Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 58
Solution of Higher Order Non - Homogeneous Linear Equations

So, welcome back and this is lecture number 58. We will be talking about the solution of non-homogeneous linear equations.

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And in particular, we will look for the particular integral and the solution techniques for evaluating the particular integral.

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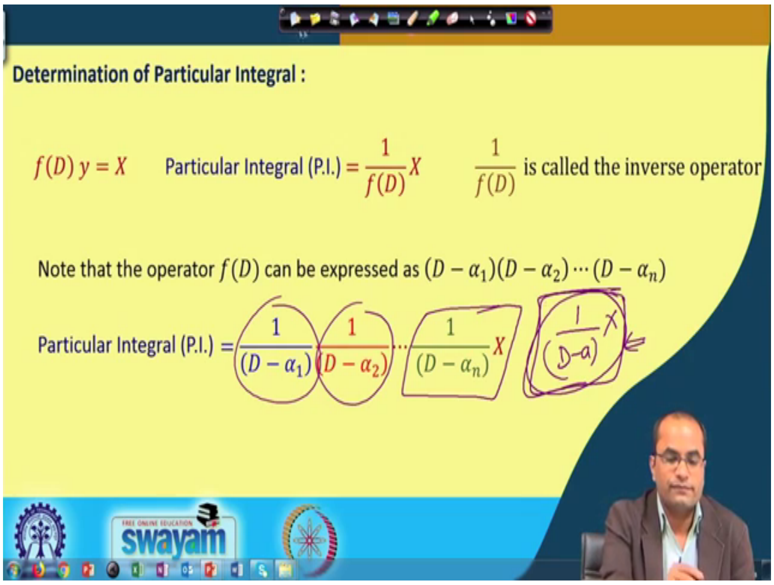

So, in the last lecture, we have already seen that how to get complementary function of the differential equation of the linear differential equation with constant coefficient. And also we have discussed already that to find the general solution of a given a differential equation, we need to find the complementary function and we need to find the particular integral and when we add the two, we will get the general solution of the given on homogeneous differential equation.

So, in today's lecture, our focus will be how to find a particular solution of the given differential equation. So, the given differential equation retain in this operator form we have this f D y is equal to the X. X is a function of small x and then this expression here is written by this f D which has been already discussed in previous lecture. So, the particular integral of this equation or a particular solution of this equation, we will denote here like 1 over f D and operated on X.

So, this 1 over f D is like the inverse operator which we multiply here to this equation and then we get directly this y here as a solution, 1 over f D and this X and we will see here that how to how to operate the this 1 over f D on function X here which is a function of X. So, this f D is called the inverse operator and the idea here which was already discussed that is f D can be expressed in terms of this D minus alpha 1, D minus alpha 2 and so on D minus alpha n where these alphas are the roots of the auxiliary equation. So, the particular integral to written in this term 1 over this f D x and this f D is nothing but the product of these D minus alphas.

So, we have written here like 1 over D minus alpha 1, 1 over D minus alpha 2 and the product of this 1 over D minus alpha n operated on this X. So, what is important here to evaluate this expression, actually the important is here if we know the way to evaluate this 1 over this D minus a this operator on X. So, if you know this how to evaluate this one over D minus a on X, then we can repeatedly apply this idea and we can find the particular integral, the general form this 1 over f D on X. So, now, first we will discuss here that how to operate this 1 over D minus a on X or what is the value here when we operate this 1 over D minus a on X.

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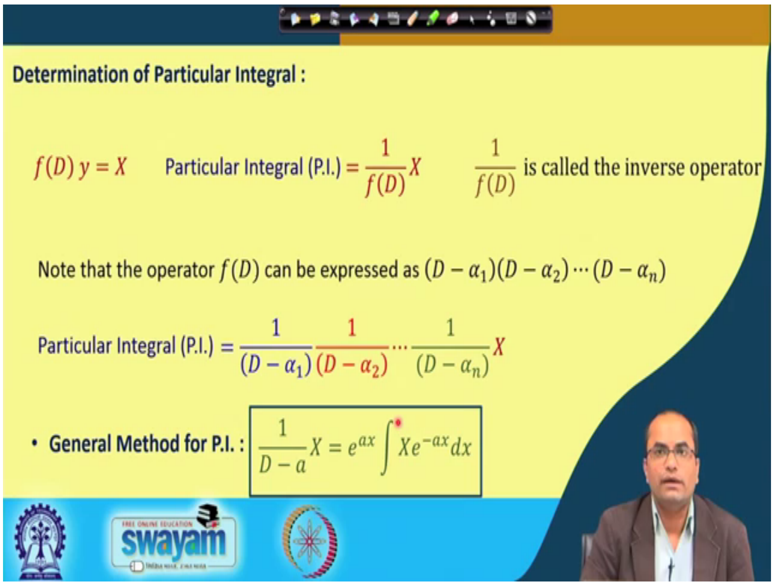

So, this is the way general method for finding this particular integral which we are discussing at first later on we will go for some special forms of this X and they will be some direct evaluation techniques which will be also discuss later. So, here first we will prove this result at when we operate this 1 over D minus a on this X, we will get actually this exponential a x an integral of this X multiplied by e power minus a x and then over this D X.

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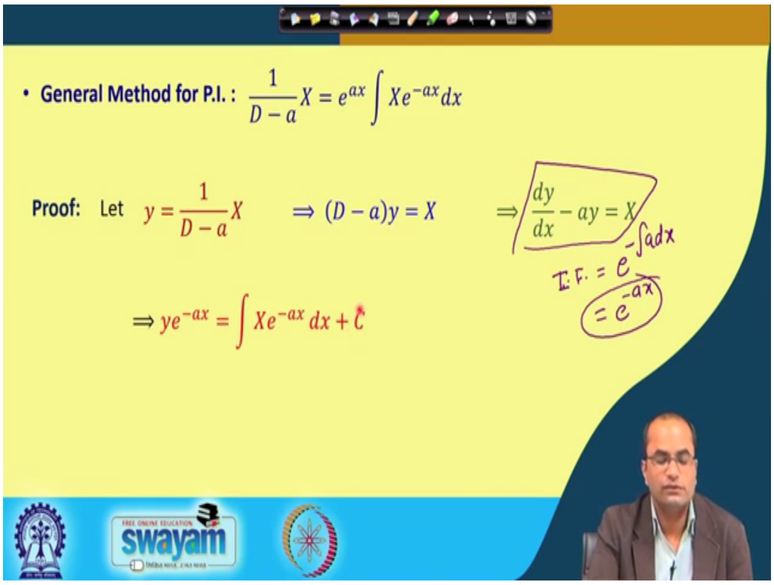

So, to see this how this result we are getting in terms of the integral, we need to consider here let this y is equal this one over D minus a X. So, we have assume that this 1 over D minus a is X and we will find out that what is actually y.

