Engineering Mathematics - 1
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Lecture - 56
Higher Order Linear Differential Equations

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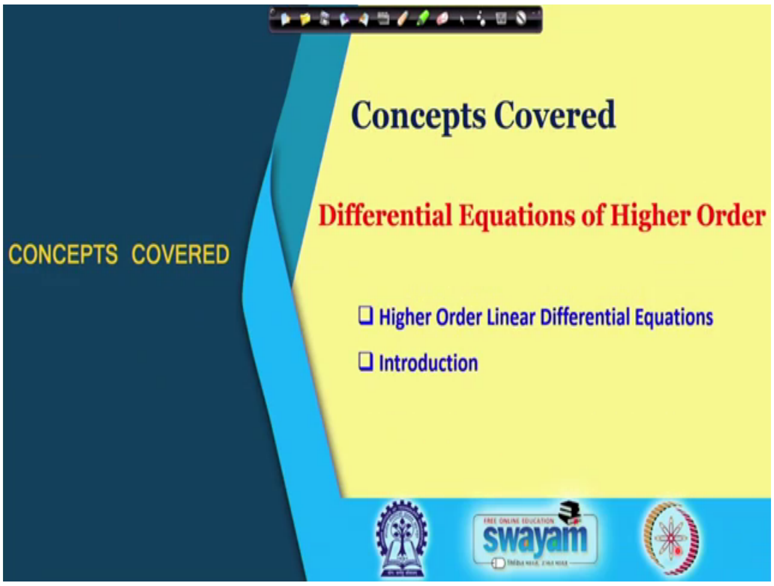

So, welcome back and this is lecture number 56 we will be talking about linear differential equations of higher order and we will go through the solution of such differential equations.

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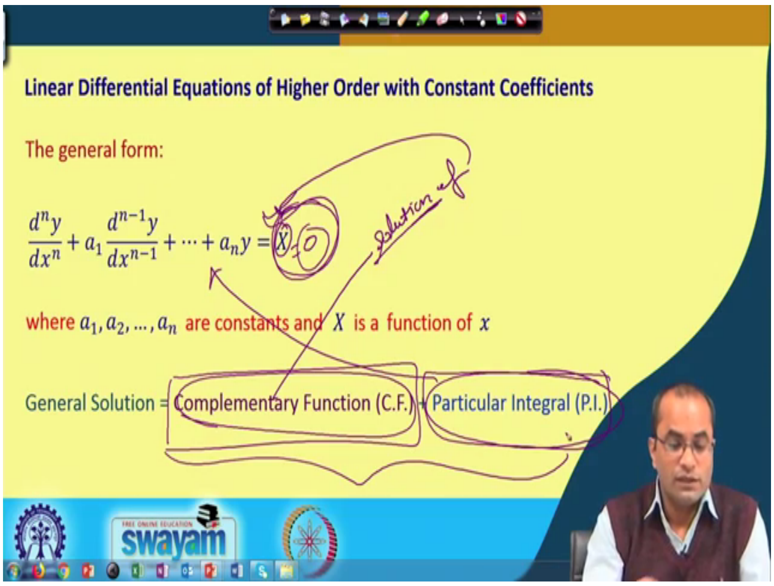

And in particular in today's lecture we will be talking about these linear differential equations of higher order with constant coefficients. So, that is a particular case of more general linear differential equations. So, here the general form of such differential equations which we will be talking about in today's lecture is this nth derivative of y plus this a 1 and that is the constant here with these constant coefficients n minus th and derivative of y and this will continue up to the y term with the coefficient here again the constant coefficient a n.

So, these coefficients a 1, a 2, a 3, a n they are the constants here and the right hand side function X that can be a function of X. So, these are the constants and X is a function of x it can be also a constant, but in general we can take as a function of x. So, such a question is called a linear differential equation because this is linear there is no product of y or y with the its derivative. So, it is a linear differential equation and nth order linear differential equation.

So, the general solution which we will be talking about today's and today's lecture will be having two factors here one is called the complementary function the other one is called the particular integral. So, this complementary function is nothing, but the solution of so this is just the solution of the homogeneous equation homogeneous means when we set here this X to 0. So, the corresponding homogeneous equation by setting this right hand side equal to 0 we call such equations as a homogeneous equation and this complimentary function here is the solution of homogeneous corresponding homogeneous equation when we set this right hand side equal to 0.

So, we will be talking about more here how to get the solution of this homogeneous equation in the next lecture and also about the particular integral. So, the particular integral we call any solution which satisfy any particular solution which satisfy the given equation with this here X. So, this is a one particular solution and then when we add the complimentary function or the general solution of the homogeneous equation, then we can get and we will see in today's lecture how this will form a general solution of the given non homogeneous differential equation.

