Engineering Mathematics - I
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Lecture – 57
Solution of Higher Order Homogeneous Linear Equations

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Welcome back, this is lecture number 57 and today we will be talking about the Solutions of Homogeneous Linear Equations or in particular we are talking about the complementary function that is nothing but the general solution of a differential equation, a homogeneous linear differential equation. And we will discuss many solution techniques how to find complementary function.

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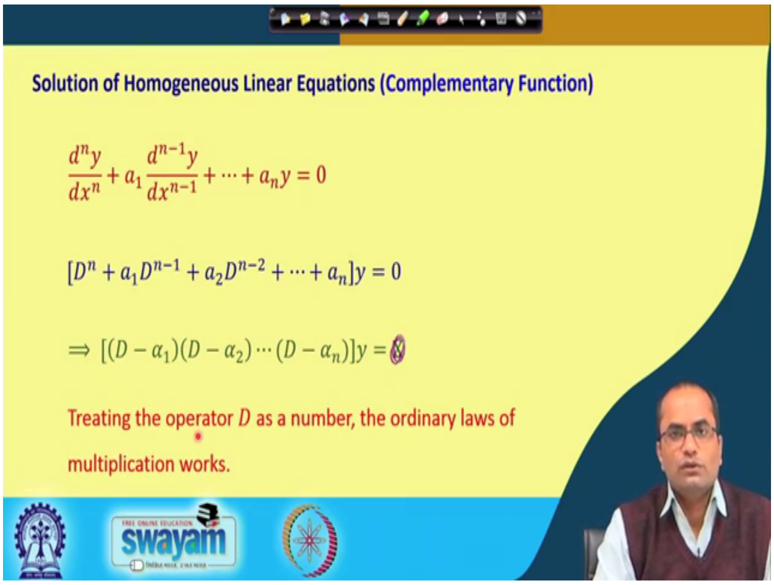

So, solution of this homogeneous linear equations the complementary function. So, here we will consider such a differential equation, the nth order linear homogeneous differential equation with constant coefficient. So, these coefficients here a 1, a 2, a 3, a n they are also taken as constants.

So, what we have also discussed in the last lecture that we can write down this equation in terms of this operator form, so D power n plus a 1 D n minus 1 so on is equal to 0. And we have also observed in the last lecture that we can work with these operators D as the algebraic operations we do with the polynomials.

So, we can factorize in the this and we can write in this form the D 1 minus alpha 1, D minus alpha 2, D minus alpha n this is the factorization of this portion here with this operator and y is equal to this X and we have proved in the last lecture that this is exactly the same as having this nth order derivativeS plus a 1 n minus 1 nth order derivatives and so on.

So, indeed this is 0 here, we are talking about the homogeneous equation, so the right hand side will be taken as 0. So, treating the operator D as a number, the ordinary this laws of multiplication works and this was the observation from the last lecture.

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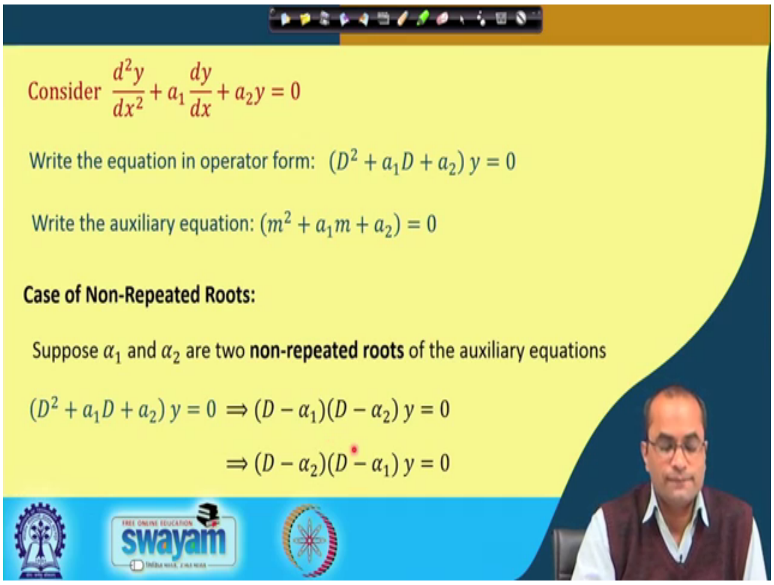

So, today we will now get to the solution of this differential equation and for example now for simplicity we are considering the second order differential equation, second order linear homogeneous differential equation with these two coefficients a 1 and a 2 and the right hand side is 0. So, how to get the general solution, how to get the complementary function of this a linear equation? So, first we need to write the equation in the operator form. So, here this will be d 2 a 1 here the d will come plus this a 2 and on y. So, here we have D square plus a 1 D plus a 2 and operator on y is equal to 0.

