Engineering Mathematics - 1
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Lecture -59
Solution of Higher Order Non - Homogeneous Linear Equations (Contd.)

So, welcome back and this is lecture number 59 we will be talking about we will continue our discussion on solution of non-homogeneous linear equations.

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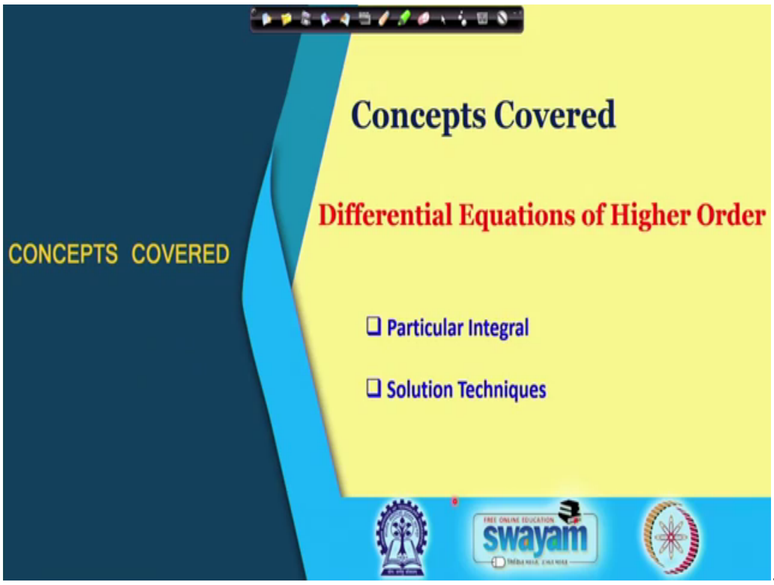

So, in particular we were talking about the particular integrals and the solution techniques for finding the particular integrals and just to recall from the previous lecture.

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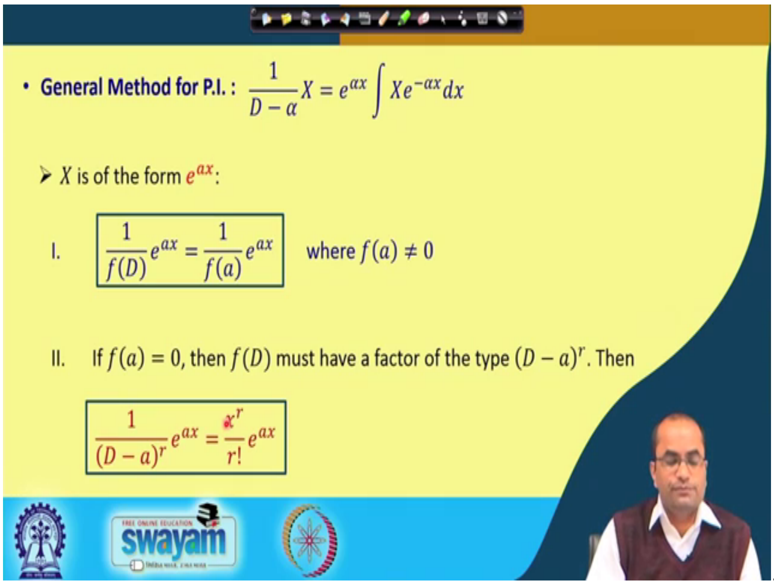

So, we have the general method for getting this particular integral was this 1 over D minus alpha when we apply on X, X is a function of small x, then we get with this integral here so, outside the integral e power alpha x and then integral x e power minus alpha x dx. So, this was a very general form which can be used for getting any particular integral, but then we have also discussed this special form when X is this exponential function meaning the e power ax.

In that case we have realized that this 1 over this function which is a function of this operator D and e power a x. When we apply this inverse operator on this exponential function only change will come that this D will be replaced by this a and that will be the value of this of this operation here when we apply the inverse operation here 1 over f D on this exponential function. But this was the point here that this f a must not be equal to 0 otherwise we cannot do that.

So, in the case when this f a is equal to 0 then this f D what we realize that this f D must have a factor D minus a power r, because that is the reason here we are going getting this f a is equal to 0. So, there must be a factor here D minus a power r, r can be any integer 1, 2, 3 and so on and in that case we have also derived that 1 over D minus a this power r when we apply one exponential function we will get this x power r over factorial r and exponential e ax. So, that these are the direct formulas which we can apply to find the particular integral in case the right hand side is an exponential function like e power ax.

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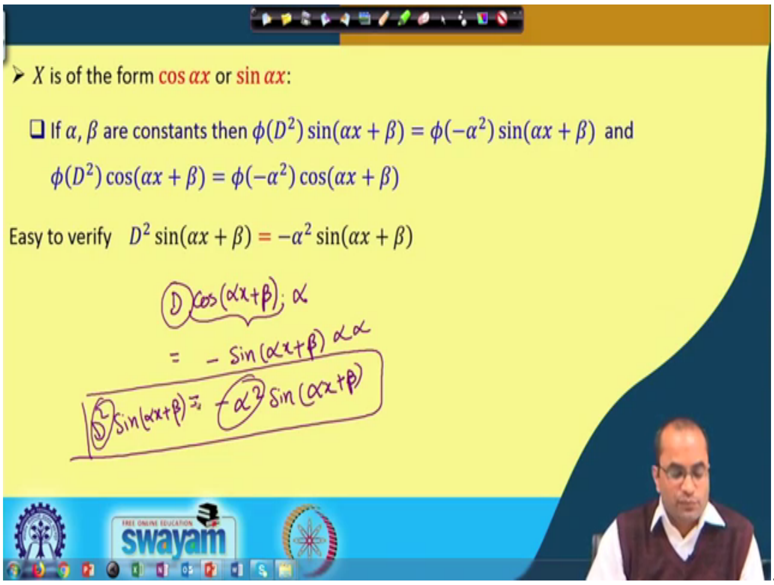

Now, we will discuss today we will continue our discussion that when x the right hand side is of the form the cosine function on the sin function in that case also can we derive some direct results which can be used to get the particular integrals.

So, to derive this result to get the particular integral when the right hand side is either sin function or the cosine function we first look at this small result here that if alpha, beta are constants then the phi D square so, here this operator the function of this operator D square. So, we if it is not given in the D square for example, in our differential equation we can I will try to rewrite it as a D square. So, here when this phi D square so, here function of the D square applied on the sin alpha x plus beta.