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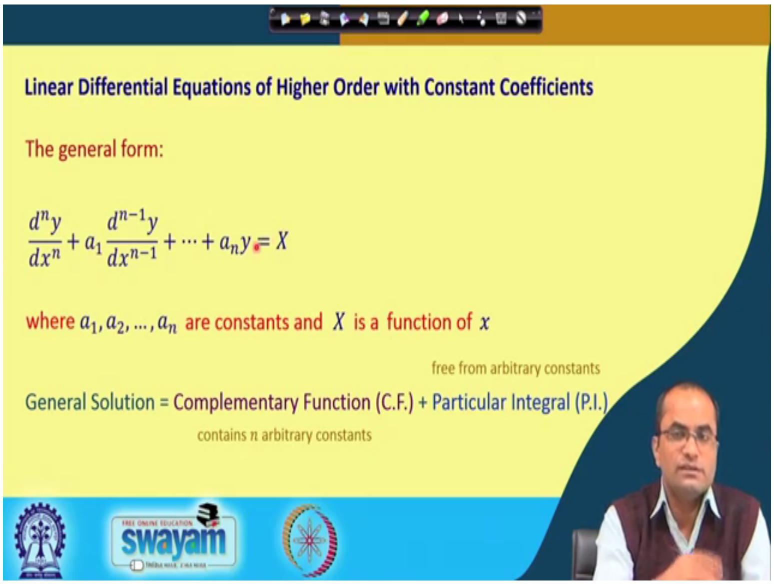

So, here since this is the general solution of the homogeneous equation this will have n arbitrary constants as discussed before that this is the nth order differential equation. So, we must have n arbitrary constant in the general solution of the given differential equation here.

And this particular integral which is a particular a solution of this given differential equation will have will not have a constant. So, it will be free from arbitrary constant or if there is any constant appears here that will be much automatically in the complimentary function. So, this is one way of getting the solution of this such a non homogeneous. So, when this X is not 0 we call the non-homogeneous equation, when this X is 0 we call as homogeneous differential equation. So, this general solution and this is one method or the easy method to get the general solution of the of the given non homogeneous equation, that if we can get this complimentary function which is the general solution of the homogenous equation and we will see that this is very easy to get the general solution of the homogenous equation.

And then we need to find just one particular solution which satisfies the given differential equation and when we add the two so we will have naturally these n arbitrary constants and today we will see that this will become a general solution of the given a differential equation.

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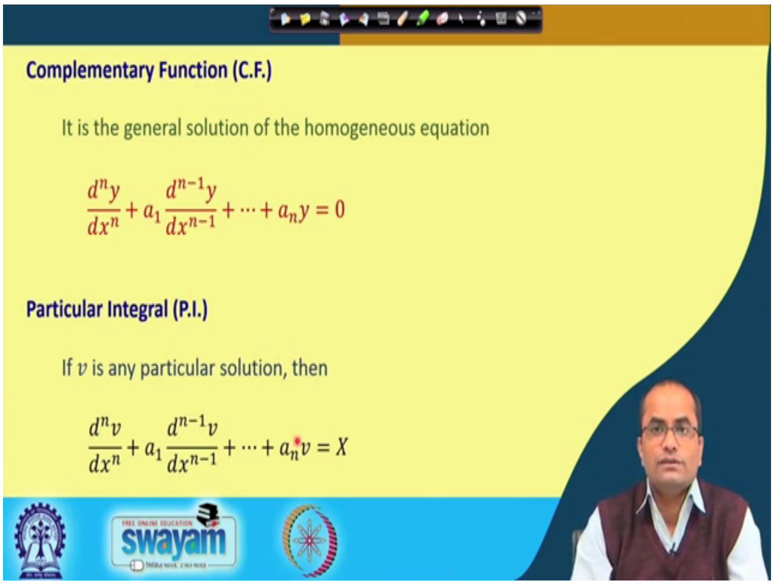

So, here what is the complementary function and how to get this idea we will; we will give now. So, this is the general solution as I said of this homogeneous equation; homogeneous equation means this setting this right hand side equal to 0 and the particular integral will have a very particular solution of the given differential equations meaning that if a is any particular solution then this will satisfy the given differential equation. So, when we substitute here in the left hand side, this we should get the right hand side X.

So, this will satisfy the given differential equation and that is how we defined the complimentary function and the particular integral and we have seen in the earlier slide, but we will prove later when we add this complementary function and the particular integral we basically get the general solution of the given this non-homogeneous differential equation.

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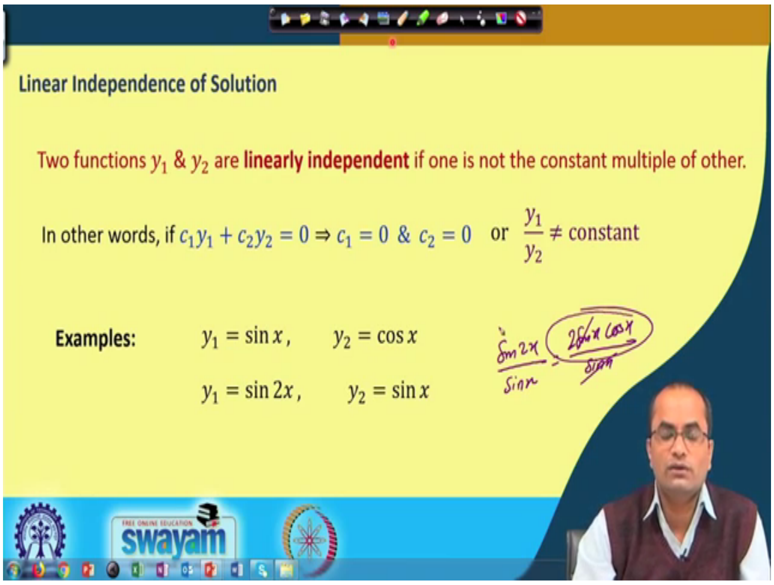

So, before we go to that discussion we need to also introduce here like linear independence of solution though we have talked about linear dependence in an independence in linear algebra and a similar concept we will just revise again here. So, two functions y 1 and y 2 are linearly independent, so first we are introducing for two functions because this is much easier to see the linear independence for two function, if one is not the constant multiple of the other. So, if one function we cannot write as a constant multiplication of the other, for example, we have sin x and we have 2 times sin x.