So, we write the auxiliary equation out of this operator equation, so the D square meaning here m square plus a 1 this m plus a 2 and is equal to 0. So, the case of non repeated roots, first we will consider when the roots of this auxiliary equations. So, how to get this auxiliary equation? This is from just from the operator equation. So, instead of working with this D which were operator is better to replace this D by this m and now let us work with this polynomial equation here m square plus a 1 m plus a 2 is equal to 0.

So, first case we will consider here for non repeated roots; that means, the roots of this auxiliary equations root of this equation here are non repeated. Meaning so we consider here suppose the alpha 1 and alpha 2, these are the two roots or two non repeated roots. So, they are distinct roots here alpha 1 and alpha 2 of this auxiliary equations, so we have the second order polynomial. So, we will get this two roots here alpha 1 and alpha 2 and we assume that these roots are non repeated, then what will be the solution of the given homogeneous differential equation.

So, we have this equation which we can write down now because we know the roots here of this equation alpha 1 alpha 2, meaning we can factorize we can write down this operator equation in this form also D minus alpha 1 and this multiplied by D minus alpha 2 on this y is equal to 0 or this is the observation again from the last lecture that this order here of this product is immaterial, so we can take this D minus alpha 2 first and then D minus alpha 1 operated on y this is equal to 0.

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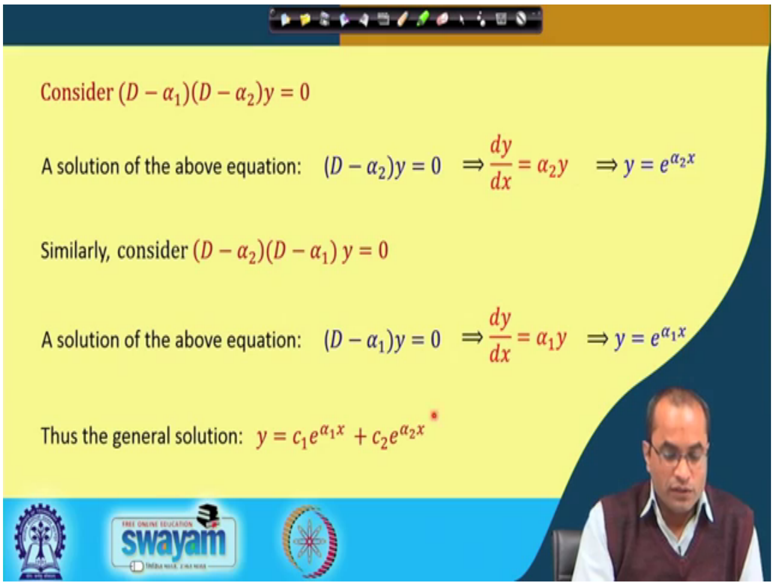

So, we consider first this D minus alpha 1 into D minus alpha 2 y is equal to 0; a solution of the above equation will be a solution here. So, if you set if you find solution here of this D minus alpha 2 y is equal to 0, then this will also satisfy the given equation because once we know that this y here is giving D minus alpha 2 y is equal to 0, so if you operate this D minus alpha 1 on 0 so we will get 0.

So, that is the idea here that we look a solution of this differential equation here D minus alpha 2 y is equal to 0 and whatever solution we get here that will be also a solution of this given differential equation.

So, how to get the solution of this differential equation that is easy it is a first order differential equation we can write down this form this is dy over dx and minus is alpha 2 y is equal to alpha 2 y because this was minus alpha 2 y, so we taken the right side and this we can solve it is a very separable here so dy in we can take y here and then dx can go to this side and then make it integrate.

The solution of this dy over dx is equal to alpha 2 y is giving here as y is equal to alpha 2 x. So, we have the solution of this part here of the given equation, but this y is equal to e alpha 2 will satisfy the given equation because D minus alpha 2 y here if this is 0 and when we operate this D minus 1 on 0, so the answer will be 0 meaning that equation will be satisfied by this solution.

The same exercise we can repeat now for once we take for instance here D minus alpha 2 first and then D minus alpha 1 if you operate on this y is equal to 0 then what will happen? That a solution now of this above equation will be also the solution of this equation D minus alpha 1 y is equal to 0, the same idea what we have applied above.

