Engineering Mathematics - I
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Lecture – 06
Limit of Functions of Two Variables

I welcome to the 6th lecture on Engineering Mathematics I and today, we will discuss Limit of Functions of Two variables.

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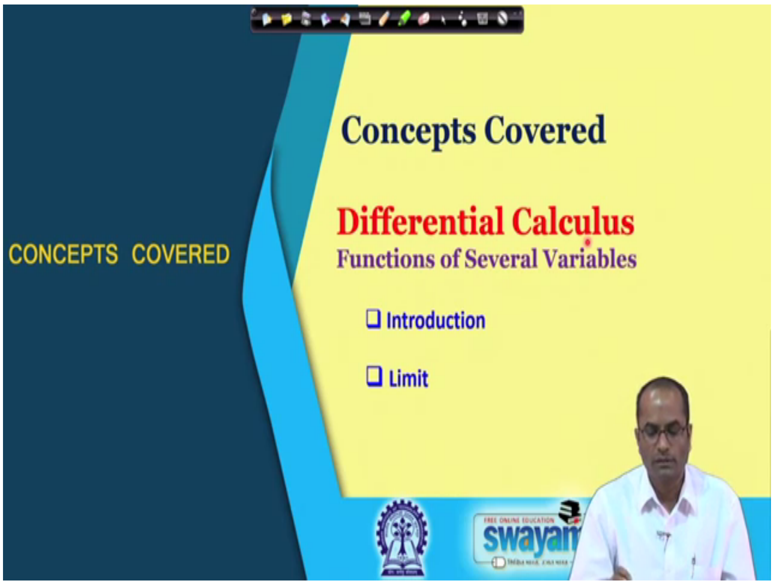

So, today in particular we are talking about now the Differential Calculus and Functions of Several Variables and before I introduce the limits, I will also discuss what type of these functions we mean for the several variables.

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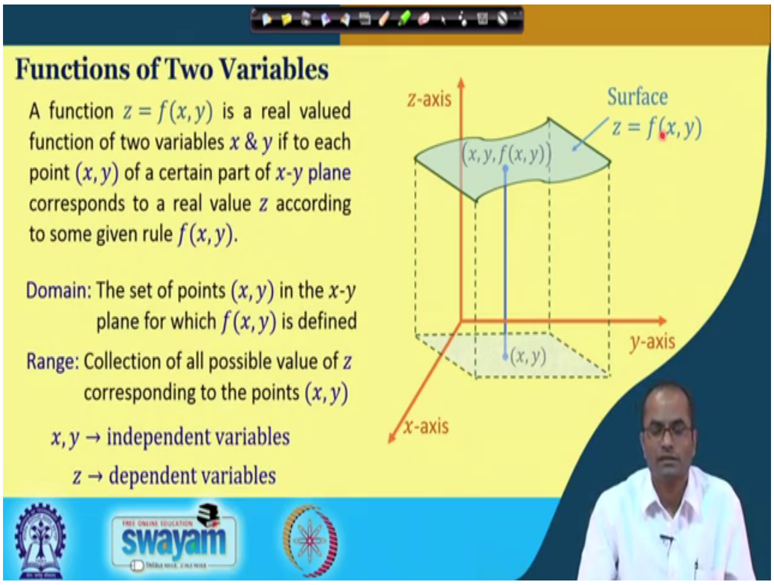

So, here first let me introduce you the Functions of Two Variables. So, a function z is equal to f x y so its a similar to what we have the function of 1 variable, but there we consider like y is equal to function of x. So, they are used to be only 1 variable. But now here the z depends on 2 variables x and y.

So, its a function of 2 variables and this is a real valued function of 2 variables x and y if to each x y of a certain part of x-y plane which is called the domain of the function and if we assign this value using this rule f is equal to z is equal to f x y to a real value z according to some given rule f x y; then, we call this function as a Function of Two Variables. So, what is the Domain? Domain is basically the set of points x y the x-y plane for which f x y is defined and the Range, Range is the collection of overall possible values of z corresponding to the point x y.

So, here we will call this x y as dependent independent variables and z as dependent variable. So, in this situation let us consider the coordinate system of x y and z axis and for example, this is the point in the x-y plane denoted by this point here and the coordinate of this point in the x-y plane are given by x and y. So, corresponding to this point, if we compute this f x y, then for instance this is the point here the height this in along the z axis at this point x y of this function is f x y. So, this is the point here in 3-dimensional system x coordinate comma y coordinate and the z coordinate which is f x y.

So, now if we collect all these points corresponding to the point in the domain, we this surface will be formed. So, this a function of two variables in 3-dimensional plane here represents a surface. So, this is the function z is equal to f x y which represents the surface in this 3-dimensional coordinate system.