So, they are linearly dependent because there the second function is just the 2 times of the first function so, but for linearly independent functions we cannot write as a constant multiple of the other. In other words or more formally we discuss and that was already discussed in linear algebra that if the c 1 y 1 and plus c 2 y 2 the c 1 c 2 are some constants. So, if this combination when set to 0 if this implies that or this is true only when c 1 is equal to 0 and c 2 is equal to 0, then we call that this y 1 and y 2 are linearly independent.

And if we get any nonzero solution which satisfy this equation then they are linearly dependent because one function 1 can write in terms of the other function. So, there will be dependency on each other, but when we talk about that when this combination is 0 this is only possible when c 1 is 0 and c 2 is 0 there is no other possible values of c 1 and c 2 then we call that these solutions are linearly independent.

Or for the case of two functions this is very easy to check because you can take just y 1 by y 2 is divide the two functions and if this is not equal to constant then we call that this is linearly these functions are linearly independent and this is equal to constant then the functions are linearly dependent.

So, some examples of this linear dependence; so, for example, sin x and cos x when you divide y 1 by y 2 we will get tan x, so which is not constant, so these sin x and cos x they are linearly independent functions. The another example we can talk about sin 2 x and the sin x, so here again if we take the ratio here sin 2 x over sin x.

So, this will be sin 2 x over sin x or 2 sin x and cos x divided by sin x, so sin x sin x cancel and we have the 2 times cos x which is again not constant and hence these combination here with y 1 y 2 they also form a linearly independent set. So, sin 2 x and sin x they are linearly independent.

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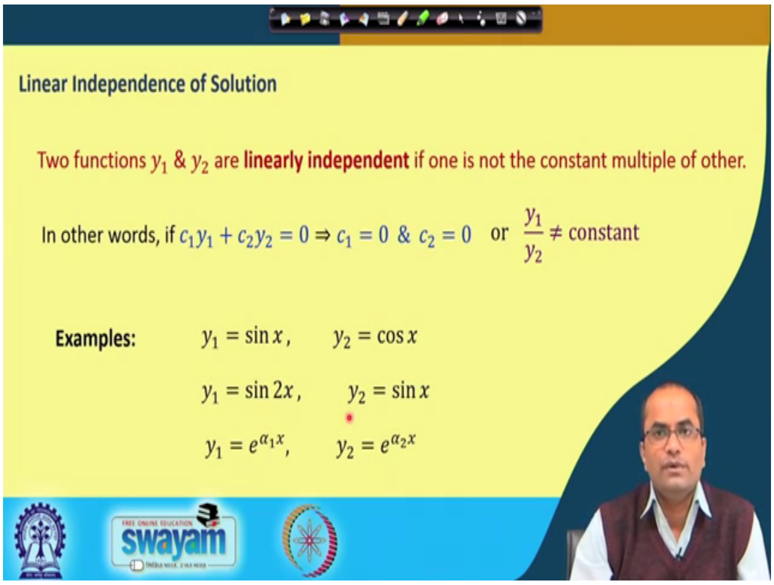

Also the functions exponential alpha 1 x and exponential alpha 2 x, so here also if we take the they take such a ratio here. So, when alpha 1 and alpha 2 these are two different numbers, two different real numbers, so then these functions are linearly independent.

So, when alpha 1 is not equal to alpha 2 then we have that e power alpha 1 x and e power alpha 2 x, they are linearly independent.

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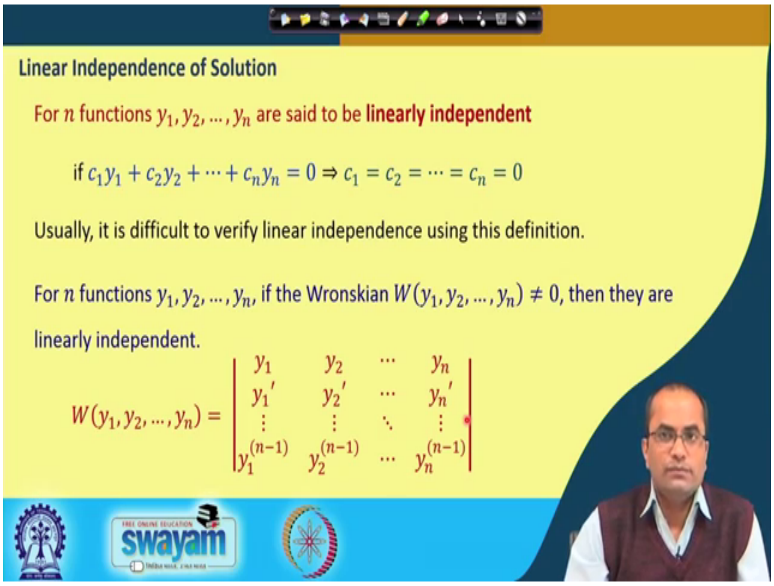

Coming to the linear dependence for many functions; that means, for n functions y 1, y 2, y n and they are said to be a linearly independent and then we can generalize this definition which we have also used for two functions, that if this linear combination c 1 y 1 plus c 2 y 2 plus c n and y n is equal to 0, if this combination when set to 0 is possible only when all these c's are 0.