So, here now, if you solve this D minus alpha 1 y is equal to 0 we will get this y is equal to e power alpha 1 x and now e power alpha 1 x and this e power alpha 2 x. So, we have two solutions now of the given this differential equation of this given differential equation, we have two solution the one is y is equal to e power alpha 2 x another 1 is y is equal to e power alpha 1 x and these two solutions are linearly independent solutions.

What we also know once we have two linearly independent solutions than this linear combinations, so alpha 1 e power also c 1 e power alpha 1 x and plus the another constant times e power alpha 2 x that will also satisfy this given differential equation. And in this case what we have now, thus the general solution we can write down because the our equation was the second order linear differential equation, we have two linearly independent solutions and if we write down this c 1 e power alpha 1 x c 2 e power alpha 2 x that will be the general solution of the given second order differential equation.

So, what we have learn? If we have a second order linear equation here, we can find out the roots of the auxiliary equation we can find the roots of this equation which was alpha 1 and alpha 2 here it is written already in the vector form. And then, the solution will be c 1 e power alpha 1 x plus c 2 e power alpha 2 x, but this was the case when we have two distinct roots because in that case only these two solutions e power alpha 2 x and e power alpha 1 x these are linearly independent solutions, once we have that this alpha 1 and alpha 2 are distinct.

If alpha 1 alpha 2 are the same value it's a repeated root then we are not forming these two linearly independent solutions in this way. So, there should be another way to compute the general solution when we have alpha 1 and alpha 2 the same number or we have the repeated roots.

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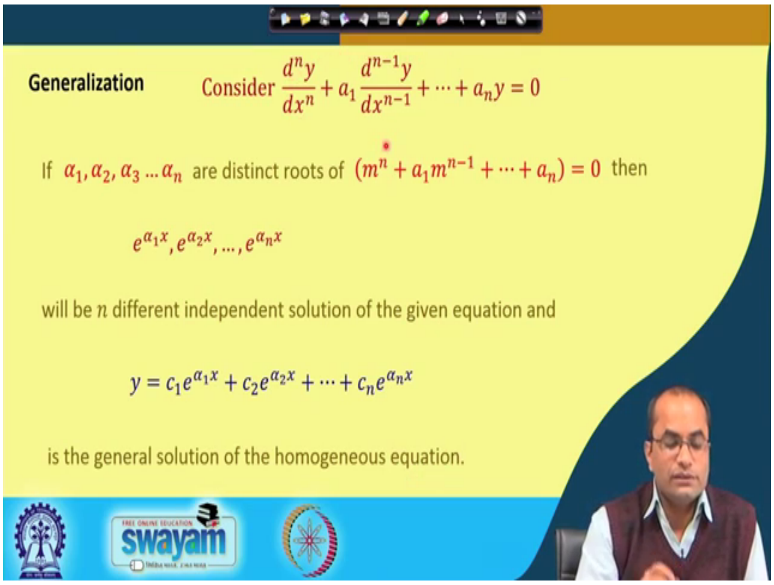

So, first getting back to the distinct roots, so when we have distinct root we can also generalize this. So, if you have a nth order differential equation and if this alpha 1 alpha, 2 alpha, 3 alpha n are the distinct rots of this so called the auxiliary equation which coming exactly from this equation you placing the operator d by m.

So, if you have these n distinct roots of this equation we can write down the solution as I mean these are the n linearly independent solution e power alpha 1 x e power alpha 2 x and so on e power alpha n x. These are the n different independent solutions of the given differential equation and then when we write down as the linear combination of the solutions meaning y is equal to c 1 e power alpha 1 x c 2 e power alpha 2 x and so on. So, this will be the solution will be the general solution of the given differential equation a given homogeneous equation here. So, that is the generalization when we have the distinct roots.

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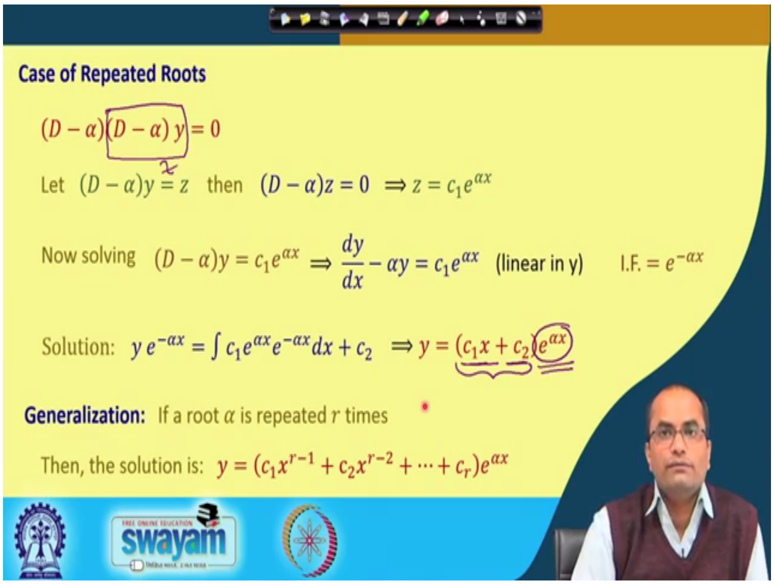