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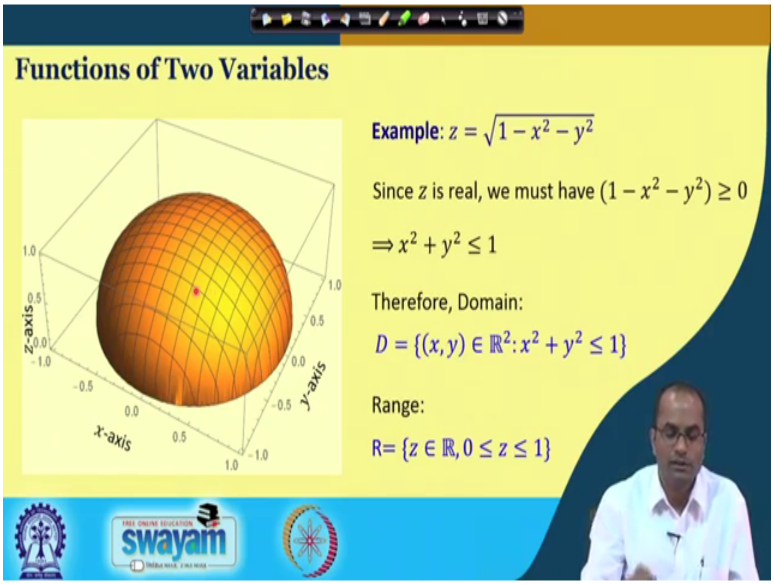

So, now let us consider this example here z is equal to a square root 1 minus x square minus y square. So, since this z is real, then the argument here 1 minus x square minus y square must be greater than or equal to 0 and this will imply that x square plus y square must be less than or equal to 1. And therefore, we get the domain D which is all contains all the points here in R 2 means both are the real. So, in x-y plane, where the x square plus y square is less than or equal to 1.

So, this is a circular disc in the x-y plane which forms the domain, for this function z is equal to square root 1 minus x square minus y square and the range. So, the values of the possible values of z for the points from this domain, is the Range. So, if you notice here that x square plus y square is less than or equal to 1, this value of the square root here will belong to 0 and 1. So, that is the Range here all the points z from the real axis and where the z is between 0 and 1.

So, this is the graph of this function which is the sphere basically the half sphere above part of the sphere. So, here the z axis, the x axis and the y axis. So, this domain here which is the circular disc x square plus y square less than equal to 0 and at each point of this circular disc, we have computed the value of z and the surface is plotted. So, this is the interpretation the geometrical interpretation of the functions of two variables.

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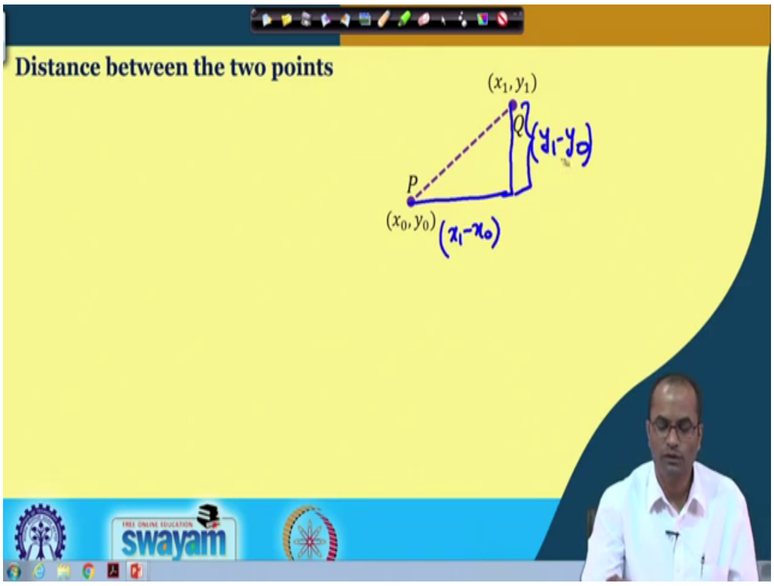

Now, before me move to the limit and continuity part later on, we need the concept of the distance between the 2 points in x-y plane. So, if we have 2 points P and Q having 2 coordinates here x 0, y 0 and x 1, y 1; then, what is the distance between these 2 points? So, which we can easily find out if we from this triangle, then this here the height will be like y 1 minus y 0 and this is going to be x 1 minus x 0 and now, this distance P Q will be the square root of x 1 minus x square and plus y 1 minus y square.