So, to find out this y, what we will do here we will multiply this equation by this D minus a operator so that we will get back to the differential equation which we know already have to solve because when we multiply here by D minus a; we will get this D minus a applied on this y is equal to this X. And this is a first order differential equation which we know how to solve indeed this is a linear equation in y because dy over dx minus this a y is equal to the right hand side function this X. Here, we can find out the integrating factor now. So, the integrating factor of this equation will be e power minus a dx meaning this e power minus a x that is the integrating factor.

So, once we know the integrating factor, we can write down the solution which was already discussed in previous lecture. So, here the y into the integrating factor which is e power minus a x is equal to the integral the right hand function here X and multiplied by again this integrating factor e power minus a x integrated over this X plus a constant of integration. Just a note here, this constant of integration is not important.

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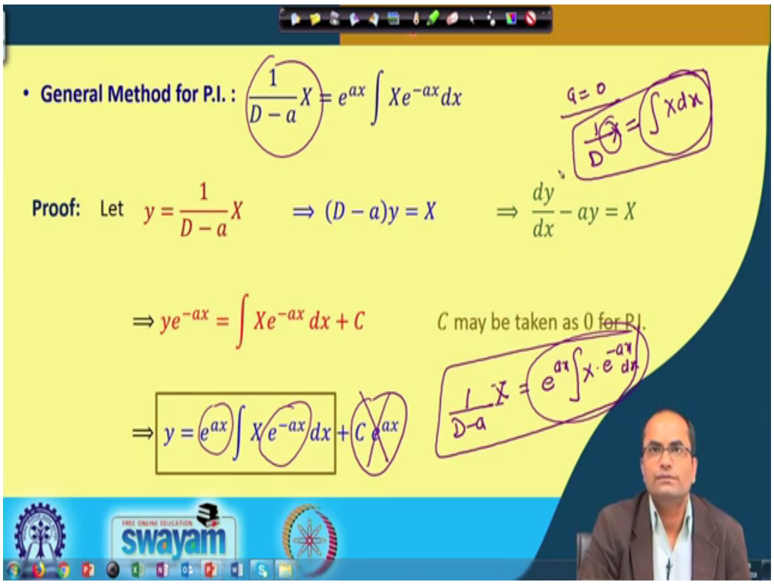

And we can set this constant of integration as 0, the reason is because we are talking about here or we are discussing how to find a particular solution of the differential equation. So, while finding the particular solution, we do not want this constant of integration also.

For any value of C, that will be the solution. So, we are looking for a particular solution, in that case we can set just this C to 0. If you want to keep and continue and finally, we will add into the complementary function to find a general solution of the given homo non homogeneous equation, in that case this constant will be must to the constants we appeared in complementary function. So, this is not important here we can assign any value in principle, but the simplest case would be that we assign this C to 0.

So, now this solution of this equation, so y will be e power a x and this X e power minus a x dx plus this constant e power a x. So, when this constant is 0, this term will disappear and basically we get this the value of this 1 over D minus a X is equal to this. So, this is very important now here the important result which will be very useful now to find the particular integral. So, we got this result that when we apply the D minus a on a function here X, then it is value is nothing but e power a x and integral this function x itself, e power minus a x dx. That is a particular solution when we said this to 0 here this is not important. So, we can get this particular solution of this 1 over D minus a when operated on this X.

In simple case, what we can consider here for instants this a 0. So, what this result is suggesting. So, when we set this a is equal to 0, what is this result here that 1 over D operated on X is equal to; so, a is 0, this is 1 here and this is also 1. So, we are getting simply the integral X dx and this is what expected we know this that D was a differential operator and this 1 over D operated on X. So, this 1 over D is like integral operator. So, this is precisely the integral of this X over dx. So, 1 over D operated on X is nothing but the integral of this X with respect to small x.

So, this is more general formula here other than 1 over D we have this 1 over D minus a operated on X and the value is e power a x integral X e power minus a x. So, this is very important for this lecture now.

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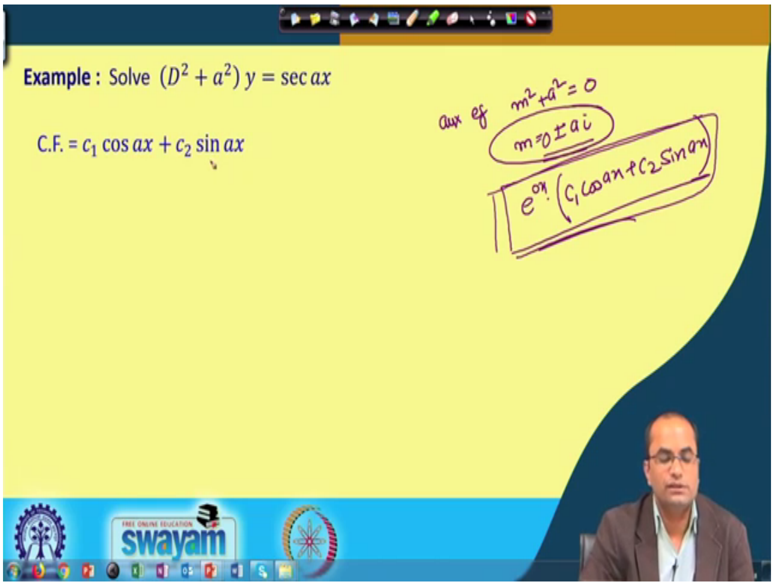

And we can use this formula for instance to compute this D square plus a square y is equal to the sec a x. So, what do we have here, first we need to find the complementary function and for that we need this equation here the auxiliary equation. So, the auxiliary equation in this case will be m square plus this a square is equal to 0 and we will get the roots as a i plus minus this complex conjugate and then we know how to write down the solutions. So, it is like 0 plus minus e power a i. So, it is a e power 0 x and then we have the c one and cos this a x plus c 2 sin a x. That is the solution of the homogeneous equation when this right hand side is set to 0 or this is rather to say it is a complementary function here.

So, this is c 1 cos a x plus c 2 sin a x we have the complementary function. So, to find this general solution or this we need the complementary function and we need a particular solution.