Then we call that these set here is linearly independent or these functions are linearly independent. So, again this is a very formal definition and very important for any functions not only for two functions we can generalize for given n functions, but what is the problem here usually this is difficult to verify the linear independence using this definition because we have to set when there are n functions set to equal to 0 and then to realize that all these c's are 0, that is the only solution here we are getting out of this equation it is little bit difficult to show.

So, there are some other ways to prove the linear independence of the solutions when they are more than two functions. And one concept which is very useful to prove this linear independence since the Wronskian which is W this is the notation for the Wronskian we will define in a minute what is Wronskian. So, if the Wronskian here of this y 1, y 2, y 3, y n function is non zero; if this Wronskian is non zero; then they are linearly independent and what is this Wronskian? This is a determinant here which is evaluated in this way.

So, the first row here this y 1, y,2, y 3, y n the second row will have their derivatives third row again second order derivative and this nth row will have n minus 1th order derivatives. So, when we compute this determinant and we see that this is not equal to 0, then these functions are linearly independent.

So, this is 1 way of checking linear independence when they are more than two functions we can do for the two functions as well this test, but the earlier one we have seen therefore, two functions one can easily see by taking the ratio of the two functions. So, this was about the linear independence and dependence of the functions because we need to introduce before we use these terminology now in the in setting of the solutions of linear differential equations.

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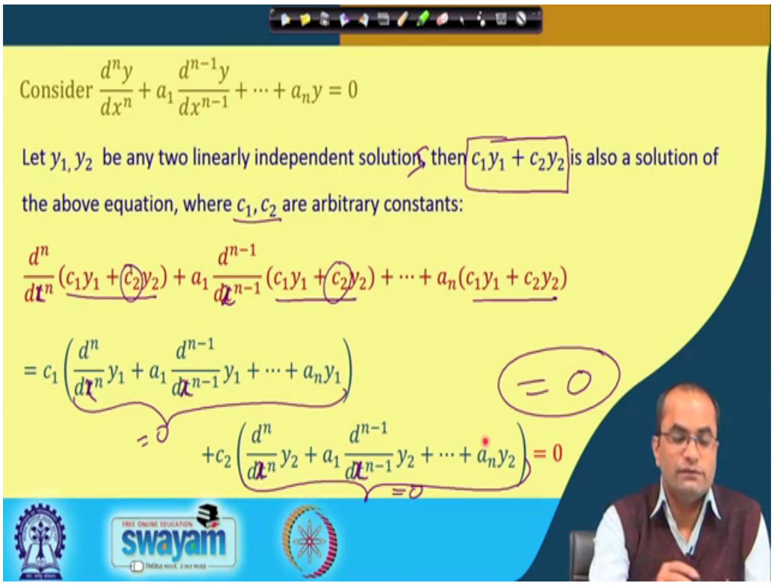

So, here let us consider this homogeneous equation first, so d this nth term, n minus 1th order term and this constant is a n and this y term equal to 0, so this is the homogeneous equation because this right hand side is said to be 0. And we let y 1 and y 2 be any 2 linearly independent solutions what we want to show here then the c 1 y 1 and c 2 y 2 is also a solution of the above equation and this c 1 and c 2 are arbitrary constant.

So, what we will show here that if we have two solutions y 1 and y 2 and two linearly independent solutions because if they are not linearly independent the c 1 y 1 plus c 2 y 2 itself does not make sense with having two arbitrary constants here we can have only one constant in that case because if one depends on the other we can write down again one in terms of other, so this will not make much sense.

So, here y 1 and y 2 be two linearly independent solutions and in that case, then the c 1 y 1 plus c 2 y 2 is also a solution of the above equation this we will see now here and for any arbitrary constant c 1 and c 2. So, what we have to do? We just substitute the c 1 y 1 plus c 2 y 2 into the equation. So, we have this nth term an nth order term with this x here not the t; we have used x there.

So, by substituting this c 1 y 1 plus c 2 y 2, c n c 1 y 1 plus c 2 y 2 in place of y and we will check if this is equal to 0, if this is the case then we call that this is also a solution of the given differential equation. So, what we can write down now? The c 1 we take common from here; from here from every term and then write down this d n over d x power n and d n minus 1 over d x n minus 1 d n minus 1 order derivative with respect to x.

And this plus the other one because with c 2 also we can take common because c 2 is sitting also in each term and now this will be with y 2 everywhere. So, the plus here with this c 2 and d the nth order derivative with respect to x n minus 1th order derivative with respect to x and so on. And then what we will observe because this y 1 is the solution of this homogeneous equation, so this will be 0 and y 2 is the solution of this homogeneous equation, so this will be also 0 and c 1 into 0 plus c 2 into 0 will be 0.