So, little more general here alpha x plus beta, this result will be equal to phi minus alpha square. So, the D square will be replaced by minus alpha square and the sin will remain as it is and the same result holds when we have this cos alpha x plus beta. So, in the case of cos as well we have the same result the D square will be replaced by this minus alpha square and this is very easy to verify that this D square when we applied this operator D 2 times on the sin alpha x what will happen. So, once we apply here once this D on this sin alpha x plus beta so, what will happen; so, D into this D we have plights then we will get this cos so, D is the differential operator. So, here alpha x plus beta and we will get also this alpha and once we have done this now we can apply this D as well so, that will be minus with the sin alpha x plus beta.

And we have another alpha from this differentiation and the one was sitting before. So, what do we get minus this alpha square and the same function sin alpha plus beta. So, this was a result of this D square and sin this alpha x plus beta and as a result what we see that this D square is replaced by simply minus alpha square the same we can prove for cos function.

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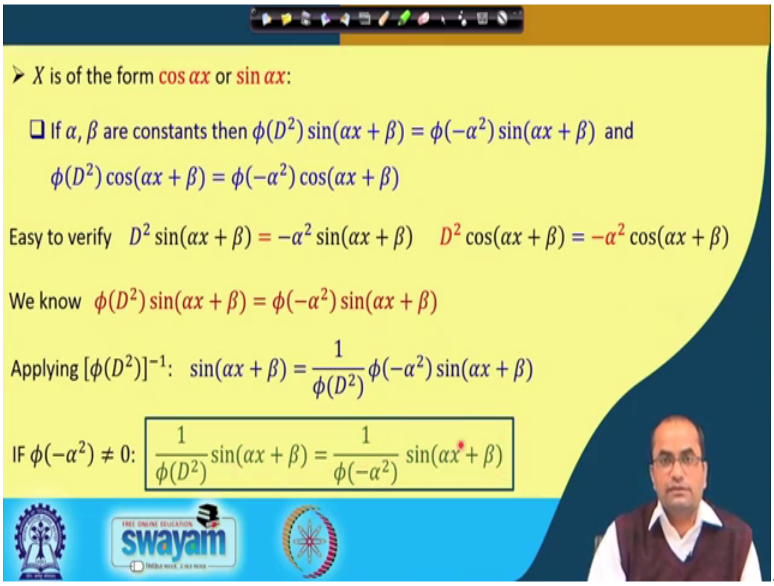

So, having this result now what we realize for same for the cos here well D square will be replaced by minus alpha square. So, whenever we have an expression in the form of D square meaning this phi of D square. So, what will happen this D square will be replaced by minus alpha square. So, that is the general rule which we have just realized by noting these that D square operated on the sin function just this D square is replaced by minus alpha square. So, anything of function D square also the same thing will happen the D square will be replaced by minus alpha square. So, now, we know that this phi D square sin alpha x is equal to phi minus alpha square sin alpha x plus beta.

And now we can operate here both the side this inverse operation inverse operator of this phi D square meaning this phi D square and this inverse. So, when we apply both the side here so, the left hand side will remain with the sin alpha x plus beta, the right hand side when we apply here. So, it is we can write down like 1 over phi D square and then we have phi l phi square and sin alpha x plus beta and now what we do here if this phi minus alpha square is not equal to 0 so, it is free from x naturally.

So, this operator will not do anything on this phi and we can bring to the left hand side and the final result what we have it is 1 over this phi D square applied on the sin alpha x plus beta gives us here this is 1 over phi l minus alpha square and then sin alpha x plus beta. So, this is the result which we can use directly, then when we apply this 1 over this phi D square on this sin alpha x plus beta or on the cosine function of alpha x plus beta we get 1 over this phi and D square will be replaced by minus l phi square and we will get this sin as it is.

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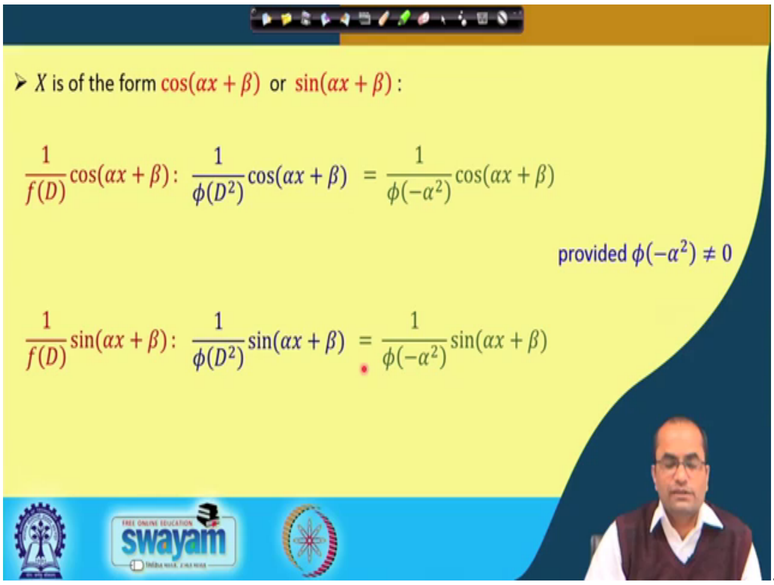

So, just to summarize again so, if this X is of the form this cos alpha x plus beta or sin alpha x plus beta in both the cases we can deal these 1 over f D here cos alpha plus alpha x plus sin alpha x plus beta, what we have to do this, f D we have to write in the form of phi D square because what we have learned that, D square will be replaced by minus alpha square. So, we have to rewrite this our given function f D as the function of this D square which may not be always possible and there will be a way out how to how to handle if we do not have this f D given in terms of this D square.

So, in that case also we will learn with the help of examples that what can be done for those cases, but let us first go through these results here. So, once we have this function of D square and we are operating on this cos alpha x plus beta term then this D square will be replaced by minus alpha square and same thing happened. So, just to note that this phi minus alpha square is not equal to 0 otherwise we cannot divide that. So, in case of sin also you will have the same result so, whenever we apply 1 over anything this function of D squared, we operate on the sin alpha x plus beta the result will be 1 over phi and minus alpha square sin alpha x plus beta.