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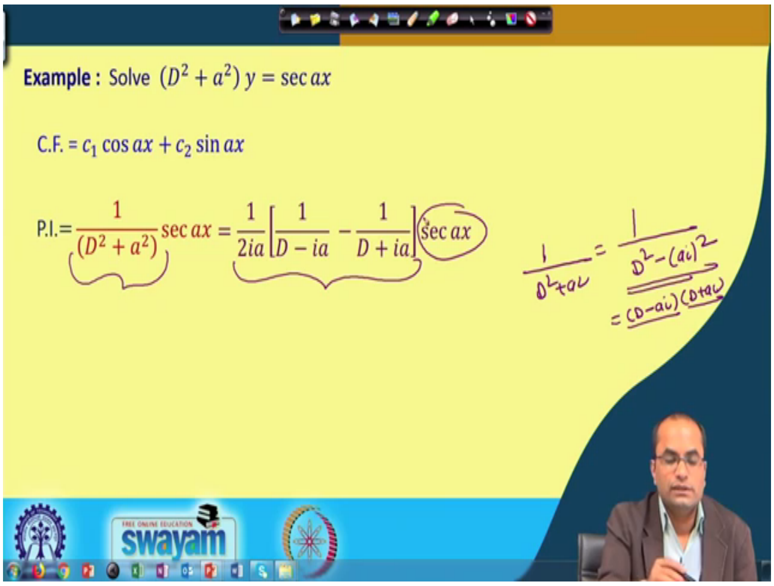

So, to find this particular solution, we will apply the idea which is discussed in the previous slide. So, we have one over this inverse operator, 1 over D square plus a square and this operated on this sec a x. So, what we can do here, we can write down this D square plus a square as D minus; so, 1 over this D square plus a square is written as one over here the D square and this minus a i square; that means, here D plus i a and D minus i a this is the product of this D minus a i and D plus a i and then so, 1 over D minus a i D plus a i and then we can do this partial fractions. So, we will get this expression which is given here 1 over D square plus a square and now we want to operate on this sec a x.

The idea is because we know already this formula here for 1 over D minus i a or 1 over D plus this constant.

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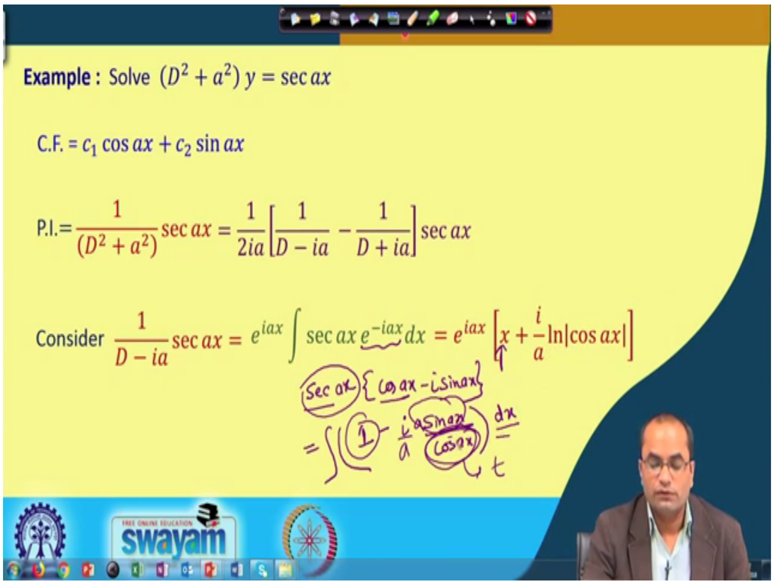

So, now we need to compute here, we need to consider for instance the first one, 1 over D minus i a operated on this sec a x and the formula which was derived before. So, just that this value will be equal to e power this constant times x integral this function sec a x and exponential with minus sin minus i a x and d x, or we can leave the constant of integration to find this particular integral. So, here we have e power minus i a x and then this sec x. So, here we have this sec a x and this e power i a x which we can write as cos a x minus i sin a x.

So, when we multiply this to sec there, so this will be 1 over cos a x. So, this will give 1 there and then we have this minus i and we have sin a x divided by this cos a x. So, this is the integrand here and we are integrating over x here. So, this 1 will give the x here, the integral of this and then this cos a x will be substituted as t and then the sin a x is there. So, we need again a there. So, we have exactly this a sin x the differential of derivative of this cos a x and then with this minus sin because cos a x will give minus sin a x when we take the derivative of this.

So, the value of this integral will be this with plus sin i over a and this logarithmic because this is like 1 over t. So, the logarithmic of this cos a x which is written here. So, this is nothing but the value of this integral here x plus i over a and the logarithmic of the cos a x with the modules.

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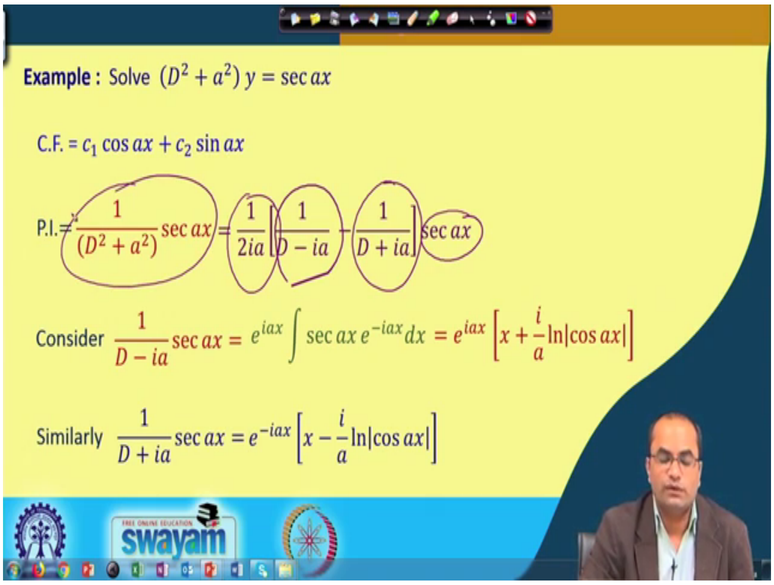

So, similarly once we have computed this 1 over D minus a sec a x which is coming to be this one here, what we can do we can also compute this 1 over D plus i a. So, 1 plus 1 over D plus i a over this sec a x, what is the difference now in the two here, the i is replace by minus i, nothing else.