So, this satisfies the given differential equation hence the c 1 y 1 plus c 2 y 2 is also a solution of the above differential equation. Indeed we can generalize this not only for two functions we can take more functions and with the same idea we can prove that that combination when we have for instance y 1 y 2 y 3 y n linearly independent solutions then their combinations c 1 y 1 plus c 2 y 2 plus c n y 1 will also be the solution of the given differential equation.

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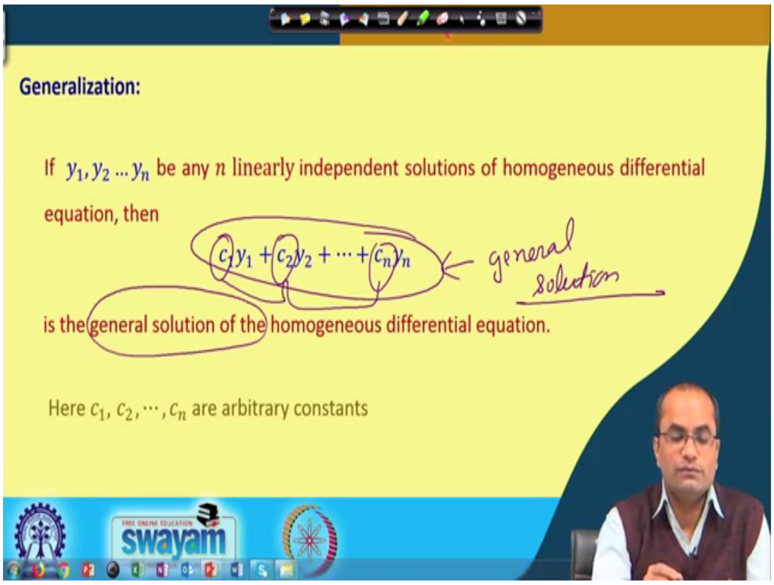

So, here the generalization that this y 1 y 2 y 3 be n linearly independent solutions of the homogeneous differential equation and in that case the c 1 y 1 plus c 2 y 2 and so on c n y 1 c n y n will be also the solution of the given differential equation and not only the solution we have used now this term the general solution. The general solution of the given homogeneous differential equation because this is a solution of the given differential equation that we can prove like we have done for two functions and this solution has n arbitrary constants.

So, that was our definition also for the general solution which was introduced earlier and in this case when we have these n arbitrary constants in the solution and it satisfies the equation meaning this is the solution, so we can call this as the general solution. And these are the arbitrary constants c 1, c 2, c 3 and arbitrary constants.

So, with this now what we have seen that if we can find these n linearly independent solutions of the homogenous equation; homogeneous differential equation then just their linear combination will be also the solution of or in fact, it will be the general solution of the given differential equation.

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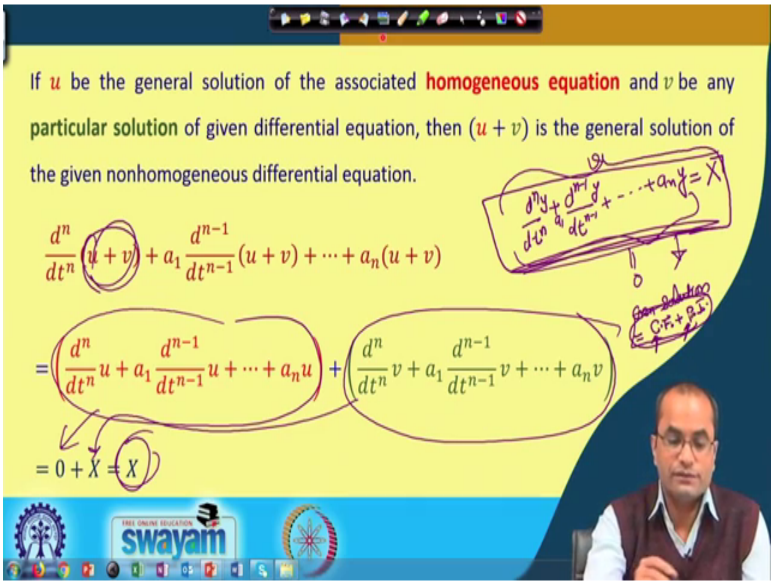

Now, another important result which we will consider now we will go through now, So if u be the general solution here of the associated homogeneous equation, so u we are talking about the solution of the homogeneous equation again recall that homogeneous equation is when the right hand side is said to be 0. So, the u satisfies that homogeneous equation and this v another function v be any particular solution of the given differential equation.

So, here we have taken this two solutions the u is a solution or is the general solution which we have already discussed before when we have n linearly independent solution; their combination exactly c 1 y 1 plus c 2 y 2 and so on that will be their general solution of the homogeneous equation and here we have taken another function v be any particular solution of the given differential equation. So, the given differential equation means the given non-homogeneous differential equation when the right hand side is not equal to 0.

Then what we will observe here that this u plus v when we add these two, so this u the for the homogeneous equation which we call actually the complimentary function and a particular solution of the given differential equation we call the particular integral. So, when we add the C F and p I the complimentary function plus this particular integral like here this will form a general solution of the given differential equation how to check that this is a general solution or not the solution should have n arbitrary constants that u itself has arbitrary n arbitrary constants and it should satisfy the given differential equation.