Next we will discuss the case of the repeated roots, that what will happen if we have the repeated root and again we will consider here the case of the second order differential equation. So, in case of the second order differential equation we can write down then we can factorize that differential part here with D minus alpha D minus alpha because we have considered now the same roots here alpha.

And now what we will use the trick here, so the D minus alpha y we will treat as z, D minus alpha y as z and now this given equation here by taking this as z here. So, we have this given equation we can write down at D minus alpha operated on z is equal to 0. And this we can solve first for z once we have z, then we can solve for y, so in that way we will get the solution y of the given differential equation.

So, here z is equal to c 1 e power alpha x that will be the solution of this first order differential equation the constant time c 1 alpha x. Now solving the equation here D minus alpha y is equal to z, so D minus alpha y is equal to z; z is now c 1 e power alpha x. So, now, we need to solve this equation because finally, we want to get this y this is linear equations, so here we have the differential of y so this D of y is a dy over dx minus this alpha times y is equal to the c 1 e power alpha x.

So, this is linear equation in y and the integrating factor which we get here is a e power minus this alpha times x. So, the solution of this differential equation here will be because it is a linear equation, we have already discussed in the last lecture. So, y into this integrating factor e power alpha x the right hand side here over the integrals c 1 e power alpha x multiplied by the integrating factor which is here e power minus alpha x and this dx plus c 2.

So, this will be the solution here for this given differential equation this linear differential equation. So, e power alpha x will cancel out with e power minus alpha x and we have c 1 when we integrate we will get x there. So, y is equal to c 1 x plus c 2 times this e power alpha x and the generalization of this we can also get when suppose we have this alpha as a repeated r times see the alpha root is repeated r times when we have a general end nth order equations, so we will get n roots so some of them may be repeated. So, here we assume that we have this alpha root r times repeated.

So, here this for example, in the second order equation we have discussed this alpha was repeated two times. So, what we get here when alpha was repeated two times we got this factor e power alpha x like the earlier distinct case also you will getting e power alpha 1 x e power alpha 2 x, so e power alpha x we got. And then these constants the 2 constants are given by this relation to c 1 x plus c 2.

Now if this root is repeated r times then what will be the solution is exactly will follow from here, we will have c 1 x power r minus 1 this was two times repeated, so we got this linear term here if we have r times repeated root we will get r minus 1th polynomial here plus c 2 x r minus 2 and so on plus the c r and e power alpha x. So, if alpha is alpha root is repeated r times then we will get the solution in this form.

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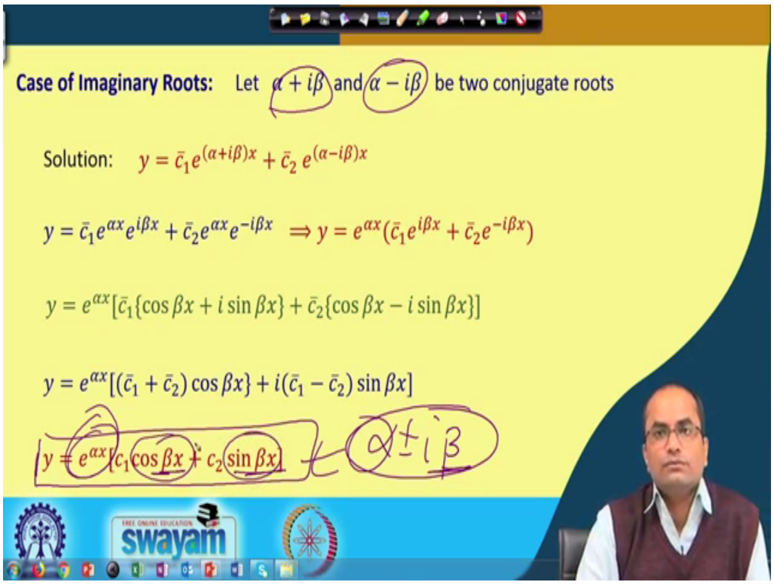

Case of the imaginary roots, so we have discussed already the real roots real distinct and also real repeated. So, in the case of the imaginary root we have the same idea, so we assume that alpha plus i beta and this alpha minus i beta is the is the roots here are the roots of the given auxiliary equations. So, or two conjugate roots.

