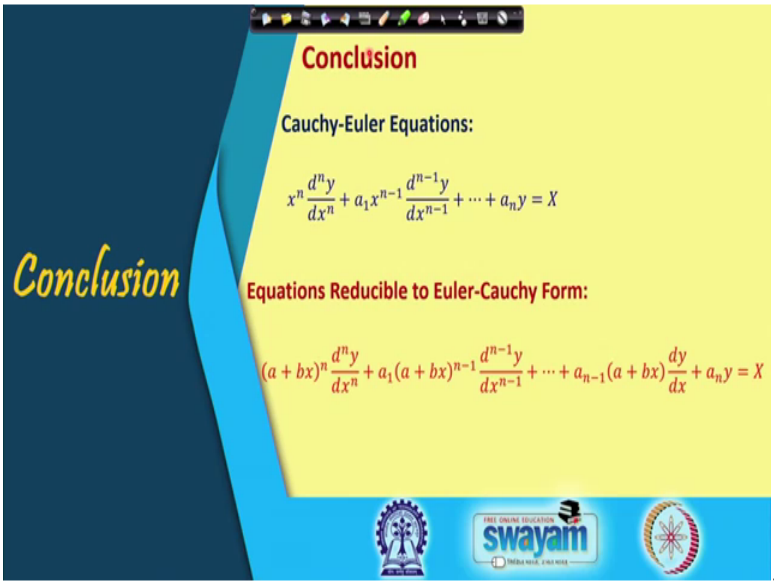

Coming to the conclusion here so, we have discussed today the Euler cause equations and that is a special form of the equation in terms of the coefficients here x power n x power n minus 1 with the n minus 1 and derivative and so on and we have also discussed the equations reducible to Cauchy - Euler form and in particular we have taken this form where instead of this x power and we have a plus bx power n, but the idea was simple that if we substitute this a plus bx as a new variable then this equation can be reduced to this Cauchy - Euler equation and then again one more substitution will lead to the differential equation a linear differential equation with constant coefficient.

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So, these are the references we have used to prepare these lectures and.

Thank you for your attention.