So, what we will check now to prove that this is a general solution that this u plus v satisfies the given differential equation, if this satisfies the given differential equation and then naturally it has n arbitrary constants because u itself has n arbitrary constants. So, then this will be a general solution of the given differential equation. So, for that we will consider this equation the left hand side of the equation and substitute this u plus v into the equation.

So, here we have this nth order derivative with u plus v and n minus 1th order derivative of u plus v and this a n u plus v. So, here we have taken this t as linearly as the independent variable here t; one can take x also. So, now, in this case since we have taken the t here, so let us assume the u is a function of t and v is also a function of t. So, here we want to check whether this u plus v satisfies the given differential equation or not remember this u is the solution of the homogenous equation.

And v is any solution of the non-homogeneous different on homogeneous differential equation. Then we can use this linearity here, so the derivative when applied on u plus v we can have this derivative on u and plus derivative on v and then we can actually break into two parts here one with this u here and the other one with the v. So, we have written into two parts this u with this differential equation in the v again with the same this left hand side of the differential equation.

And now if we look at because this u satisfies the homogeneous equation meaning that this part here will be 0 because u is the solution of the homogeneous equation; that means, this is equal to 0 and plus this v is a particular solution of the given non homogeneous equation meaning that here we will get the right hand side that is the x of the given differential equation, so we have 0 here and we have x from here.

So, this will add again to this right hand side of the given differential equation. So, what we have observed that this u plus v; u plus v satisfies the given differential equation because our differential equation was this nth order term here now with y and then n minus 1th term and with coefficient a 1 and so on a n y and is equal to X this was the given differential equation then we have taken these two solutions one was the u which satisfies this homogeneous equation and the other 1 was v which was satisfying the full equation.

So, here therefore, this u with u it is coming to be 0 and when with v this differential equation is giving X here and they are adding 2 X again. So, this u plus v is satisfying the given differential equation and it has n arbitrary constant because this u has n arbitrary constant. So, this u plus v is the general solution of the given differential equation.

So, that is the trick we are using their to get the general solution of the non-homogeneous differential equation and we write down the general solution here for this differential equation as this like we have written u plus v or we call this complimentary function that is u here the solution of the homogenous equation and we call the particular integral that is there one particular solution a particular solution of the given non-homogeneous equation and we will observe now that finding the C F is easy and also finding one particular solution of the given differential equation is also easy and when we add the two we get actually the general solution of the given differential equation.

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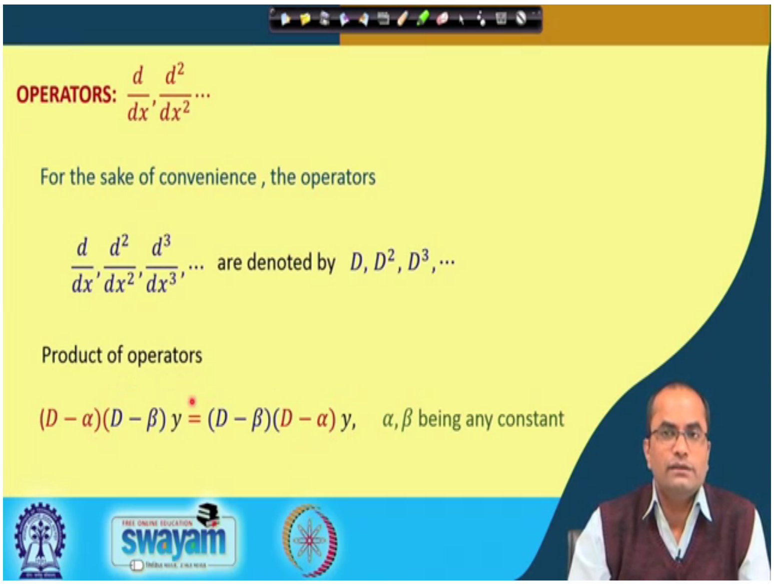

So, now because we will be talking about these solutions the complimentary function and particular integrals in the following lectures, but to here we will prepare for all the all the requirement all the knowledge we need to discuss the solution here. So, one concept which will introduce now its operator, so these are the basically the differential operators and in our differential equation we do see all these derivative terms.

So, here we have the first order this differential d over dx the derivative. So, this is the operator is a differential operator here the second order this differential operator. And for the sake of convenience these operators we will denote these operators we will denote by this symbols here D here we will take this D square and D cube means this third order derivative.

So, with this introduction of these operators here what we will see now that this product of the operator's product means here D minus this alpha; alpha is a constant beta is another constant. So, D minus alpha D minus beta when we operate on y it is same as that first we operate D minus alpha and then we operate this D minus beta. So, here we will see that this product of these operators is same whether we first apply here the D minus beta or we apply D minus alpha it does not matter the value of this left hand side and the right hand side are the same, so here alpha beta any constant. So, this is a very important effect here which will be used in following lectures to discuss the solution of the differential equations.