So, we will do the same here this, i will be replace by minus i in here also this minus i. So, only this change will come the rest everything will be the same. So, we have evaluated this 1 over D minus i a operated on this sec a x and also 1 over 1 plus 1 over D plus i a operated on this sec a x. So, we can we can get this difference now and multiply this 1 over 2 i a to see this value of 1 over D square plus a square sec a x. So, that is what we will do now.

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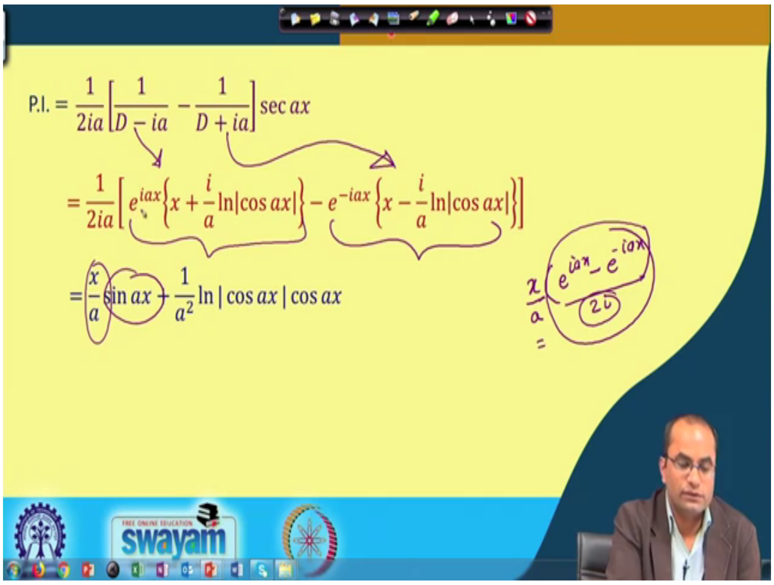

So, this particular integral which was given here, so we can now substitute the value of this 1 over D minus a operated on sec x which was given here and then minus sin with 1 over D plus i a which is written here.

So, this is corresponds to this 1 over D minus a and this corresponds to second one here 1 over D plus i a and this factor remain as it is 1 over 2 i a. Now, we can just simplify this. So, this will be the x here also x here. So, e power i a x and minus e power minus i a x divided by this 2 i will give the sin a x and then we have x over a here. So, this first term here with e power. So, x common and then e power i a x and then we have e power minus i a x and this will be when we divide here 1 over 2 i was there already.

So, this x over a and this 2 i is here. So, this will give this sin a x from x over a. So, this is the first one here.

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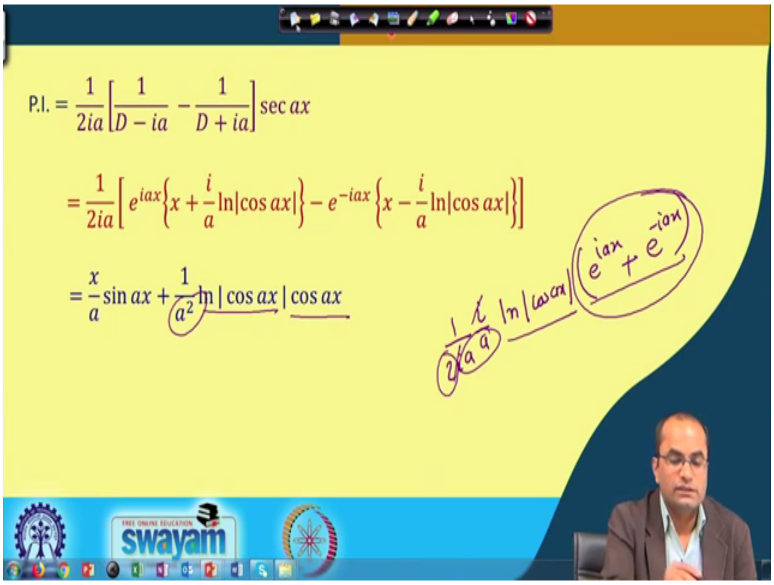

And then, the second one when we have again this common term as the log of this cos a x, we take this common from the two. So, this will be also plus here i over a and this is also ii over a. So, here i over a and the outside is 1 over 2 i a and then we have when we have taken this common e power i a x and this will be with plus sin e power minus a i x. So, this i gets cancel and this 2 will be merged here. So, this is nothing but the cos a x term with a logarithmic of this one and 1 over this a square is here.

So, this is just the simplification of what we got here this, 1 over a square l n cos a x and cos a x again.

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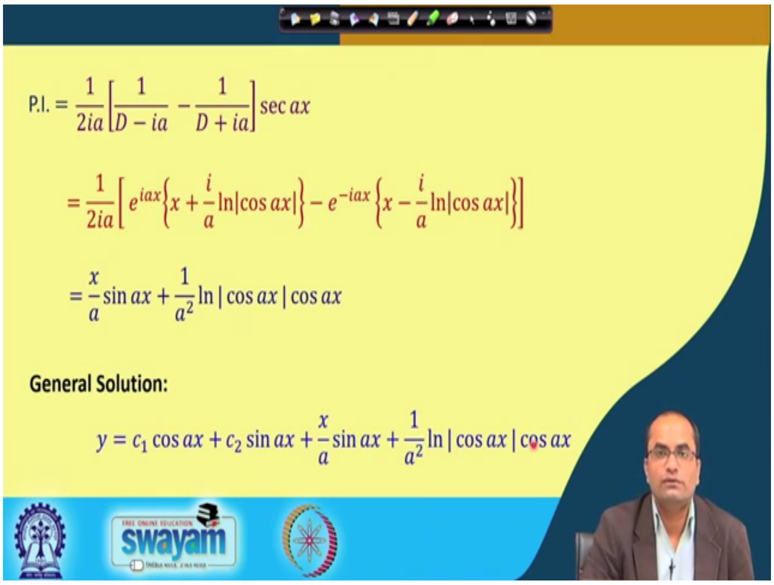

So, the general solution now will be the complementary function which we have evaluated earlier that was c 1 cos a x plus c 2 sin a x and this particular integral. So, this is the particular solution of the given differential equation, the given non-homogeneous differential equation, this is the general solution here and when we add the two, so we have this the general solution of the given differential equation.