And then the solution it is exactly followed from the case 1 when we discussed the distinct roots and here these roots are distincts they are not the same roots. So, we have written exactly our solution in that form, some constant here c 1 and exponential the first root alpha plus i beta x plus c 2 another constant e power alpha minus i beta x because the case 1 which we have discussed for the distinct roots we have not assumed them whether they are real distinct or the imaginary distinct roots.

So, exactly we can use that idea here to get the solution this two independent solutions that e power alpha plus i beta x and e power alpha minus i beta x to; to linearly independent solutions and with these constants they are the linear combination will be also the solution or the general solution. But what we do because this is not common to write down in terms of these conjugates or in terms of these imaginary roots what we will be simplify further here.

So, e power alpha x and e power alpha x we have the common term in both. So, what we will do now? So, e power alpha x and this e power alpha x we can take this common and what is they are now the c 1 e power i beta x plus the c 2 e power minus i beta times x.

And then this exponential with this i beta x or exponential with this minus i beta x we can also expand it in this form that this cos beta x plus this i sin beta x and the for c 2 was also we can have this cos beta x plus this i times sin beta x that is the we can expand this e power i beta x here in terms of the cos and sin and here also e power minus i beta x in terms of cos and sin.

So, now what we will do here y is equal to e power alpha x the same and her this cos beta x here also we have cos beta x. So, these constants here c 1 and the c 2 bar we have combined it and then with the sin beta x also we have this I terms with the c 1 bar minus this a c 2 bar. So, we have now these 2 new constants c 1 bar plus c 2 bar and also the i times c 1 bar minus c 2 bar and here the y is equal to e power this alpha x. And we have now renamed these coefficients here c 1 plus c 2 bar as c 1 and here c 1 minus the c 2 with the i itself, so the another name we have given as c 2 because these two are the arbitrary constants now.

So, we will write down the solutions when we have two roots here the conjugate roots as the complex number we will write down the solution of our homogeneous equation as e power alpha x c 1 cos beta x and c 2 sin beta x whenever roots are alpha plus minus i beta type. So, this alpha will go to the exponential part here in this beta will be with the cos and the sin term.

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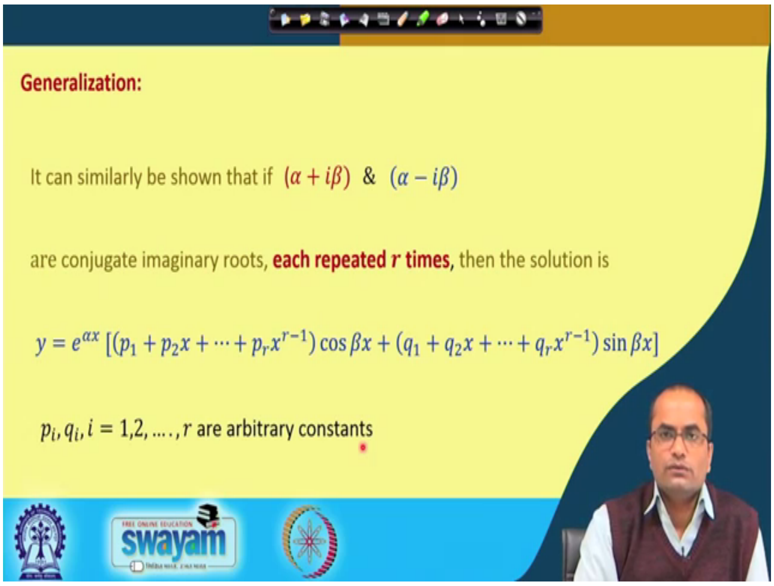

Next we will consider now the generalization of this one, that when we have a like alpha plus i beta alpha minus i beta are the conjugate the imaginary roots and they are repeated r times, in that case what would be the solution? So, these conjugates are repeated r times now. So, again the same idea like we have discussed further for the real roots that will be applicable now.

So, e power alpha x that will be the common term here, only with that constant term will now get into the polynomial terms. So, p 1 plus p 2 x plus p r x r minus 1 with the cos beta x and this q 1, q 2, q x, q r x power r minus 1 with a sin beta x and all these ps and the qs they are the arbitrary constants now.

So, again the idea of this repeated roots or the imaginary roots they exactly follow what we have derived for the distinct roots and the repeated roots in case of the real numbers.