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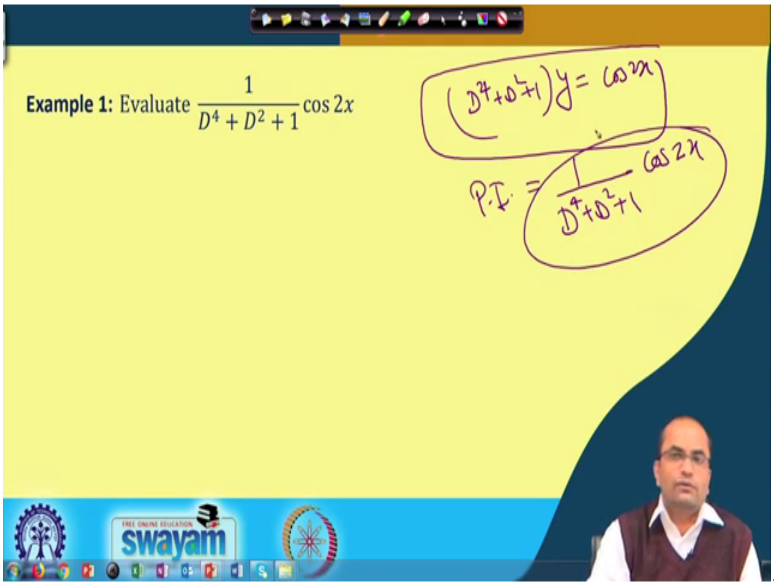

So, just to take an example based on this. So, we would like to evaluate here 1 over D power 4 plus D square plus 1 cos 2 x. So, what we realize that here this ah, this is a particular integral of a differential equation.

So, here this 1 over D 4 plus this D square plus 1. So, our differential equation will be like D power 4 plus this D square plus 1 operated on y is equal to cos 2 x. So, this was the differential equation and we want to get just the particular integral. So, this will take this form here D 4 and D square plus 1 and then we have this cos 2 x so, you want to just evaluate this particular integral nothing else.

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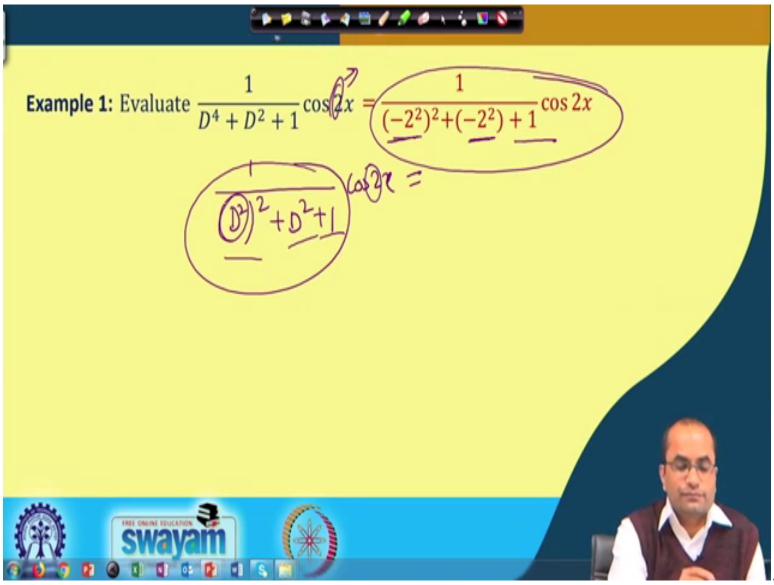

So, here what we will do, this is a very simple case because 1 over we have everything in D square form. So, D power 4 we can write like D square whole square and then we have D square 1 and this cos 2 x and then what we will do this D square will be replaced by this minus 4 here so, the minus 4 here also minus 4 and then this plus 1. So, in this case we could write everything given there in terms of the D square and then D square is replaced by this minus a square minus alpha square and we got the value of this operation which we can simplify here.

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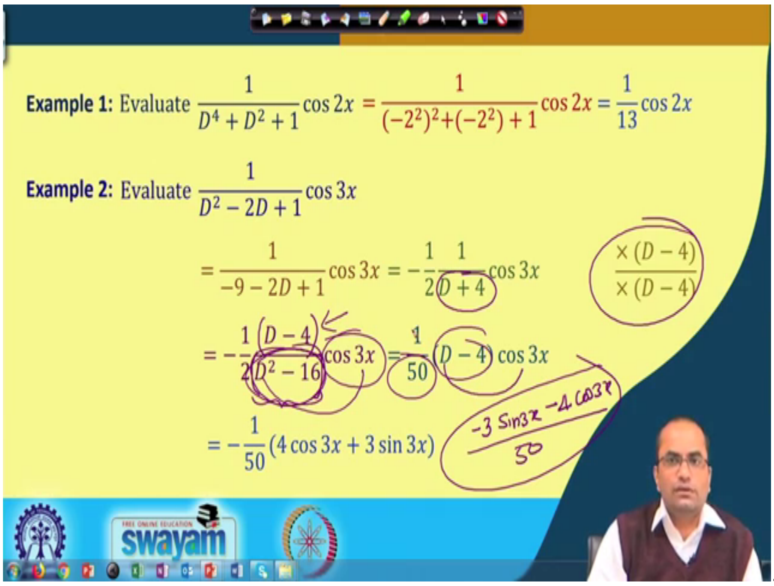

So, this will be 4 and then 16 and here we will get this minus 4. So, it will be 12 and plus 1; 13. So, 1 over 13 and cos 2 x will be the value of this operation here on cos 2 x. We take another example where we will evaluate this 1 over D square minus 2 D plus 1 and this we want to evaluate on this cos 3 x the cosine function of this 3 x. So, now, in this case also we cannot use actually that trick that everything cannot be made as D square like in the earlier case here, but here we have this D square.

So, again we will apply that trick here. So, whatever D square wherever we have this D square that we will first remove with minus this alpha square term. So, meaning here that D square we will replace by this minus 3 square; that means, minus 9 at this place and then we have minus 2 D because we cannot touch this D here plus 1 and cos 3 x. So, that is that is the rule we are applying here that D square is replaced by minus alpha square meaning this minus 9 here.

And now what we have atleast this D square terms are simplified and we will get the terms in D only. So, we have minus this half here because this is minus 8 and minus 2 D so, which can be written as minus half and then this D plus 4 form. So, D 1 over D plus form on this cos 3 x and now we can either use the formula which was derived for the general case in the form of integral or there is another trick which can be used here.

So, if you multiply by D minus 4 and divide by this D minus 4 to this operator, then what we will get the numerator will have D minus 4 and this denominator will have D square minus this 4 square. So, here 1 over 2 was already there and D plus 4 and D minus 4 will give us D square and this minus 16 here. So, we have the D square minus 16 in numerator we have D minus 4. So, now, we have in the denominator also like D square term which we can operate again on this cos 3 x.