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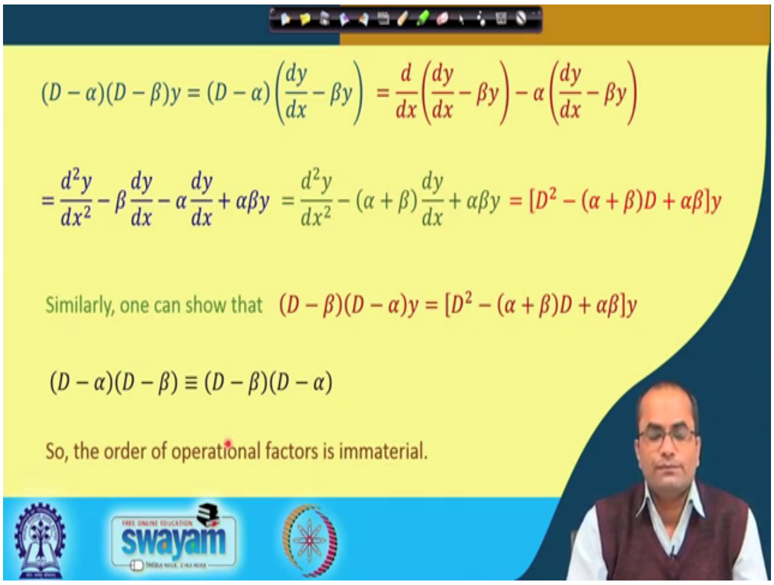

So, let us consider the left hand side here D minus alpha and D minus beta into y. So, what do we have here? D minus alpha; so first we operate this D minus beta meaning that the derivative of y minus beta y, so the derivative of y with respect to x and minus this beta times y and then we operate this D minus alpha also on this meaning this derivative term will be applied on this or d over dx on this minus this alpha will be also operated on this, but that is a constant here so nothing will happen.

So, in this case when we apply this operator d here or so this will be a second order derivative minus this beta and this first order derivative. So, that is here d 2 y over dx 2 minus this beta dy dx and this minus this alpha times dy dx and plus this alpha times beta y. So, this is the result of this operation which we have made first we have operated this D minus beta here and then we have operated this D minus alpha operator. And as a result we got this one which we can write down in this form as well the second order derivative term here we have this common term dy over dx.

So, if we take this common we got minus this alpha plus beta term and plus this alpha beta y, which we can write down again in terms of the operator because this is d 2 here operated on y here minus alpha plus beta d operated on y and here this alpha beta times y. So, if you y we can take this common to the right hand side then we have here the D square coming from this second order derivative minus this alpha beta this D the first order this derivative d over dx and this alpha beta from here and this y goes the right hand side of this operator.

So, similarly one can also show that when we operate this D minus alpha first and then D minus beta because the same thing will happen now, we will end up at this expression which will be the same whether we operate first D minus alpha or D minus beta. So, meaning that we can again write down the same thing when we apply first D minus alpha and then D minus beta.

So, what we have observed here said it does not matter that what sequence you take here for the operations when we have such differential operators. So, meaning this operator is same as this operator, so we will not care much that in which sequence we should apply this operator to our y here, we can first apply D minus alpha then D minus beta or the other way around.

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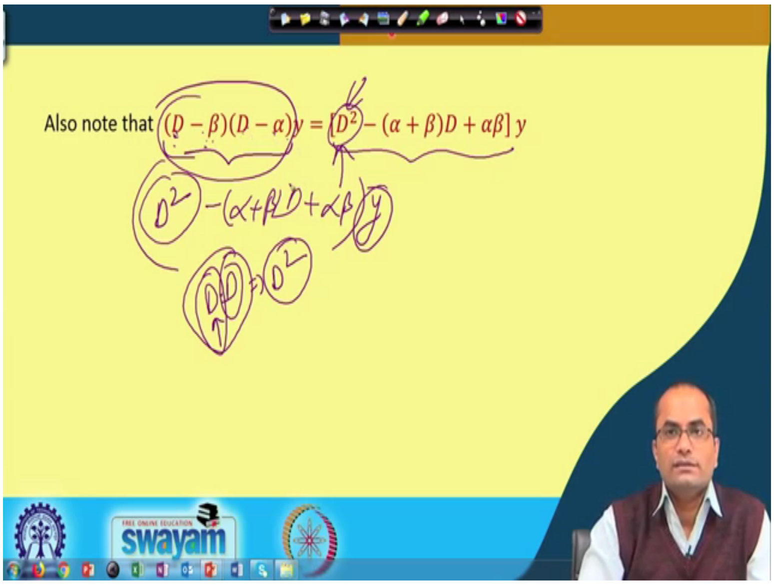

So, the order of the operation null factors as immaterial again also we will note that just in the previous slide that d minus beta D minus alpha or other way around D minus alpha D minus beta is equal to this one. D square minus is alpha plus beta D plus alpha beta. What is the observation here that this product it is working like the product here though this D are operator they are not the numbers, but here they are being treated as the numbers here because if we make this product of this D minus beta D minus alpha what will happen? Here we will have D D so that will be D square.

Though this D square is not the product of the D and D it is a second order derivative and these are the first order derivatives. So, here the D multiplied by D, so we have like D square with this giving D square there and then we will have minus this alpha times D terms minus beta times D which we have written this alpha plus beta D. And then this product of alpha beta will give alpha beta and this everything applied on y.