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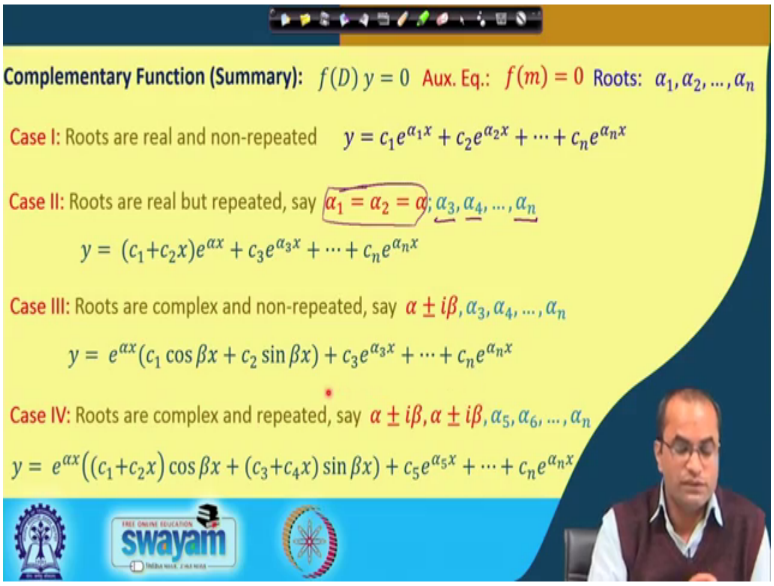

So, the complementary function or the summery now to summarize everything, so, we have the equation they given differential equation f D y is equal to 0, we need to write first the auxiliary equation; that means, this f m is equal to 0 we will just replace this D by m and we will have the auxiliary equation. And it is roots of the auxiliary equation will be given by alpha 1, alpha 2 and alpha n. The case I when the roots are real and non repeated that is the case I.

So, we have y is equal to c 1 e power alpha 1 x plus c 2 e power alpha 2 x and so on c n e power alpha n x, that was the case when the roots are real and they are non repeated because each of the term here e power alpha 1 x e power alpha 2 x e power alpha n x they are the linearly independent solutions of the given homogeneous differential equation and their linear combination will give the general solution of the given differential equation.

Second we have considered the roots are real and non repeated say, this alpha 1 is equal to alpha 2 is equal to alpha. So, this is the repeated root here that first 2 roots are repeated and the alpha 3, alpha 4, alpha n are non repeated. So, in this case what would be solution now the general solution will take this form, for the repeated root you will have this linear polynomial, so when the root is repeated 2 times we will get this linear term here with the constant? So, c 1 plus c 2 x and e power this alpha x and all other are non repeated here, so c 3 e power alpha 3 and c n e power alpha n x. So, that will be the general solution of the given homogeneous differential equation.

The case III when roots are complex and non repeated say this alpha plus minus i beta and all others are real roots for instants alpha 3, alpha 4 and alpha n. So, these first two roots are the complex roots alpha plus i beta and alpha minus I beta. So, in this case what will be the solution the complementary function of the given differential equation?

So, here y will be e power this alpha x and the c 1 cos beta x plus c 2 sin beta x and then these are the real non repeated roots. So, we will get these terms as c 3 power alpha 3 x plus the c 4 e power alpha 4 x and this will end to the c n e power alpha n x. The case IV when the roots are complex and they are repeated. So, in that case let us say the roots are alpha plus minus i beta and again this is repeated as alpha plus minus i beta. So, these are the four roots here and then the root number fifth alpha 5, alpha 6 and so on alpha n are the real non repeated roots.

So, now for this case we have to write down the expression here in terms of the linear terms with c 1 and c 2. So, e power alpha x will remain as it is and inside of the c 1 it will take now c 1 plus c 2 x and here this another constant will be c 3 plus c 4 x, so the 4 constants will be introduced here corresponding to this repeated complex roots. And then we have corresponding to non repeated distinct real roots we have these terms here with c 5 e power alpha x plus and so on c n e power alpha n x.

So, this is the whole summary, so we have the real non repeated root that is the case here roots are real, but repeated then we will have this polynomial term with the constants and when we have the complex roots we have exactly c 1 cos beta plus c 2 sin beta terms and when we have the complex and the repeated roots again we have this e power alpha x and then the c 1 and c 2 x will be there as c 3 plus c 4 x term will be there as we have in the case of the repeated roots for the real.