And as a result what we will get here 1 over so, D square is replaced by this minus 9 minus alpha square. So, minus 9 and minus 16 so, we will get minus 25 and then we have minus 1 over 2. So, we got here this 1 over our 50 and then D minus 4 operated on this cos 3 x and then we can operate now this operator because once we have this D operator in the numerator then it is a simple differentiation here. So, the D on this cos 3 x we will get this D on this cos 3 x will be sin 3 x with minus sin and the 3 and then we have minus this 4 cos 3 x and then this 50 here.

So, this is the value which we have written in this form minus 1 over 50 here the cos 3 x plus 3 sin x. So, the trick here was that once we have this D plus 4 in the in the D in the denominator here then we can apply with this conjugate here D minus 4 and D minus 4 which makes again this D square term in the denominator meaning this D square minus 16 and in the numerator we have this D minus 4 term. So, D square minus 16 will be now again we can apply because D square term is there.

So, D square will be replaced by minus 3 square and then we can proceed with the numerator once we have the operator in the numerator that simply means the differential operator and when we can operate that easily on the given function.

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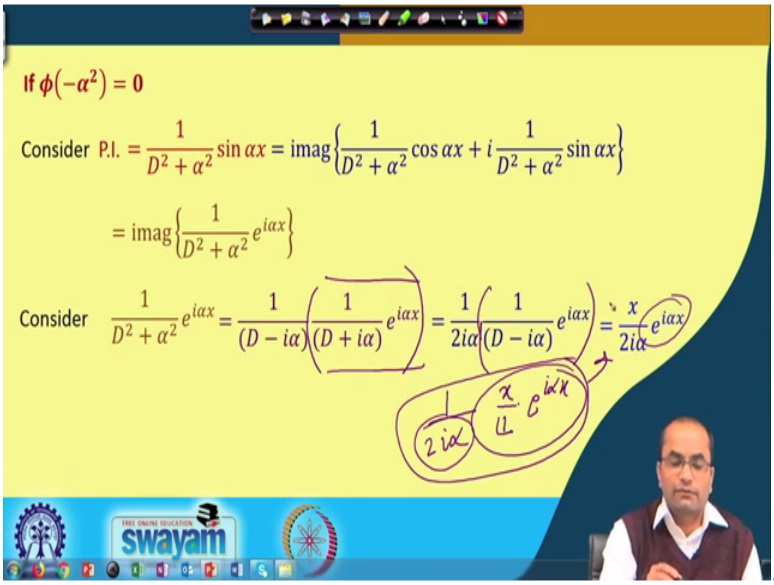

So, here we have this the other case if this phi minus alpha square is equal to 0 in that case the particular integral which we will learn now that 1 over this D square plus alpha square and when operated on this sin alpha x. So, here D square if we replace by this minus alpha square this is becoming 0. So, this is not allowed to do; we are not allowed to perform this formula which was derived earlier, but we have to now rewrite this and then see what finally, we get.

So, what how do we rewrite this now, the sin alpha x so, the sin alpha x we will take as the imaginary part of this 1 over D square plus alpha square that is a operator. So, this sin alpha x we have written actually this cos alpha x plus i times the sin alpha x and we are taking the imaginary part of this meaning we are talking about 1 over D square plus alpha square sin alpha x which is the operation we want to do here.

So, writing this in terms of this plus introducing this i and again the same function which we want to evaluate; the benefit we are getting that now inside this. So, we will collect this imaginary part of whatever we get here, because the imaginary part is precisely what we want to evaluate, but what simplification we have made in this way that if we combine this. So, we have 1 over D square plus alpha square and cos alpha x plus i sin alpha x. So, cos alpha x plus i sin alpha x will be exponential function with i alpha x.

And then we have this operation 1 over D square plus alpha square and we know the operation on the exponential function we have already learned from the last lecture. So, we will consider now this 1 over D square plus alpha square operated on exponential i alpha x. This we can rewrite now 1 over D square plus alpha square as this 1 over D minus i alpha and into 1 over D plus i alpha.

So, D square plus this alpha square we have written like D square minus i alpha whole square and then we can write down this D minus i alpha and D plus i alpha form and we know already that how to apply this 1 over D plus i alpha, because here when D will be replaced simply by i alpha term. So, we will have 1 over 2 i alpha as result of this operation here and then the rest is 1 over D minus i alpha operated on the e power i alpha x and this 1 over 2 i alpha we have taken this out of this operation.

Now finally, this one 1 over D minus i alpha e power i alpha x and this is precisely the case which was also discussed separately there that this is becoming 0 when we directly substitute this D here, but we have derived the formula and that says the value of this 1 here will be this x over factorial 1 and this same e power i alpha x.

So, now, this the value of this will be x over 2 i alpha because x over factorial 1. So, out of this here this operation we will get x over this factorial 1 and e i alpha x and then together with this one we have 2 i alpha. So, this we have written here x over this 2 i alpha and exponential i alpha x.

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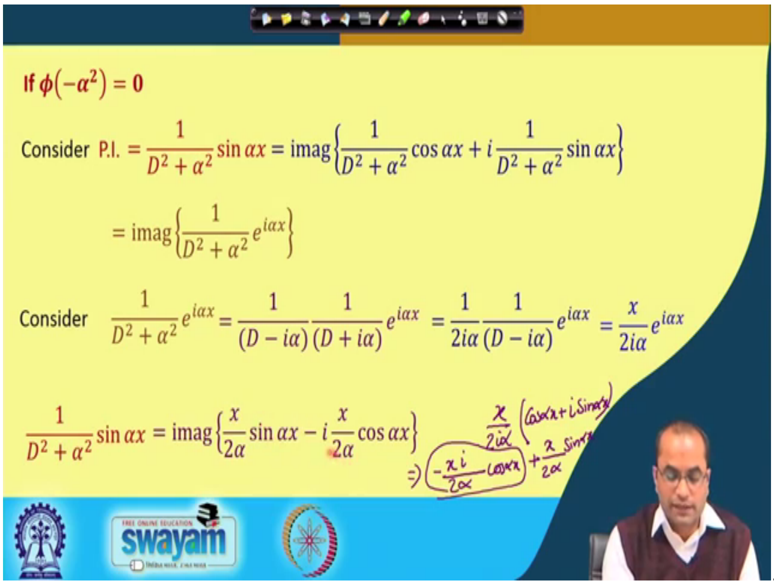

So, now we want to get this 1 over D square plus alpha square operated on the sin alpha x and we want to take the imaginary part as the derivation suggests here.