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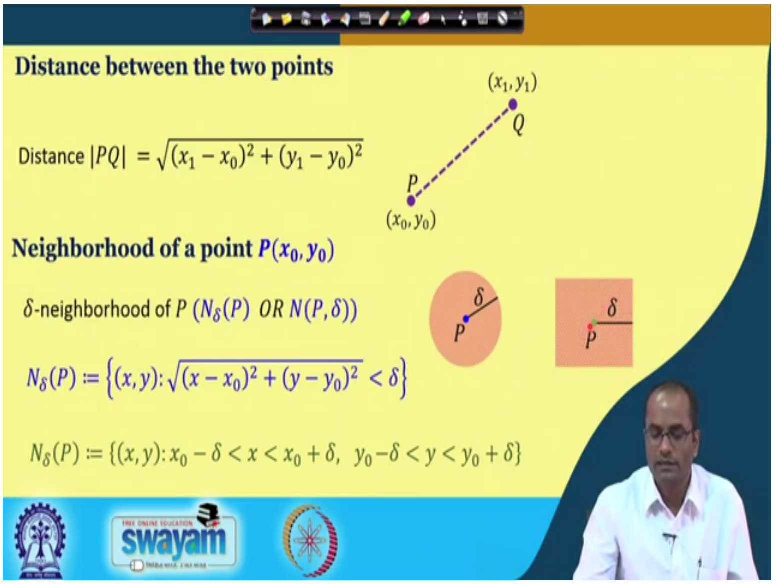

So, this is the distance x 1 minus x square plus y 1 minus y 0 whole square and the square root. We will also introduce now the neighborhood of a point P x 0 y 0. So, if a point has given in the x-y plane as x 0 y 0 which is noted by P here. Then, we will introduce now what is the delta neighborhood of this point P which is usually denoted by N delta P or N P delta. So, here N delta P is given by all the points here x y whose distance from the point x 0 y 0 from the distance from this point P is less than delta.

So, naturally now this one here is again open circular disc which is represented here; that is the point P and the radius here is delta and now all the points here in this disk satisfies this relation here. So, this is the neighborhood of this point P. Note that this is a open neighborhood because the boundary is not included. If we include here less than equal to delta, then the boundary of this disc that means, the circle will be also included in the point here. So, this is the open neighborhood of P or with radius this delta.

This is the most common definition for the neighborhood, but we can also introduce neighborhood as the square around this point P introduced by the x which lies between x 0 minus delta and the x 0 plus delta; y also lies between y 0 minus delta and y 0 plus delta and this is represented by this is square here, with this half of the side is this delta and the P is the point.

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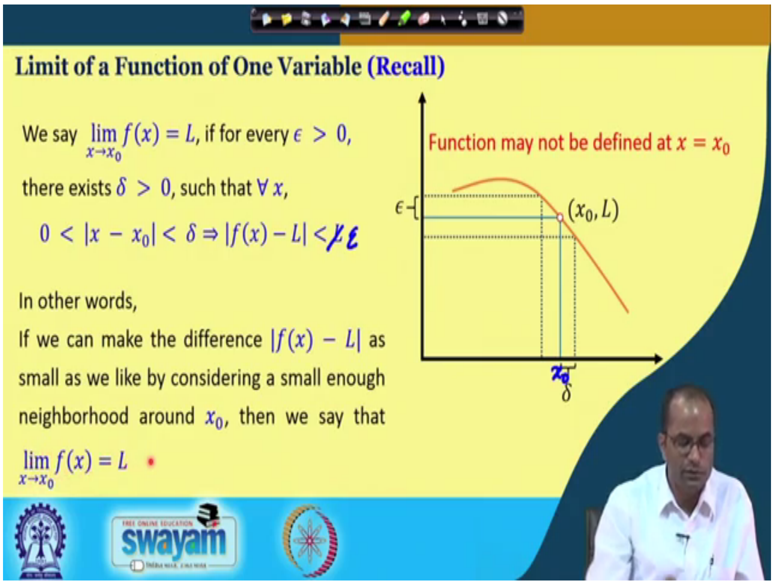

So, Limit of a Function of One Variable. So, we recall now before we introduce the limit the concept of the limit for the functions of two variables, it is important to first discuss the limit of a function of one variable. And mainly, we will now focus on this delta epsilon definition of the limit which is very important to discuss the limit for the functions of several variable.

So, here we say for the one variable case that this limit x goes to x naught f x is equal to L, if for every epsilon greater than 0. If for every epsilon greater than 0, there exists a delta greater than 0 such that for all x; so, all those x which belongs to this neighborhood of this x 0 implies that the difference between f x and minus this L is less than the given number here L.

