Engineering Mathematics – I
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Lecture – 55
Linear Differential Equations

So, welcome back this is lecture number 55 and we will be discussing a Linear Differential Equations of first order.

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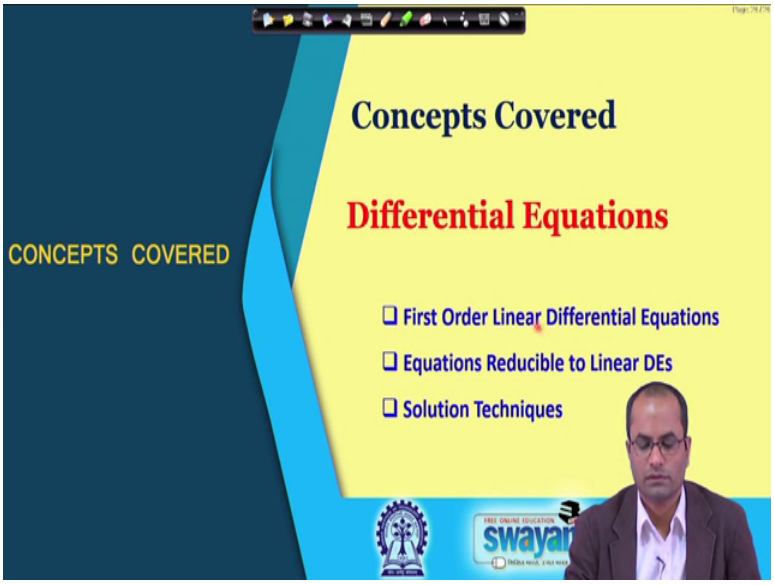

And, we will go through the not only to the linear equations, but also the differential equations that are reducible to linear differential equations and also their solution techniques.

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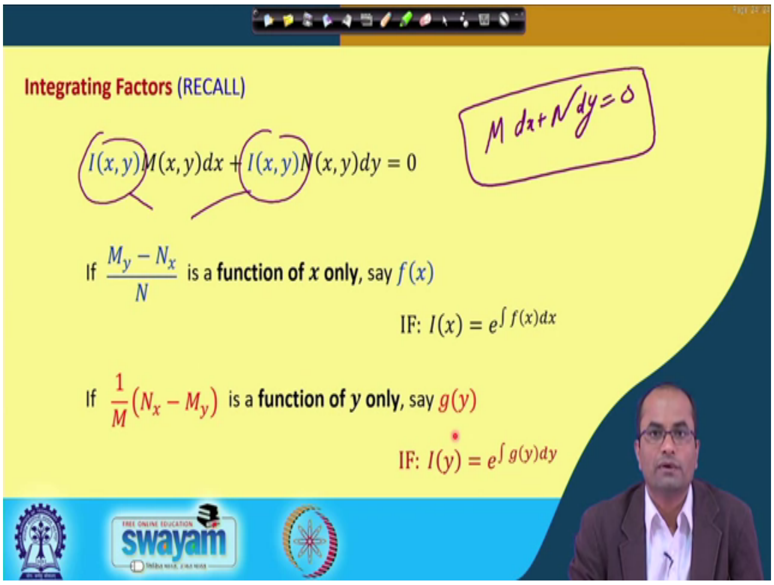

So, just to recall from the previous lecture where we have discussed integrating factors. So, if our equation has given like M dx M is a function of xy and N is also a function of xy. If this is a this is a given differential equation which is not exact and if you multiply that equation by this function I x, y and then this equation becomes exact then we call such a function here the integrating factor. And, what we have also seen that how to evaluate integrating factor in some special cases.

So, one of them was that if this M y minus N x over N is a function of x only say f x in that case the integrating factor we can evaluate simply by this e power integral f x dx and also this case here 1 over M N x minus M y if this is a function of y only let us call it a g y in that case also we can compute this integrating factor again as I y is equal to e power the integral of this function g y dy.

So, we require these knowledge of the integrating factor how to evaluate in these special cases in today's lecture.