So, the imaginary part of this now we will get so, what will be the imaginary part. So, we have here this x over 2 i alpha and this e power i alpha x we will write cos alpha x plus i sin alpha x and this will be the first part will be minus x. So, I will multiply here minus x and this i so, this i square which is minus 1 2 alpha will be with the cos alpha x and the second term will be free from this i. So, i, iwill cancel and we will have x over 2 alpha with the sin alpha x term.

So, the imaginary part we will now extract from this one and that will be the imaginary part of this x over 2 alpha sin alpha x and minus this i times x over 2 alpha and cos alpha x. So, the imaginary part for the sin will be minus with the minus sign x over 2 alpha cos alpha x.

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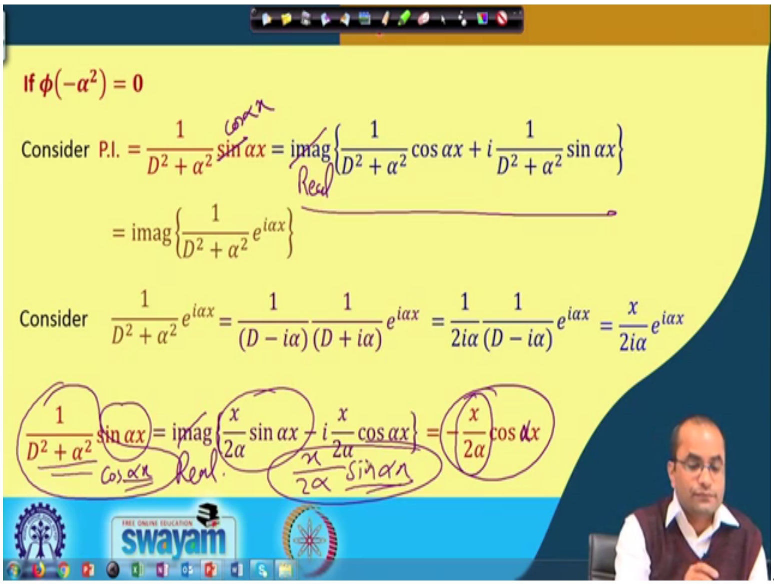

And that is that is a result we have got here the imaginary part of this is nothing but minus x over 2 alpha cos ax. So, the direct result what we have that 1 over D square plus alpha square operated on sin alpha x we will get the value here minus x over 2 alpha. So, here x over 2 alpha and the cos sin will be taken as the cos alpha x or this is alpha here. So, the cos of alpha x and minus x over 2 alpha term will come out of this.

So, just to remember because 1 over D so, this inverse operator so, the sin will be taken as the cos the integral of this sin will be the cos with the minus sign and x over 2 alpha factor will come, because when we take here cos alpha x. So, what will happen for cos alpha x this will be the real. So, if we put here cos alpha x then we want to have the real part of this means finally, the real part of this and real will be this x over 2 alpha with this sin alpha x. So, that will be a result when we apply this 1 over D square plus alpha square on the cos alpha x the value will be x over 2 alpha and sin alpha x.

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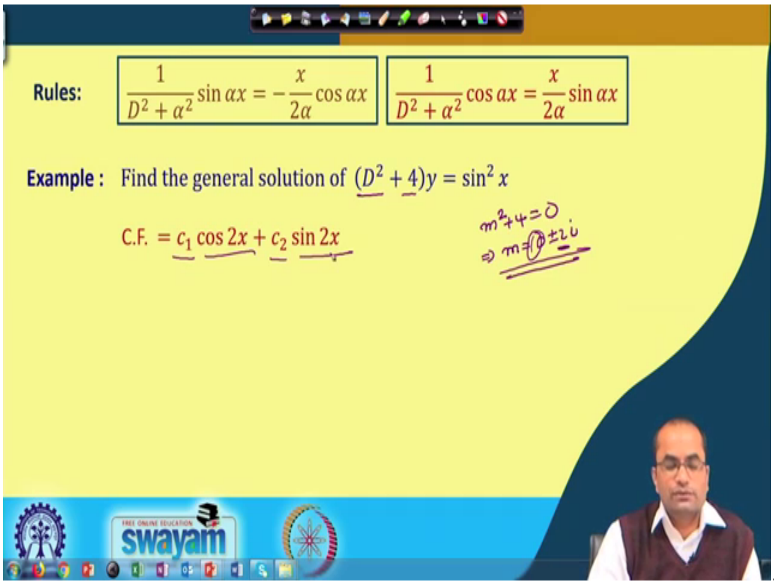

So, just to summarize now these two important results what we have when we apply 1 over D square plus alpha square on sin alpha x the result will be minus cos alpha x with this factor x over 2 alpha, for the cos will be the sin here alpha x with the same factor x over 2 alpha.

So, just an example find the general solution of this D square plus 4 y is equal to sin square x and first we need to because we need to find a general solution. So, we have to find a general solution of the homogenous equation or rather we call it complementary functions. So, we have to find the complementary functions of this where the auxiliary equation will be m square here plus 4 is equal to 0. So, m is equal to 0 plus minus this 2 i and then the complementary function takes exponential 0 x that is 1 so, c 1 cos 2 x this 2 here. And c 2 the sin 2 x so, this is the auxiliary function of this differential equation.

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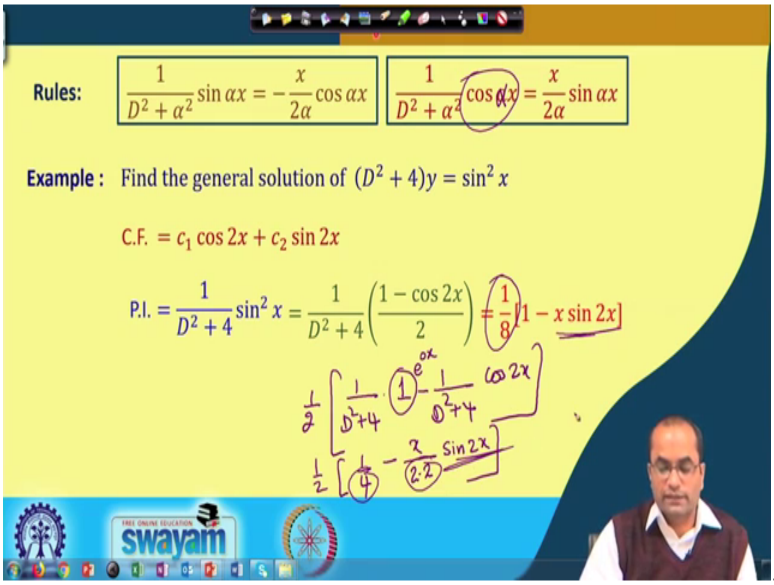

And then we will get the particular integral which will take the form 1 over D square plus 4 and sin square x and we know the results here on sin alpha x and the cos alpha x those are results we know. So, what we can apply here this we can rewrite the sin square x into this 2 x angle. So, that will be 1 over D square plus 4 and the sin square x will be written as 1 minus cos 2 x divided by 2 and now we can apply this operator on this 1 minus cos 2 x so, 1 over 2 is a constant we can take out here.