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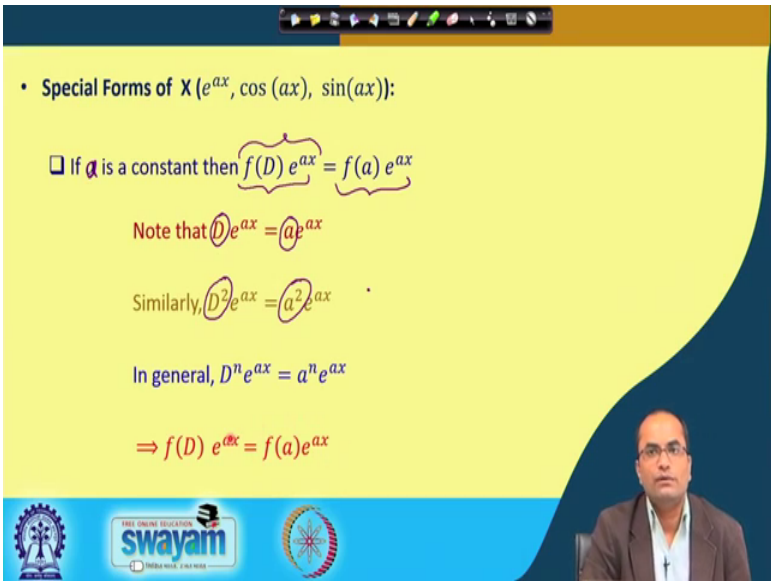

Now, we will be talking about some special forms of this X for instance e power a x cos a x sin a x etcetera. So, in those cases, instead of evaluating this like we have done earlier with the help of that integral formula, we can also directly remember these formulas and that will be much easier in many cases to evaluate when we have the right hand side the simple such simple functions. So, the first to derive, today will be deriving this when our special function is e power a x and for that we need this result here that this. If this a is a constant here, a is a this is a is a constant then this f D operated on e power a x is nothing but f a e power a x; that means, when we apply this f D on e power a x, the value the value of this operator when operated on e power a x will be nothing but f a. So, D will be replaced by a and we have this e power a x.

So, this we will see first which is very simple to realize because when we operate D on e power a x what we have getting. So, D means the derivative the difference the derivative of this e power a x which is nothing but e power a x multiplied by this a. So, what we realize here once when we have applied this derivative here the D is replaced by this a. Same thing, if we do this two times what will happen we will get a square with this e power a x and again the same rule the D square is replaced by the a square term. If you continue this, in general also we will get this D power n e power a x and this D will be replace by simply a power n e power a x.

So, when we have this operator f D which is usually D power n plus I mean kind of this polynomial. So, the same thing will happen when this D power n e power a x that will be replaced by the D is going to replace by a. So, in the whole expression here when we have this f D we apply on e power a x, what will happen that all these D will be replaced by a. So, instead of this f D, we will get f a and this e power a x will remain as it is. So, this is the general result what we will be useful now to derive when this right hand side x is the special function e power a x. So, what we have here f D e power a x is equal to f a e power a x.

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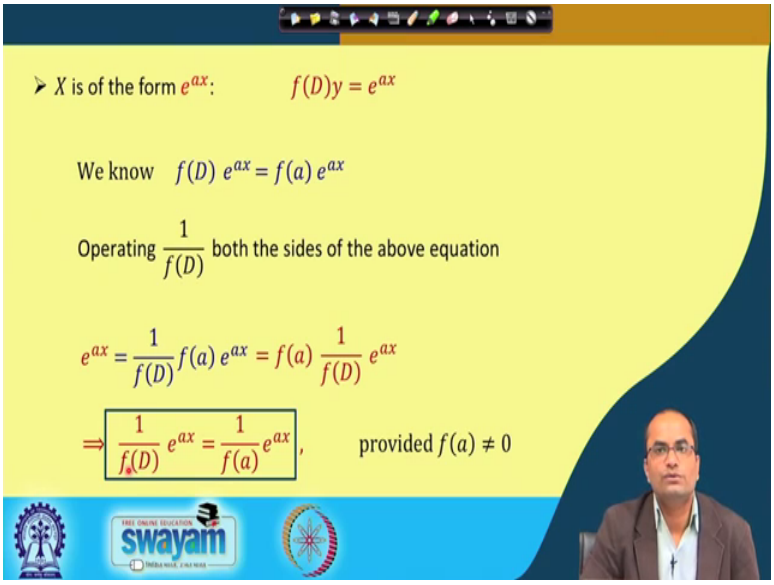

So, with this now we can derive now the first result of this special form and that is here if X is of the form e power a x, then what will be our formula to evaluate the particular integral to find a particular integral? So, this is for instants I have differential equation f D y is equal to e power a x, y right. So, the right hand side here is the special function this exponential a x, what we know already from the previous slide that f D when applied on e power a x, we get f a e power a x and now what we do here we operate both the sides this inverse operator 1 over f D and what we will get then here when we apply. So, this f D and the inverse f D will just give the will cancel each other. So, we have e power a x is equal to and this 1 over f D, we have this inverse operator applied on this f a e power a x and this f a has nothing to do with this inverse operator because this is just a constant.

So, we can take we can take this constant out and f a we have basically 1 over D and operated on this a x. So, now, from this relation that e power a x is f a 1 over f D e power a x what are we getting that this inverse operated on this e power a x here is nothing but e power a x divided by this f a. So, what is this interesting result here that when we operate this 1 over f D on e power a x, we do not have to do much we do not have to use that formula which was derived earlier.

What we can do just the D will be replace by a and that is the value of this operator 1 over f D when we have this right hand side function e power a x. But it should be noted here that this f a should not be 0, otherwise we can a divide here and this formula will not be valid. So, this is again important point that f a should not be equal to 0, otherwise this does not make sense.

Now the question arises that what will happen if this f a is 0. So, we have this f D and when we substitute this ah a here and if this becomes 0, then what will be the formula, what will be the changes here in this particular integral.