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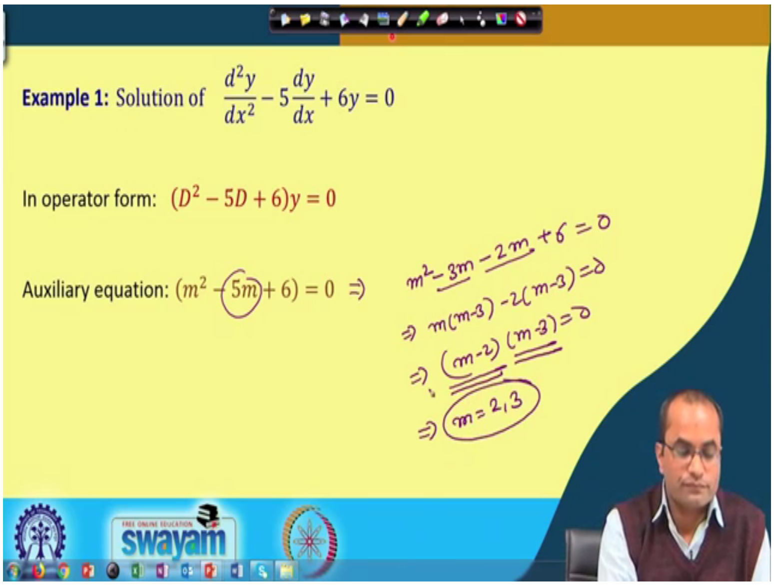

So let us go through a quickly a two examples, so the first example here we want to find the solution of this second order differential equation. The second order term with y minus 5 times the first order term of y plus this 6 y is equal to 0, so this is the homogeneous linear differential equations or equation of order two. So, what as a first step we need to write down this equation in the operator form. So, we have this D square here minus this 5 times is D and then plus this 6 operated on y is equal to 0.

So, once we have the operated form operated form of the equation we can easily write down the characteristic equation, so or the auxiliary equation. So, the auxiliary equation corresponding to this will be just we can replace this D by m. So, we have here m square we have minus 5 m then we have plus 6 here and we want to solve this equation which in this case it is it is a simple one.

So, we have here m square minus 5 m plus 6 so; that means, this m square and will be minus 3 m minus 2 m and plus 6, so, this minus 5 m we have written as minus 3 m and minus 2 m. So, this will give now m into m minus 3 and minus 2 this m minus 3, so we will get this factors as m minus 2 and m minus 3 is equal to 0, so we got the solution here m is equal to 2 and m is equal to 3.

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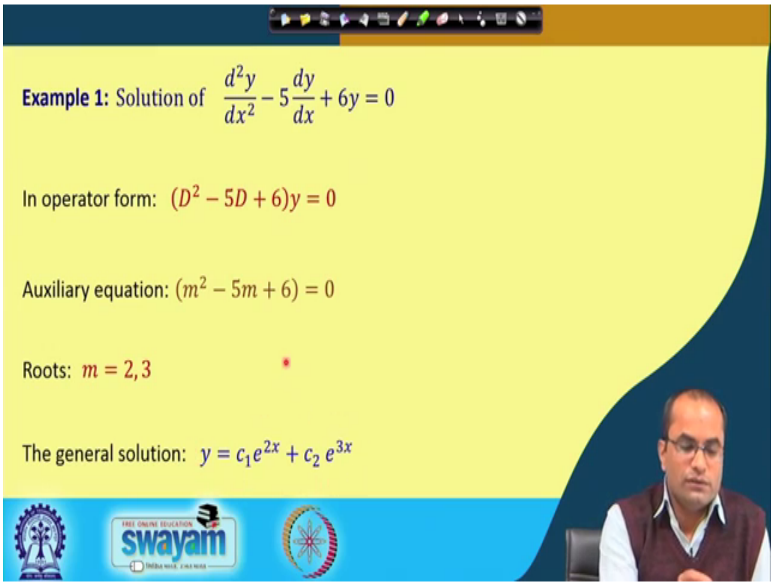

So, that is the solution the roots of this auxiliary equation as 2 and 3. So, we have distinct roots and in case of the distinct root the general solution is c 1 e power this 2 x and c 2 e power 3 x that will be the general solution of the given differential equation one can also verify by substituting this into the equation, whether this satisfies the given differential equation or not.