So, 1 over this D so, 1 by 2 we can take out and we have 1 over this D square plus 4 operated on 1 minus this 1 over D square plus 4 operated on this cos 2 x here and then this 1 as the trick we have used yesterday we can take it is exponential 0 x. So, this D will be replace replaced by 0. So, we will get this 1 by 4 first and then here this cos 2 x. So, we are applying here this cos 2 x 1 over D square plus alpha square. So, that will be x over 2 a is 2 and then we have cos will be the sin 2 x. So, this 4 again here also 4 so, we have 1 over 8 and then we will get 1 minus x and the sin 2 x. So, that is the result we are talking about here.

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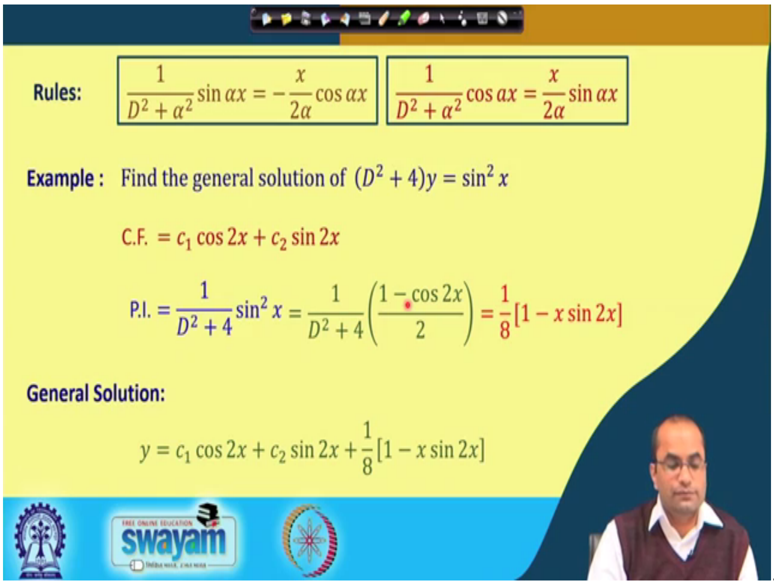

And the general solution will be just the complementary function and plus this particular integral.

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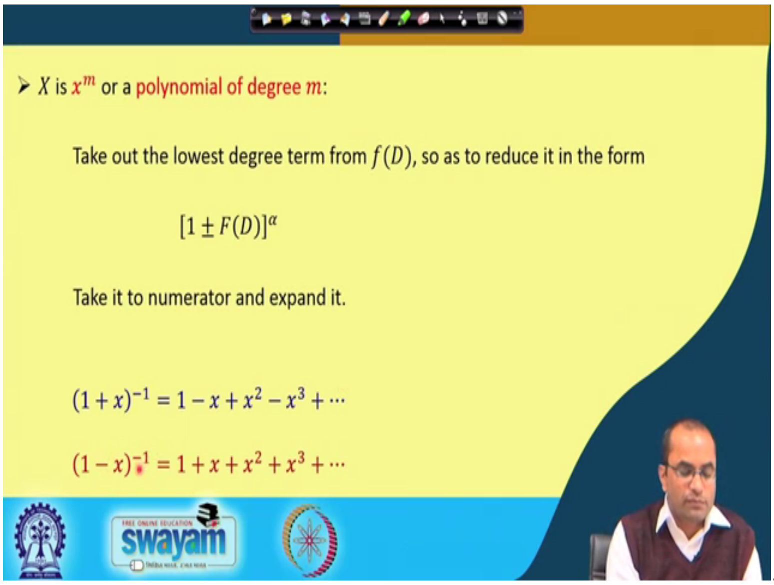

There are two more rules frequently used here. So, we will not go for the derivation, we will just write down the rules here. So, if x power m is a polynomial of degree m then how to handle this; the idea is which will be explained properly with the help of example.

So, take out the lowest degree turn font from this operator function there f D and so, as it to reduce to this 1 plus some another function of f D power alpha. So, we have in the denominator this F D from there we will take the lowest degree term. So, that we get something of this form 1 plus or minus F D and power alpha;usually this alpha is 1 most of the cases there and when we do this we will take now to this numerator and then we will expand it.

So, we will get in the expansion only the differential operator which we can handle easily, but this is easy this is possible when we have the right hand side like x power m, because the differential operator when we apply on x power m this polynomial kind of result there. So, we can exactly get those derivative terms those differentials there, mostly we have to use these expansions there when we take this to numerator. So, 1 plus x as 1 minus x plus x square minus x square or 1 minus x power minus 1. Then we have to use this formula 1 plus x plus x square plus x cube.

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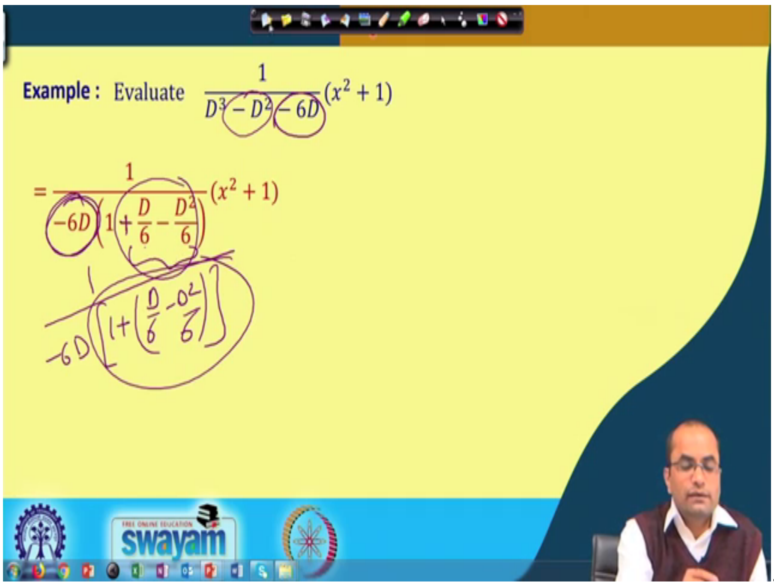

So, here we take just this example to demonstrate this idea which is again quite general. So, here we have x square plus 1. So, this polynomial term and 1 over D cube minus this D square and minus 6 D. So, what we will do, we will take this lowest order term from this denominator that is minus 6 here. So, minus 6 D we will take common and then this will be 1 and then here we have this minus D square. So, the minus has been taken out. So, will be plus there and then we have 6 over D and here minus D square by 6. So, taking this minus 6D out we are getting this 1 plus this D by D by 6 and minus D square by 6 term. So, 1 plus and this 1; so, we are getting in this denominator here with minus 6D 1 plus D by 6 minus D square by 6 and we can take this to the numerator then.