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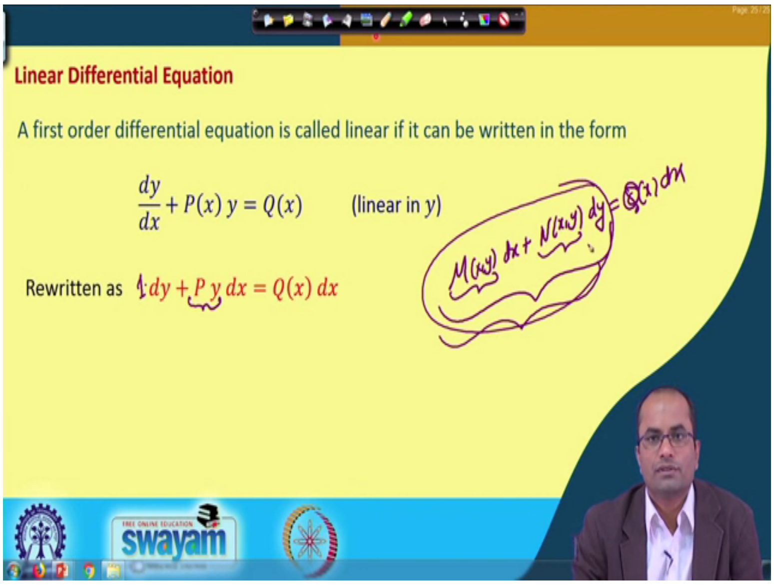

So, here this linear differential equations which is the topic of today's lecture. So, a first order differential equation is called linear if it can be written in this form dy over dx plus P x y and is equal to Q. So, here this P is a function of x only and the Q is also a function of x, but that can be a non-linear function of x.

Why we call this linear? This is basically linear equation in y because this dy over dx appears as a single term here also y is they appearing just alone. So, there is no product of y with y or with its derivative. So, this equation is linear in y the function p and the Q may have powers of x. So, x square or any other function. So, no restriction on P and Q, but this y has to be like dy over dx and here is y. So, that is a general form of first order linear equation which we will be considering in today's lecture.

And, this we can also rewritten as. So, this differential term here dx we can multiply the whole equation by this differential term. So, we will get this differential of y plus P into y and this dx here the right hand side also we will have Q x dx and now this is the equation which is in this form that M function of x, y dx plus N function of x, y and dy is equal to 0. So, here this is M with this dx and N x, y is just 1 here and then the right hand side instead of this 0 we have the sum function of Q x dx.

So, the important point is here this M dx plus N dy the question is if we can find an integrating factor here for this so that if we multiply by that integrating factor. So, we get the total differential at the left hand side of this equation.

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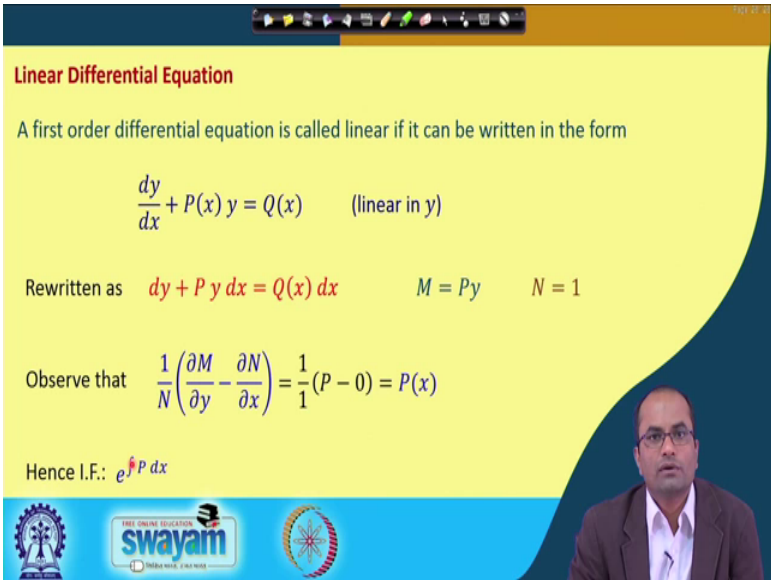

So, let us look at it here. So, M is Py and this N is 1, the coefficient of this dy and if we compute if this 1 over N and d M del y. So, del M del y minus del N del x this term. So, what we are getting it is just the function P which is a function of x alone. So, what is the point here we can use this now the idea which was explained on earlier slide the when this is a function of x only then we have the integrating factor for this M dx N dy term.