So, let me just explain you this concept with the help of this plot here. So, we have a function f x whose limit as x approaches to this x naught is L. So, this is the point x naught here. This is the point x naught and then corresponding to this, this value is L there and now.

So, what this definition says is that corresponding to every epsilon; so this could be a very small number or anything. So, this epsilon positive that means, which is denoted by this distance here around this L. So, for every epsilon, however small, there always exist in neighborhood of this x naught which is the case here and now, if you take any x in this neighborhood.

So, this is the delta. So, if you take any point in this neighborhood now, the value will be or the difference of the value of this f x and minus this L will be less than the epsilon. Yeah, this is epsilon here. So, let me just write it. This is epsilon. So, that difference will be less than epsilon and this function may not be defined at x is equal to x 0. So, define the limit of function at point x naught, the function may not be defined at this point x naught and therefore, here that x is equal to x naught is excluded.

So, in other words, if we can make the difference this f x minus L, this difference here f x minus L; if we can make this difference as a small, as we like by considering a small neighborhood around this x naught, then we say that this f x is equal to the limit of this f x naught; f x is goes to x naught is L.

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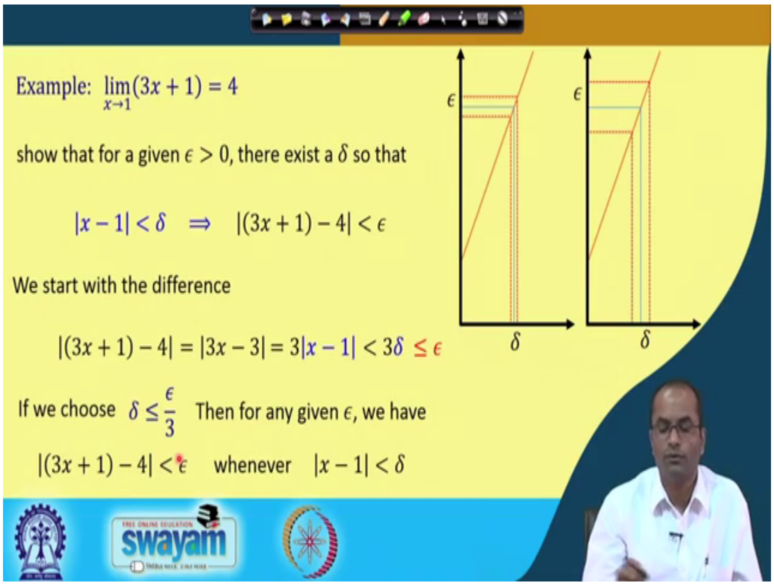

So, here now let us take this example that what is the limit of this 3 x plus 1 as x goes to 1 and its clearly visible here that the limit is 4 in this case and we will prove now using this epsilon delta approach that this indeed is the limit of this function 3 x plus 1. So, what we need to show? We need to show that for a given epsilon that there exists a delta. So, that the x minus 1 less than equal to delta. So, this is the neighborhood around this one delta neighborhood around 1. If we take any point from this neighborhood, this will imply that this difference between the function and minus this 4 is less than epsilon.

So, to prove this, we will start with this difference here 3 x plus 1 minus 4 which is equal to 3 x minus 3 and then, 3 and we have the modulus x minus 1. So, this modulus x minus

1; so, all x from this neighborhood satisfies that x minus 1 less than delta. So, here 3 was there; so, less than 3 delta and what is our aim now? To make this difference is smaller than the given epsilon. So, if we set here less than or equal to epsilon now, so we get a relation between delta and epsilon. For given up epsilon, we can choose now the delta such that this difference will become less than epsilon. So, what is the relation?

Relation is 3 delta less than equal to epsilon. So, if we choose this delta less than or equal to epsilon y 3; for given epsilon if you choose this delta less than equal to epsilon by 3. Then, for any given epsilon, we have this relation that the difference between this 3 x plus 1 and minus 4 is less than epsilon which we have just seen here, this difference is less than epsilon. Because of this relation we can choose this delta less than epsilon by 3 and then, this difference will become less than epsilon whenever for all x from this neighborhood of this one delta neighborhood of this 1 here.

We can see the situation in this plot. So, this is a for instance and at x is equal to 1 the value of the function is 4 and now, you take any neighborhood. So, for example, we have chosen this neighborhood here. So, there exist a delta neighborhood again correspond around this one in this case. So, usually when the epsilon is a small, the delta is also small and when the epsilon is big, we can have a big neighborhood around that point.

So, for instance here we have chosen a big neighborhood of around this point 4 and then, correspondingly this delta is also big. So, all these x in this delta neighborhood satisfies this relation that the difference of this function minus 4 is less than the epsilon.