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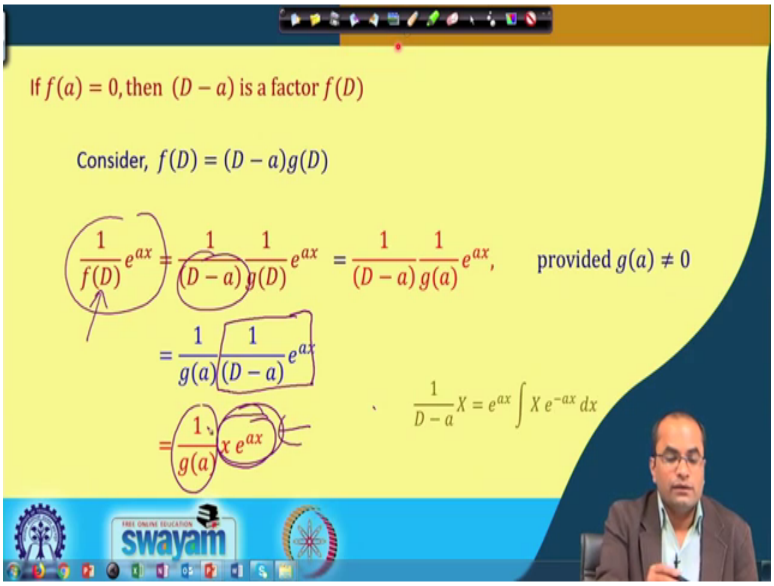

So, now this is precisely the case here when we take that if this f a is equal to 0 then definitely D minus a is a factor of f D because something is making this 0 this function here is a polynomial function. So, there is a factor D minus a which is making 0 either D minus a or D minus a square or D minus a cube. So, there is a factor D minus a it may appear in some power, but there is at least one factor with power 1 here D minus a in this f D.

So, let us just consider that f D is nothing but D minus a and this g D. So, another function the left over here when we have taken this D minus a out, so in that case what will happen now, 1 over f D e power a x when we operate here this we can write down as 1 over D minus and 1 over g D because f D was this D minus a and g D. And now, we can operate first this e power a x on this 1 over g D suppose this g a is not equal to 0. If g a is also 0, then we have certainly another factor ascertain here in this g D and that we can again take out from this g D and this maybe the square in that case and then whatever left here, they we will substitute, we will replace D by a.

So, that is the trick we have to do here. So, here we have this 1 over g D, we assume that g a is not 0 because we have already taken this D minus a out here and suppose this g a is not 0, then what will happen that this value of this value of this operator here one over g D operated on e power a x we have already seen if g a is not 0 the value here will be nothing but this one here, 1 over g D operated on e power exponential a x, then we have 1 over g a e power a x and again this factor 1 over D minus a is sitting as it is. So, provided here that this g a is not equal to 0 that is the assumption again. If this is 0, we will handle again in a similar fraction. So, we will take again this factor D minus a out of this g and in the remaining we will again handle by replacing this D by a.

So, what now, so we have this 1 over D minus a to be operated now on this 1 over g a e power a x ; that means, this 1 over g a we can take out because this operator will not do anything on the constant it is like the Integral operator which we have already seen before. So, 1 over D minus a operated on e power a x. Now, we will use this formula which was derived in general that 1 over D minus a x is nothing but e power a x integral X e power minus a x D x. So, here when we apply this one, so our the right hand side this X is e power a x. So, here e power a x and this X is nothing but e power a x. So, e power a x and e power minus a x will cancel out each other and we will have under this integral only the 1.

So, the integral of this 1 dx will be just x. So, we get here the x. So, the inverse operation here inverse operator 1 over D minus a e power a x will give us x e power a x and this 1 over g a is sitting as it is. So, what we got now that if 1 over f D e power a x and this has a factor if f a is equal to 0. So, it must have a factor like D minus a and the value will be when we have 1 minus 1 over D minus a the value will be x e power a x of this operator.

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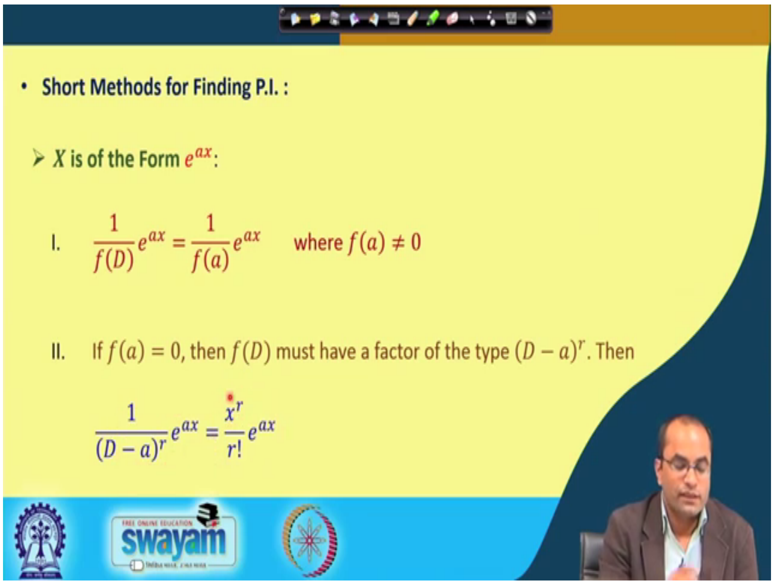

Well; s, now what are the short methods for finding this particular integral yes if this X is of this form e power a x; what we will do?

So, 1 over f D e power a x, 1 over f a e power a x this D will be replace just by a. So, we get the value directly without using that general formula of the integral. So, directly this D will be replaced just by a if I mean provided this f a is not equal to 0. If that is 0 that was a second case we have already discussed. If this f a is 0, then f D must have a factor of this type D minus a power r and we have already seen that what will happen when we have 1 over this D minus a operated on this e power a x, we are getting this x e power a x. Suppose we have here D minus a square, so there was a factor 2 here sitting inside this f D.

So, D minus a square, so that means, we were again operate on this x e power a x. So, what will happen if you operate again, this 1 over D minus a on this x e power a x. So, this formula again will be use. So, instead of this X, we will replace now x e power a x. So, e power a x will get cancel we have X there and the integral of this X will be x square by 2. So, we will get as a result when this is square here we will get x square by 2 e power a x and we can generalize this further and that result will be stating here now ah; f D must have a factor of this D minus a power r and in that case the general formula for this dealing with 1 over D minus a power r e power a x will be x power r, like incase of 2 we have seen it is x square by 2 and in case of the 3 again we will get this 3 and the 2 earlier. So, factorial will come.

So, this 1 over D minus a power r e power a x, the result of this the value of this is nothing but x power r over factorial r into this e power a x. So, we need to remember this one, this is simple when this f a is not 0 we will just replace D by a. But when we have this 0 we have to take this factor out somehow and then we can deal in this way x power r over factorial r e power a x.