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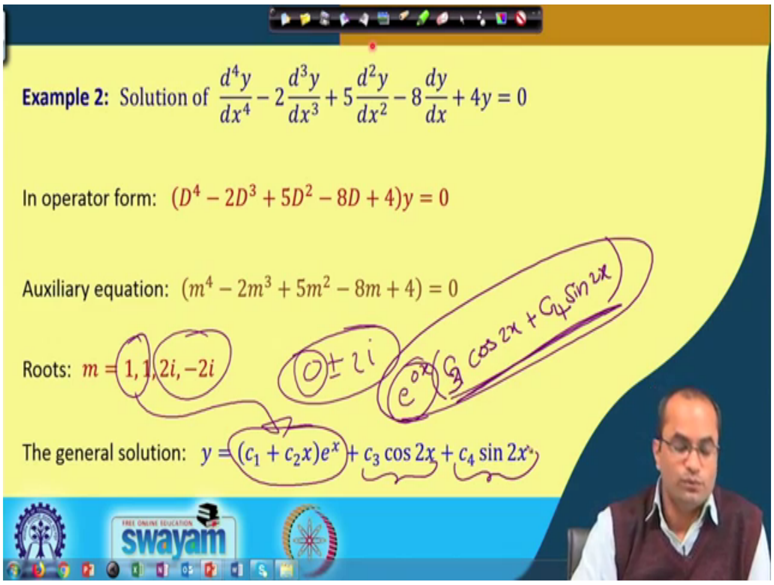

So, the example 2 we will take this fourth order linear differential equation, so again the corresponding to this differential equation we can write down in a operator form. So, this is D 4 minus 2 D 3 plus 5 this D square minus this 8 D here plus 4 operator on y is equal to 0 and then this auxiliary equation we can just replace this D again by m so this is the fourth order auxiliary equation it could be a difficult to solve.

So, once we solve this the roots are coming as 1 and 1 that is the repeated root here and then we are getting also 2 i and minus 2 i the complex conjugate. So, we need to write down the complementary function or the general solution of the given differential equation, so first this repeated root case. So, remember when we have the real repeated roots, so we will get c 1 plus c 2 x and the exponential of this 1 and here for complex conjugate we will get in terms of the cos and sin.

So, the general solution will be the c 1 plus a c 2 x and this exponential x because of this repeated root here because of these repeated root we are getting these two terms and then for plus minus this 2 i 0 plus minus 2 i that is the another root, where we are getting this e power the 0 x to alpha times this x and then we will get the c 1 I mean the constant which is c 3 here because c 1, c 2 were already appeared. So, c 3 and then we will get this cos this 2 x plus this c 4 and we will get sin 2 x.

So, here e power 0 x is 1, so we are getting this term only that c 3 cos 2 x and c 4 sin 2 x. So, we have the solution now in this case when the equation was the fourth order equation and we will have the four constants in the general solution.

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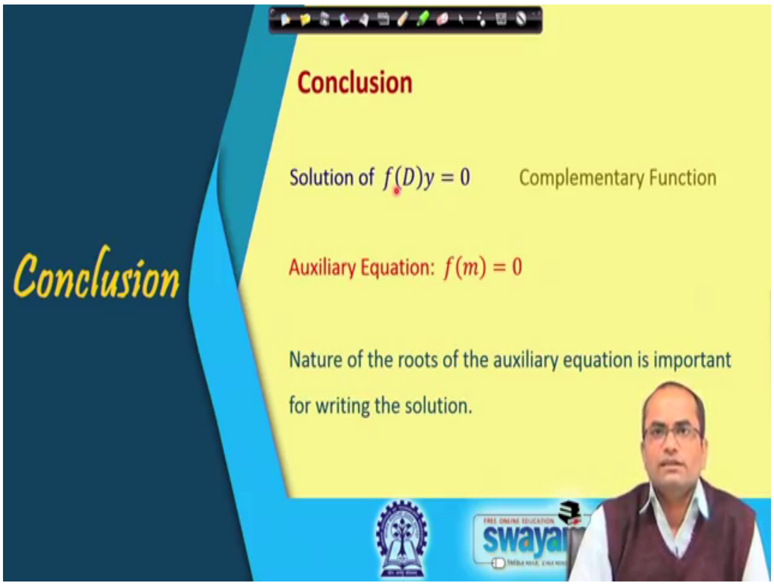

So, coming to the conclusion we have today discussed the solution of the homogeneous linear differential equation of this type f D y is equal to 0 this is just for the operator differential operator we have denoted by this f D and the complementary function which we call as the solution the general solution of this differential equation we call as complementary function.

So, the crucial part was that first we write down the auxiliary equation and find it is roots and the nature of the roots of the auxiliary equation it may be real, distinct, real repeated, complex or complex repeated roots and that is a important here for writing the solution and in each case we know now how to write the solution of the given homogeneous differential equation.

So, in the next lecture what we will learn now how to find a particular solution of the given non homogeneous differential equation and when we add that two we will get basically the general solution of the given non homogeneous differential equation.

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So, here are the references used for preparing the lectures and.

Thank you for your attention.