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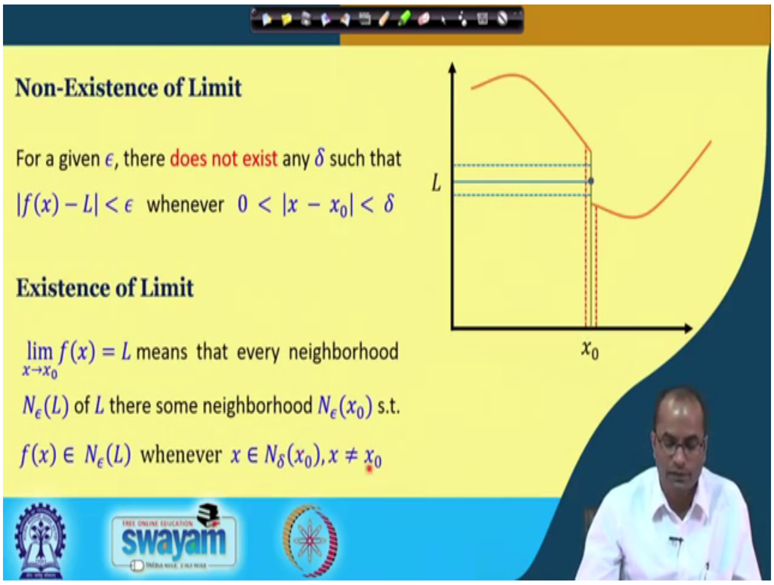

So, what will happen for the Non-Existence of a Limit if the limit does not exist for? For example, in this case as x approaches to x naught, the limit whether right or the left is not equal to L and what will happen if we try to get a neighborhood around this L here and whether the corresponding neighborhood around this x naught will exist or not?

So, here for example, this is epsilon neighborhood, but what we see in this case that there is no neighborhood of this x naught. Whatever small neighborhood you take and any x between this, the difference between the function value and this limit will exceed than this epsilon here. So, it is clearly visible that you take any point in this neighborhood here and then, the difference between the f x and this L will increase than the given epsilon here.

So, in this situation we do not have the possibility that for a given epsilon, there exists delta neighborhood around this x naught. So, for a given epsilon in the situation, there does not exist any delta such that the difference f x minus L is less than epsilon wherever this x is from this delta neighborhood.

Recall again, the existence of the limit was that limit x goes to x naught f x is equal to L and this means that every neighborhood of L. So, every neighborhood which is represented by N epsilon L of L there exist some neighborhood of x naught such that the f x belongs to this neighborhood of L whenever x belongs to the neighborhood of this is naught, the delta neighborhood of x naught.

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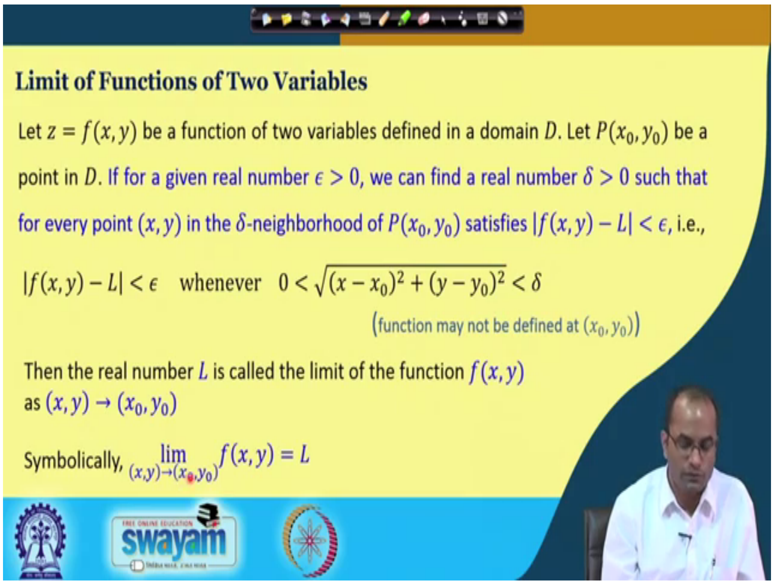

So, Limit of Functions of Two Variables, we will discuss now. So, this is just the continuation of this rather general definition here. So, this x maybe point in 2-dimensional plane for example, in x-y plane and again, this definition will be carried for generalization the definition of the limit.

So, for example, in this case we take z as f x y function of 2 variables which is defined domain D and let this P x naught y naught be a point D. So, if for a given real number epsilon however is small, we can find a del a real number delta such that every point x y in the delta neighborhood of this point x naught y naught it satisfies that f x y minus L is less than epsilon; then, we will call tell the L is limit.