So, what is the integrating factor? The integrating factor will be this just the exponential of this integral P dx.

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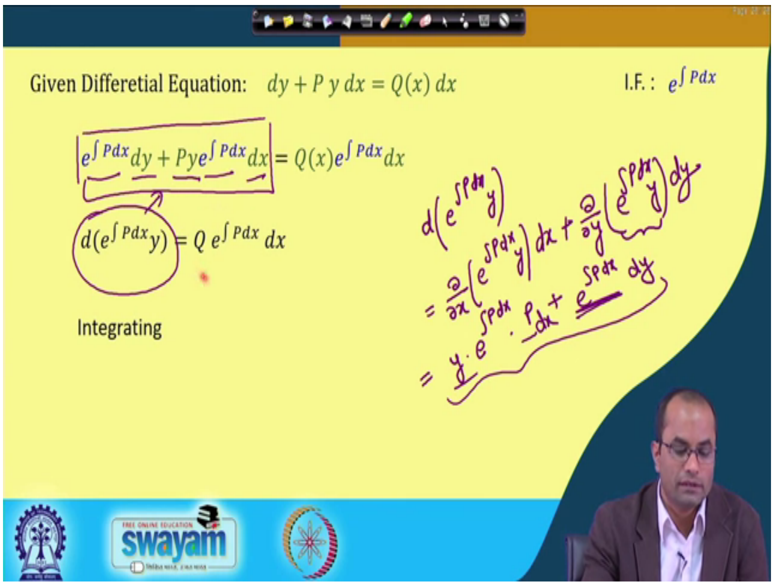

So, we have this differential equation, we have the integrating factor here e power P dx meaning that if we multiply by this integrating factor this given equation; that means, e power square root P dx is a appearing now here also in this term and the right hand side. Now, this part here the left hand side should be now the differential of some function f x, y that is the point after multiplication here this will become the total differential of some function of xy.

So, here it is clear now that this term left hand side is nothing, but the differential of this e power exponential P dx y. Why this is the differential here? So, if we just compute this e power Pdx and y so, the differential definition is that the partial derivative with respect to x of this given function and then dx plus again the partial derivative with respect to y of this given function and then dy.

So, here we have to get this partial derivative with respect to x only. So, y will be just taken as a constant. So, the exponential of this e power integral Pdx. So, the exponential the differential will be also the derivative will be the exponential only and then the derivative of this integral Pdx with respect to x and that will be coming as P and here with respect to y even we integrate so, this will become one and this e power Pdx will remain and then this dy term and here we have this dx term.

So, this is a total differential of this e power integral Pdx y. So, exactly that is what we have here the P into y if the P into y the exponential term with this dx and e power Pdx with this y e power Pdx with this dy term. So, this is the total differential here of this function e power Pdx into y and then we have this right hand side as this Q and exponential e power Pdx and dx.

So, what is the point now that we can easily integrate this equation because left hand side we have this differential of this e power Pdx y and right hand side is some function of x.