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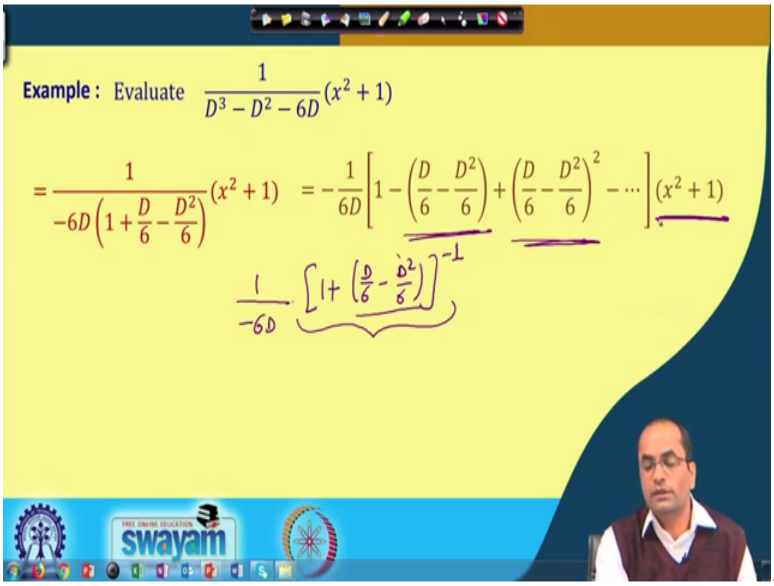

And by doing so, and when we expand it so, the numerator we have taken like minus 1 over 6 D remain as it is and the numerator we have taken 1 plus this D over 6 minus D square minus 6 to power this minus 1 which we can expand again here. So, 1 minus this D by 6 minus D square by 6 and plus the same here the square term and we do not have to consider more because our polynomial is of degree this 2 and we have here up to the D square and terms we have collected all the next term will have certainly D cube which will make it to 0. So, we do not have to now expand more because when we operate these higher order terms this x square plus 1 will become 0 so, that is enough up to this D square terms.

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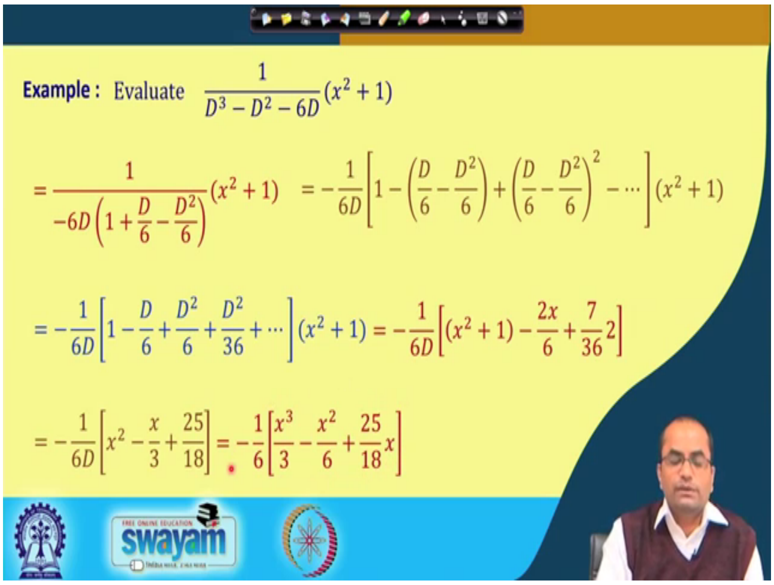

So, now we will collect only the D square term from here. So, we have minus 1 over 6 D 1 minus this D by 6 term and minus plus here D square by 6. And from this term also we will get 1 D squared term that is the square of this D by 6 so, D square by 36. All other terms of this square here will be D cube and D 4, but they will make this x square plus 1 to 0 so, we do not have to consider those terms.

And now we can apply this differential operator on this x square plus 1. So, here 1 with this x square plus 1 will be as it is when we apply on D. So, we have 1 by 6 and D of this x square plus 1. So, derivative of this x square plus 1 will be 2 x there and then we can add these 2. So, here we will have 7 by 36 D square 7 by 36 and this D square when we operate on this we will get 2 there. So, now, we can simplify this we have this result 1 over 6 D and x square minus x by 3 plus 25 by 18; 1 by D means the inverse operator so, the inverse to this differential meaning this simple integration. So, 1 over D of this means we have to integrate once which that will be the result here for this integration.

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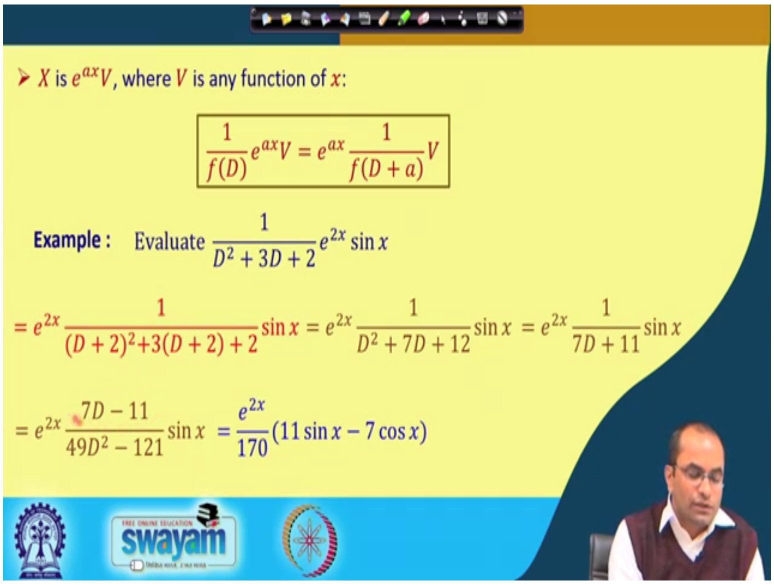

So, this direct formula for the case when X is e power ax into this V, V is any function of x takes this form. So, 1 over f D power ax V so, we can take this e power x out of this operation e power ax, but here the D will be replaced by D plus a and this V will remain as it is. So, that is a kind of shifting theorem which we use shifting result here.