So, what does that mean? That this f x y minus L is less than epsilon wherever these x y fall in the delta neighborhood of this x naught y naught. So, if for a given epsilon, we can find such a delta, then we will call that the cell is the limit of f x y.

Again, the function may not be defined at x naught y naught for the existence of this limit and therefore, we may exclude this that x is equal to x naught y naught and then, this real number L is called the limit of the function f x y as x y goes to x naught y naught. Symbolically, we denote again the limit x y goes to x naught y naught f x y is equal to l.

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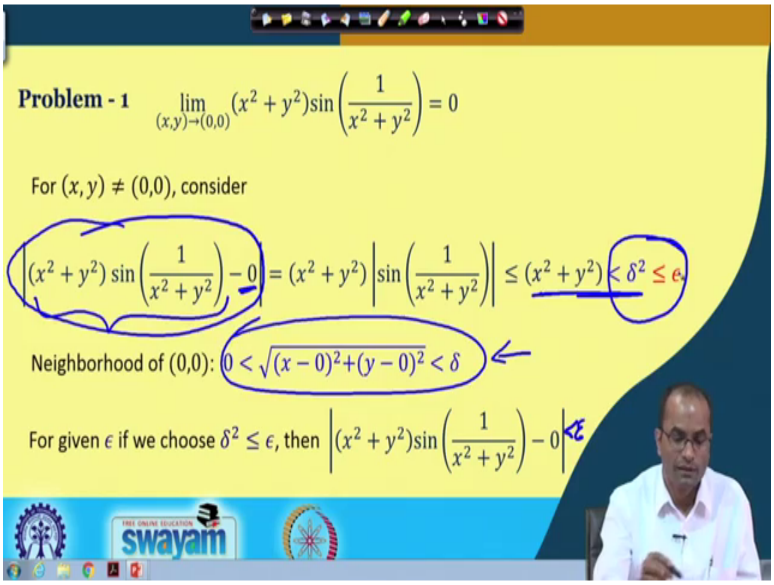

So, if you take this problem now from these functions of two variable x square plus y square and sin 1 over x square plus y square and this limit is 0. We want to prove that this limit is 0 using epsilon delta approach. So, how do we proceed? Now for x y not equal to 0 0 because at 0 0 this function is not defined and we consider this difference. Difference of the function which is x square plus y square sin 1 over x square plus y square less than the limiting value 0.

If we start with this difference now, this is x square plus y square because this is going to be positive and the absolute value of sin 1 over x square plus y square and we know that that this value of sin always lies between minus 1 and 1. So, this is this quantity is bounded by 1. So, this is less than equal to 1; that means, this difference is less than equal to x square plus y square and what we know, the neighborhood of this 0 0 point because taking this limit to 0 0.

So, what is the neighborhood of this 0 0 or rather delta neighborhood of 0 0? It is x square plus y square less than delta or x square root of this less than delta or x square plus y square less than delta square. So, we can use this inequality there. So, x square plus y square in this neighborhood is less than delta square and we want to set for a given epsilon, this difference less than epsilon and we are looking for such a delta which we satisfy that inequality.

So, here this difference if you want to make less than epsilon; then, we can get this relation between delta and epsilon. So, for given epsilon, if you choose delta such that the delta square is less than or equal to epsilon then, we are done with the limit that this difference is less than epsilon for the delta which satisfies the delta square is less than epsilon.

So, here if for a given epsilon if you choose the delta square is less than or equal to epsilon, then this difference between the function value and its limiting value will be less than epsilon. So, this here will be less than epsilon, will be less than epsilon; if we take x y from this neighborhood. So, again just to recall this so, we have started with the difference between the function value at any point x y not equal to 0 0 and its limiting value and somehow we have to write down this difference in terms of the so that we can use this delta inequality from the neighborhood of the x naught y naught point which is 0 0 in this case.

And once we reach to this delta, then we can set that this must be equal to less than equal to epsilon because we want to make this difference; this difference here of the function N minus this limiting value less than epsilon. So, out of this difference, we can say that for a given epsilon there exist a delta. The relation is that the delta square must be less than or equal to epsilon.