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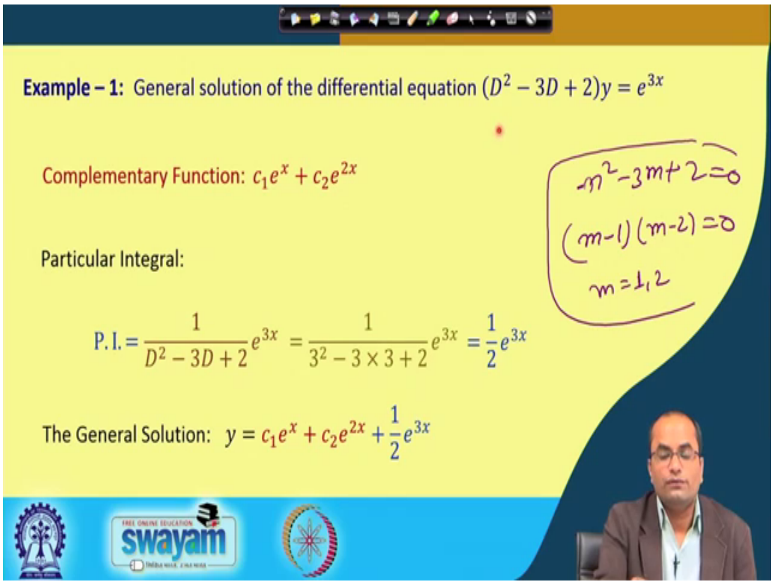

So, we see some example now. So, the first is to find the general solution of this differential equation D square minus 3 D plus 2 y is equal to e power 3 x. So, the complementary function, we write down the auxiliary equation here m square minus 2 m and plus m square minus 3 m plus 2 is equal to 0 and this will have a solution m minus 1, m minus 2 is equal to 0.

So, the roots are 1 and 2. So, having these roots here we can write down the complementary function ah, that c 1 e power x and the plus c 2 e power 2 x. Getting to this particular integral which is already discussed now, so 1 over D square minus 3 D plus 2, that is the particular integral operated on this e power 3 x and the trick is here that we replace this D by 3. So, we have 9 minus this 9 and this 2 here. So, the value will be coming as 1 by 2 e power 3 x. So, in this case this was not 0 here and therefore, we got this value directly by replacing this D by 3 the values 1 by 2 e power 3 x. So, this is a particular integral, it is a particular solution of the given differential equation here and we have the general solution of this homogeneous equation; that means, this complementary function and if we add the two so, we get this general solution of this given differential equation.

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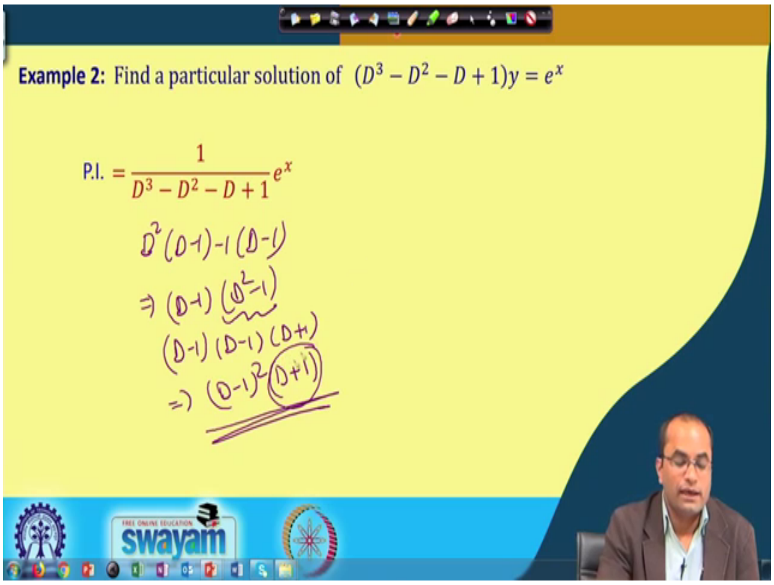

In the next problem, what we will see we will find just the particular solution of this differential equation. So, the particular solution will be 1 over this inverse operator e power x and if we substitute this here because the coefficient of this x is 1. So, we need to just replace here 1 minus 1 minus 1 plus 1. So, this is coming to be 0 that means we cannot apply here. There is a factor D minus 1 sitting here which we can clearly see also if we take this D square common. So, this is D minus 1 here minus 1, then D minus 1. So, we have this D minus 1 and D square minus 1, again here 1 D minus 1 is sitting. So, D minus 1 and the D plus 1.

So, basically what we have D minus 1 square and D plus 1. So, then we have to deal now with D plus 1 is not a problem because e power x this one is not making it 0. So, this will give simply 1 by 2 e power x and then e power x when we operate 1 over D minus 1 square, then we have to use that formula which was derived earlier.

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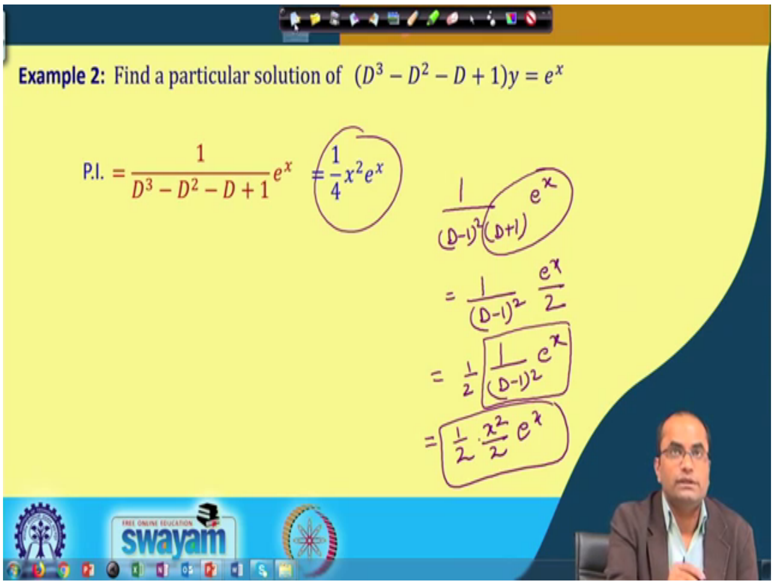

So, what we will get they this x square by 2 e power x will come because of 1 minus D square term and 1 by 2 will come because we had here D minus 1 is square and we have D plus 1 and the right hand side is this e power x.