Again just to show one example based on this we have 1 over D square plus 3 D plus 2 e power 2 x and then this sin x. So, here e power 2 x we have taken out of this operation, but then D will be replaced by D plus 2 which is done here and now we want to operate this on the sin x and thus what we have already learned here. So, just to simplify this we will get D square plus 7 D and plus 11 on this sin and then we can replace this D square by the minus 1 as discussed before. So, we will get 1 over 7 D plus 11.

And then we have to multiply here 7 D minus 11 divided by the 7 D minus 11 to get this D square again there and this sin x again here this D square will be replaced by minus 1 and then this operation we can apply on the sin x to get this result.

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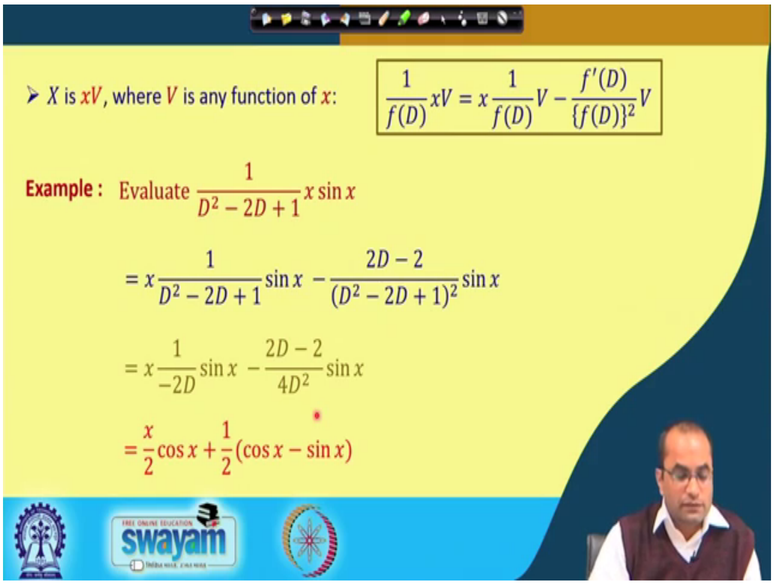

Well, so this is the final direct formula which can be used and we are not providing the proof now here again. So, 1 over f D x V so, x into some V, V is again function of x so, this formula says it is kind of product rule. So, this x you can take out apply this 1 over f D on V minus then you have this f prime D over f D whole square and apply on this V.

So, with this formula also we can get directly the particular integral like in this example 1 over D square minus 2 D plus 1 applied on this x sin x. So, we will use this formula. So, we will take this x here apply this whole operator on the sin x then minus the derivative of this f here. So, D square I mean with respect to D so, we had D square minus 2 D plus 1, the derivative here with respect to D will be 2 times D minus this 2 here then f D whole square. So, the whole square of this term D square minus 2 D plus 1 and then we have the sin x there.

So, this we have to evaluate which we have already learned how to do that. So, which can be done like D square minus 1. So, this this get cancel, this will become 4 D square and again this D square will be replaced by minus 1. So, it will be here 1 by 4 and then this 2 D again we can apply on this sin x and here also 1 over D on 2 sin x, then this will be just the again the integral of this sin x there. So, you will get this result using this formula discussed here.

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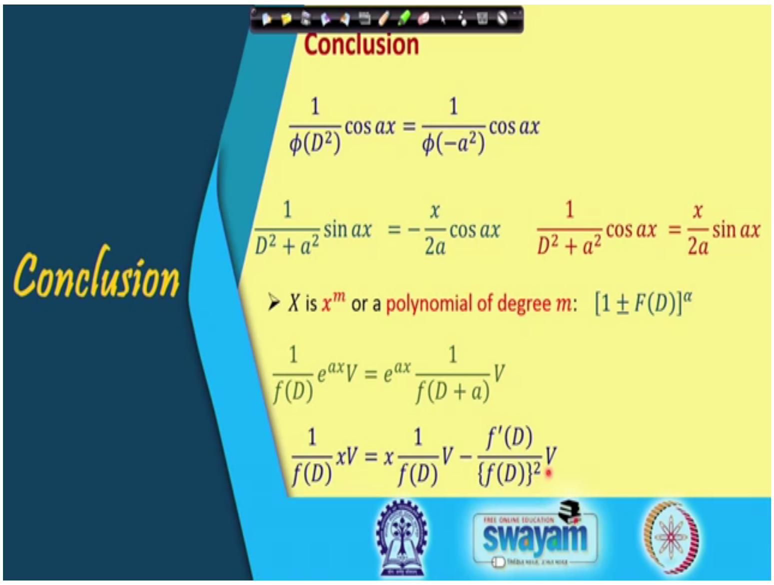

Coming to the conclusion now so, we have learned several direct formulas one was this 1 over this phi D square cos ax which was the rule says that D square will be replaced by this minus a square. And this 1 over D square plus a square sin a x when we cannot do that because the D square replacing by minus a square making it 0 the formula was that this result is minus x over 2 a cos x, same thing we have for the cos ax and in this case this will be x over 2 a sin ax.

When this x power m the right hand side is a polynomial this idea will work that we can write down in this denominator as 1 plus minus F D power alpha which we can take to the numerator and expand it. We had another direct formulas that a power ax is sitting with some function of x then e power exponential this a x can be taken out and then this will be a shift here in the D will be shifted by D plus a. Finally, we have also learned about this formula when 1 over f D we are applying on x into V then this is kind of product rule here.

The x and then you apply this 1 over f D on V then minus this the derivative of this f D with respect to D and divided by this f D whole square into V. So, we have seen so many tricks about getting this integral, particular integral we have also learned how to compute the complimentary function or the general solution of the homogeneous equation. So, to find out the general solution of the non-homogeneous equation we have to just add the two the complimentary function and the particular integral which we have learned in many situations.

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So, these are the references used for preparing the lectures and.

Thank you for your attention.