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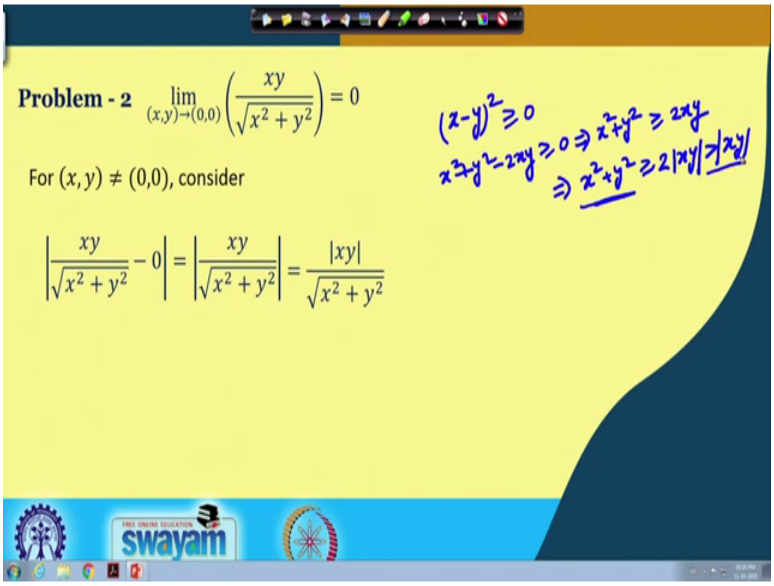

So, another example where we need to prove that this limit x y goes to 0 0; x y over x square plus y square under square root is 0. So, in this case again the function is not defined when x y goes to 0 0 because of this reason x square plus y square.

So, this will become 0, but what we will prove now that this limit is 0. We cannot just directly substitute 0 0 here because this will be like a 0 and then 0 here and there is no such an (Refer Time: 24:25) rule which we can apply in case of 1 variable we had the (Refer Time: 24:30) rule and which says that the derivative this limit will be equal to the limit of the ratio of the derivatives..

But here so we cannot substitute thus directly the value of x and y in this expression. So, we will prove that this limit is 0 using again epsilon delta approach. So, we take this x y not equal to 0 0 and we consider again as in the previous example, this difference between the function which is x y over square root x square plus y square and minus this limit 0.

And now, so this is equal to the x y divided by square root of x square plus y square and now this is positive. So, we have basically x y the absolute value divided by this x square plus y square and now, the aim is the same that you want to write down everything in terms of the neighborhood inequality which will be in this case because we have 0 0 point will be x square plus y square and the square root. So, we will convert now this expression in the form of the x square plus y square.

So, we notice here now that if you take this inequality x minus y whole square which is always greater than or equal to 0 because of the square here and then what do we get here? x square plus this y square and then minus 2 x y greater than 0 which will imply that this x square plus y square is greater than 2 x y and then, so here you have x square plus y square greater than.

We can take the absolute value. So, here x square plus y square is positive. So, we have the 2 and the x y is greater than this 2 times the x y and this is greater than the x y. So, in this case, we can use this inequality that x square plus y square is greater than the absolute value of x y.

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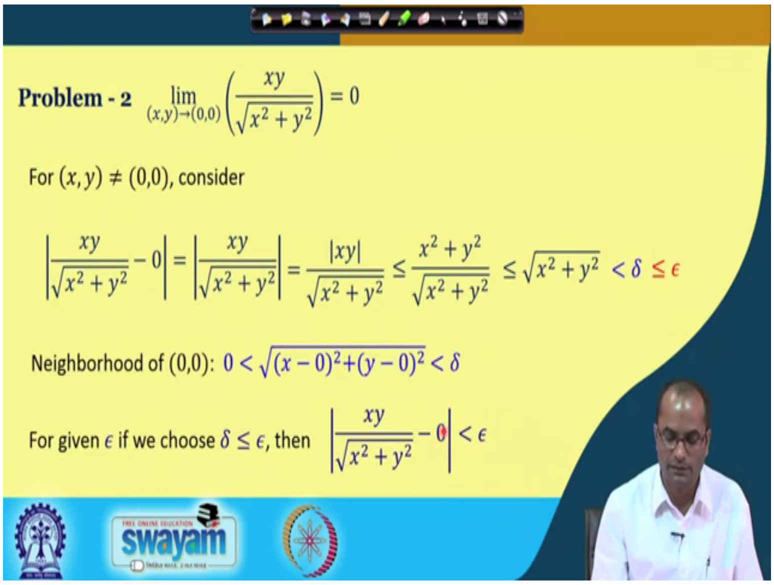

Having this relation, we can now replace this absolute value x y by the bigger quantity which is x square plus y square and divided by x square plus y square. So, here now we got the x square plus y square and we considered again this neighborhood of this delta which is x square plus y square root less than this delta. So, we said that the x square plus y square root is less than delta at least in the neighborhood of the 0 0 and now we want to said that this difference is less than epsilon.