So, what is interesting though D is a operator here and D into D which is we are writing here D square they have a different meaning because here it is a derivative and then again derivative, so two times this derivative. So, here D square is also 2 square; two times the derivative that is our symbol for second order derivatives. So, here this product is working as the same as we work with the numbers and that is the nice property we have with these operators also. So, we can without worries here we can do exactly we do the product with treating this D as a number; real number.

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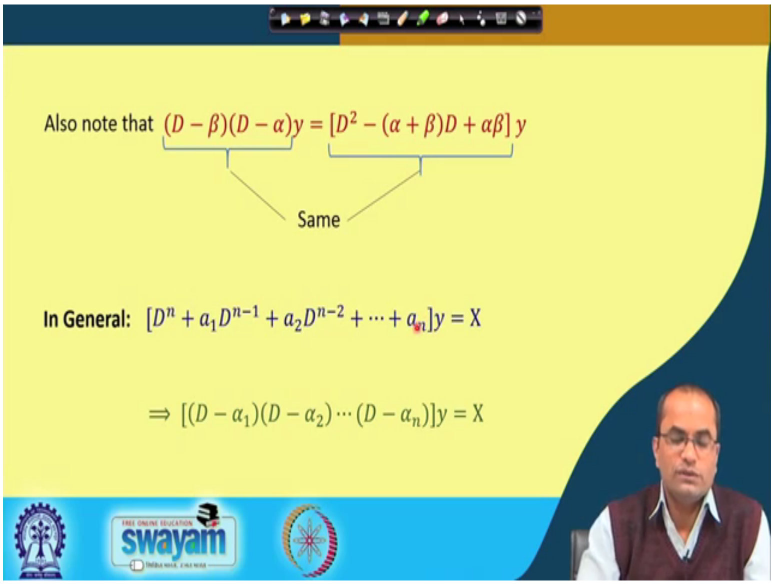

So, and because we have seen that these two are same and this is just the product of this one if we forget that these D; D is an operator here, so we can simply algebraic multiplication we can do. In general also when we have this nth order differential equation like written in this operator form the power n meaning the nth order derivative here a 1 the n minus th order derivative a 2 the n minus 2th order derivative and so on and the right hand side we have this X.

So, here also we can like factorize because this is a kind of polynomial equation we have in terms of D if we do not worry about this operator D. So, it is like a polynomial equation and we can factorize it also, so this will be equal to this D minus alpha, 1 D minus alpha 2, D minus alpha n if these alpha 1, alpha 2, alpha 3 alpha n are the roots of this different of this equation here D power n plus a 1 D power n minus 1 plus so on plus a n is equal to 0.

If we have the roots here these alphas then we can write down this as a product of this D minus alpha 1, D minus alpha 2 and so on, this y n is equal to X the right hand side. So, with this note here which we will continue in the in the next lecture it is a very useful to factorize this in this in this way though here D n means the operator which have which is the nth order derivative here it is not like D power n, but it is like the nth order derivative when we apply to this y.

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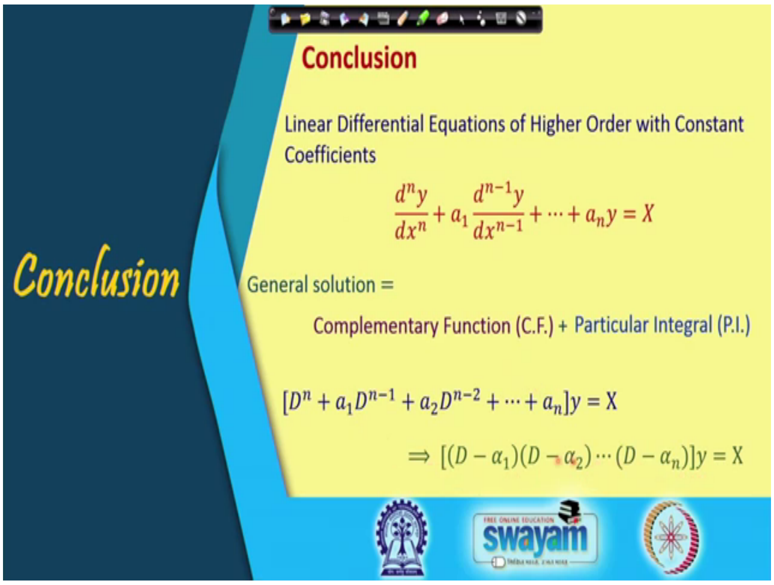

Well so coming to the conclusion here, so we have discussed about this linear differential equations with constant coefficients. So, this what treated as constant at first and in later on some lectures we will also talk about this known constants. And the general solution what we have seen the one way of getting the general solution of such a non-homogeneous equation is to find the complimentary function; the complimentary function is nothing, but the solution of this equation when we set this X to 0 and then the particular integral that is one particular solution of this given non homogeneous differential equation.

And what else we have seen that this when we write into the operator form we can treat this D power n a 1 D n minus 1 like the polynomial here the algebraic equation and we can factorize this equation once we know the root of this here equation equal to 0. So, we can then write down this operator equation in this form also which is going to play a very important role in finding out the solution this y whether it is a complimentary function or a particular integral.

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So, these are the references we have used here and.

Thank you for your attention.