So, let us first operate here 1 over D plus 1, so the D minus 1 square we keep as it is and when we apply this, this will be e power x by 2. So, this is nothing but 1 by 2 and this D minus 1 whole square e power x and this is 1 by 2 here and 1 over D minus 1 square e power x. This is the formula we will use now from the earlier slide that is x square by factorial 2 and e power x. So, we have x square by 4 e power x as the particular integral of this problem.

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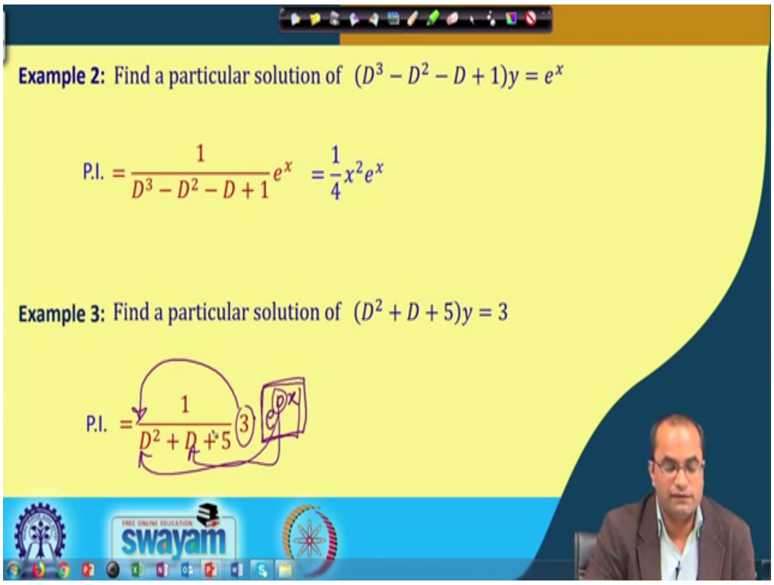

In the next problem, we have this is the last for today. So, D square plus D plus 5, y is equal to 3 and the idea here is that this particular integral will be 1 over this D square plus D plus, this 5 and we have this 3.

So, the 3 anyway we can take this is a constant we can take out of this operator, but they will be one here so that one what we can write down e power 0 x. This trick we are using here 3 we can take outside here of this operator because this operator will not do anything with the constant and here e power 0 x now which is replacing one. So, this 0 will be substituted for this D here, the same formula for e power a x we have used.

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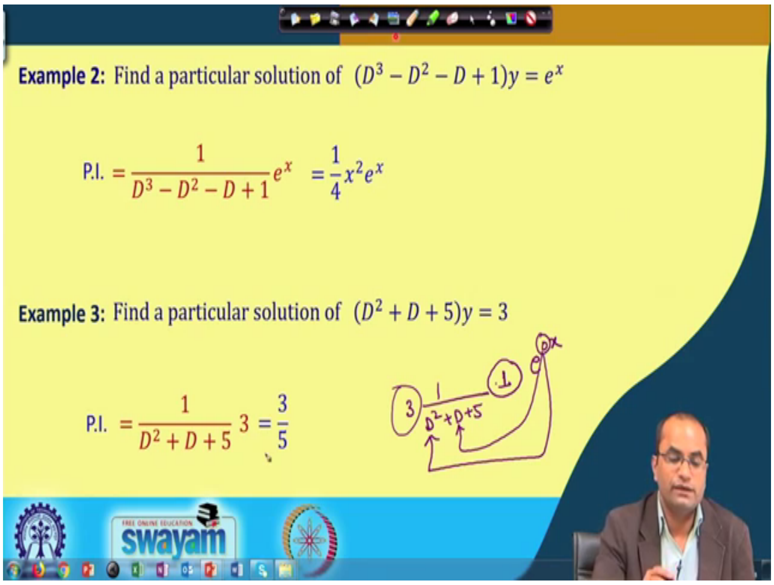

So, doing this now, what you will get the value will be 3 and then when we put this D to 0, will get this 1 over 5 there. So, the value will be 3 by 5.

So, whenever we have a constant that constant can we taken outside of this operator. So, like 3 and then we are actually operating on D square plus D plus 5 and over 1 and this 1 can be replaced like e power 0 x and then this 0 we can use the same formula which we have used earlier that D will be replaced by 0. So, we will get 1 over 5 here and this 3 will be there. So, 3 by 5 is coming the value of this integral. So, this is the particular integral of this equation, this is one particular solution of this equation.

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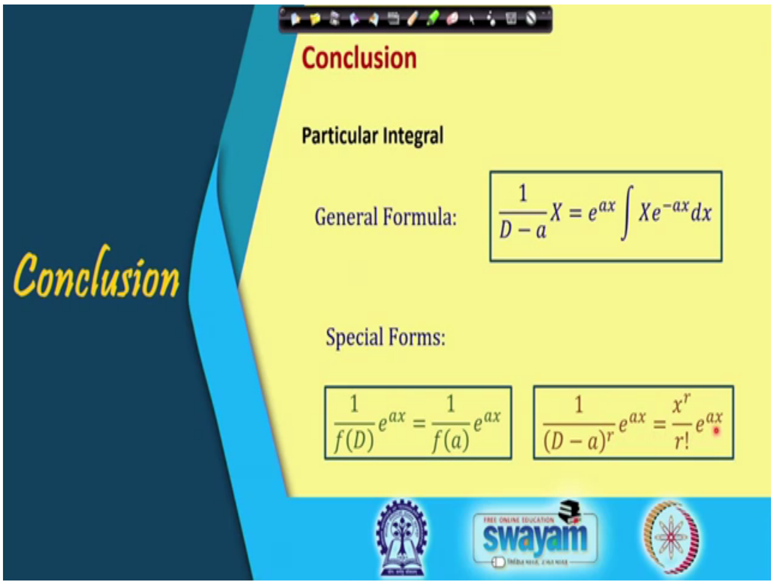

So, coming to the conclusion now that we have seen the particular integral and indeed this general formula which will be very useful to find the integral when we do not have such a shortcut method which was discuss just like special functions x power e x. So, this can be used in any case as long as this integral we can evaluate and the special forms also we have discussed today this e power a x. So, the rule was that 1 over f D e power a x we can replace this D by this a if this f a is not 0. And if this is 0, then we certainly have D minus a factor and that can also we easily evaluated as x power r here by factorial r and e power a x.

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So, with this, now we have these differences used for preparing this lectures and

Thank you for your attention.