So, again we will use the same idea. So, here it is less than equal to epsilon. So, this expression the difference between the function and the limiting value is less than epsilon. If we choose this relation delta is less than or equal to epsilon. So, for given epsilon, if we choose delta equal to epsilon or anything less than epsilon; then, the difference between the function value and its limiting value will be less than epsilon.

So, for given epsilon which use delta less than epsilon, then this difference between the function value and the minus the limiting value will be less than epsilon.

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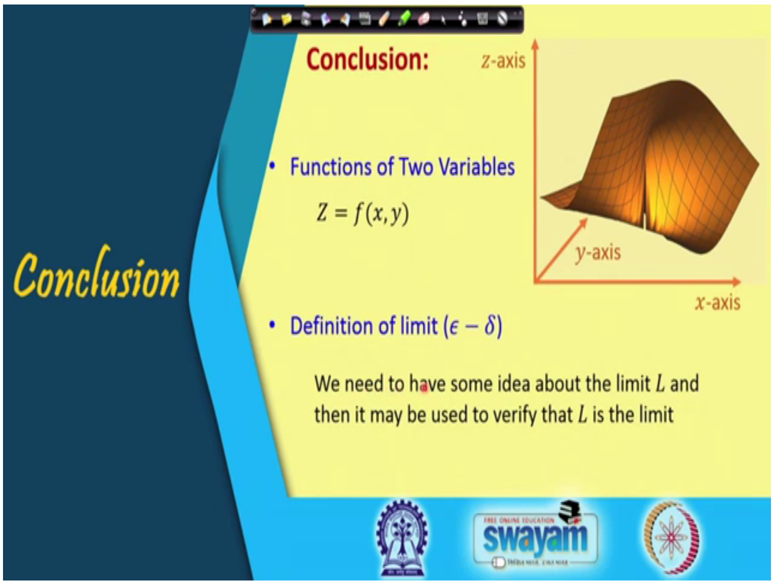

So, now to conclude, so we have discussed the functions of two variables mainly Z is equal to f x y. So, here x and y they are the independent variable. So, they can take any values and this that depends on the value of the I mean this functional value here and we have also seen that such functions represent surface in the x y z plane. We have also discussed the definition of the limit in particular the epsilon delta definition which is very useful to prove somehow that a given number L is the limit.

With the help of this epsilon delta approach, we cannot just get the limit; but if we have some idea about the limit, then this epsilon delta approach may be used to verify that L is the limit. Because what we will observe now in case of these 2 variable when we have the functions of 2 variables, then there are several parts which approaches to a particular point in the x-y plane and this is completely different what we have seen in case of 1 variable where there were 2 parts; one from the right side, one from the left side and we used to get the limit from the right side, from the left side and then, if they both are equal we say that the limit exist.

But now, in this case sins you can approach to a particular point from the several direction, it is not possible to get as the long some path because there are infinitely many parts involved in this limiting process. So, this epsilon delta approach will be very useful once we have some idea about the limit and then, this value of the limit can be used to prove that this is indeed the limit. So, in the next lecture, we will learn more on the Limits, the evaluation of the limit in particular.

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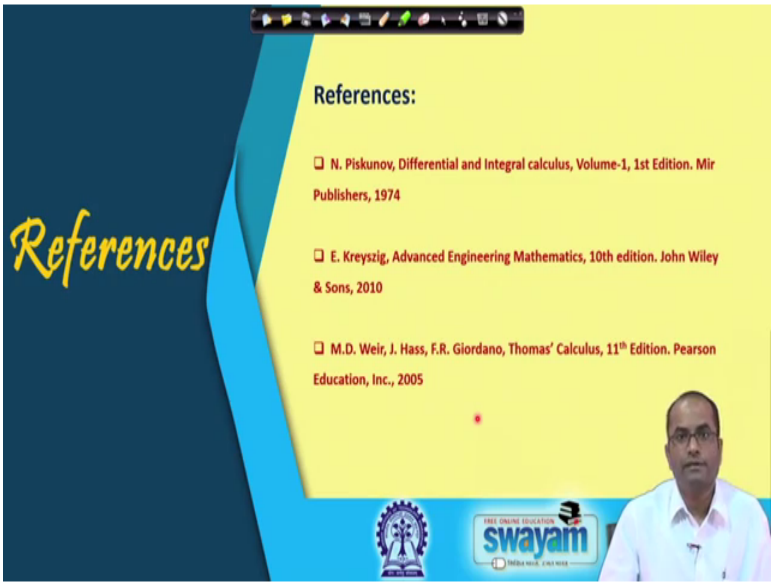

These are references which we have used to prepare these lectures and.

Thank you very much.