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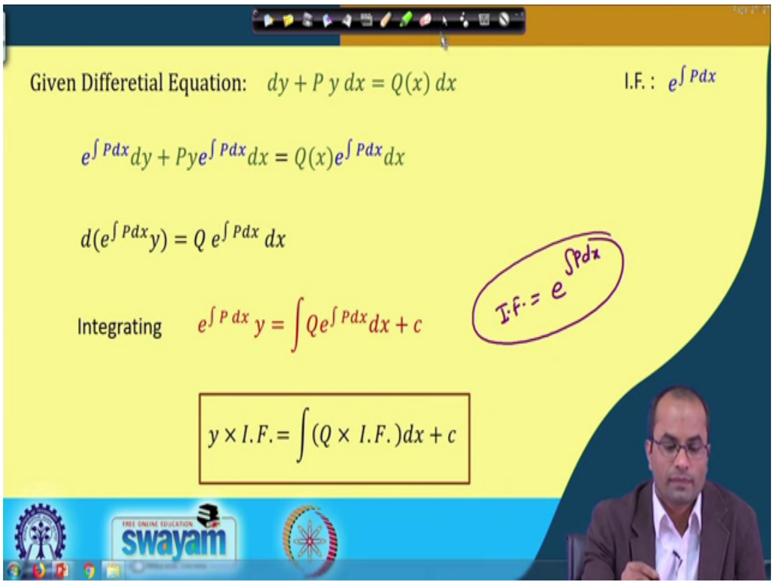

So, while integrating this; so, we will get here when we integrate this e power Pdx y only and the right hand side will be integrate so, we have Q and this integrating factor dx and we have this a constant of integration. So, this is a eventually the solution of this given differential equation, the given linear differential equation and just too easy to remember we write down as the y multiplied by this integrating factor which we have to evaluate.

And, what is integrating factor? This integrating factor is nothing, but this exponential not P dx. So, with this integrating factor what we write down the y into this integrating factor is equal to the integral Q the right hand side of the linear equation into this integrating factor and we need to integrate here and plus one constant of integration c. So, this is the formula which can be used to get the solution of the linear equation. So, this Q is coming already in the right hand side which was also in the equation and the P comes here in the integrating factor and then we can simply write down the solution of the differential equation.

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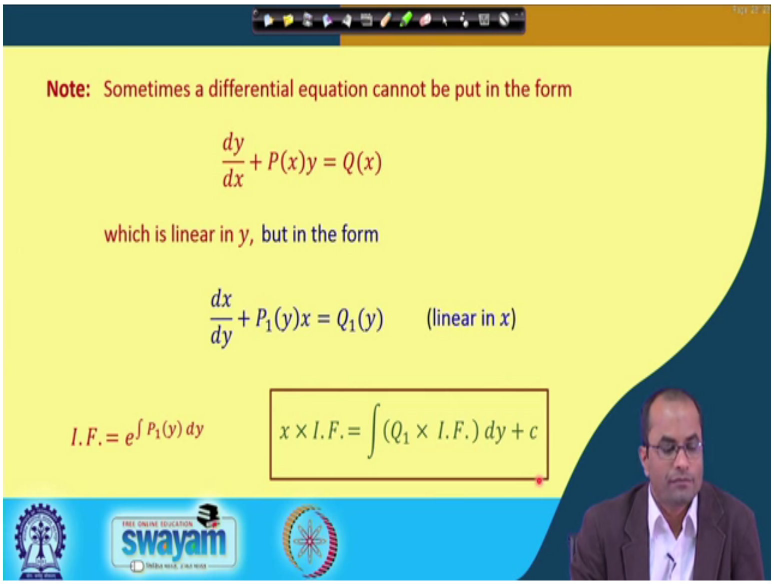

So, you just a note that sometimes a differential equation cannot be put in this form here dy over dx plus P x y is equal to Q x meaning that it may not be possible to write down this equation as a linear equation in y. This is linear equation in y, the standard equation which we have considered before, but the equation can be put in the form here dx over dy plus P 1 y x is equal to Q 1 y; meaning that instead of writing the equation which is linear in y which may not be possible, but we have to also see this possibility whether the given equation may be written as a linear equation in x instead of y.

And, once we write down this equation, so, now we will be just treating this x as we have treated y before and because this is just the linear equation in x and we know the solution. So, we have to write down the integrating factor where this P 1 y will come now here in the exponential there under the integral and then the solution is absolutely the same formula which we have used earlier. But, now instead we have x into the integrating factor and the right hand side this Q 1 y again with the integrating factor and this needs to be integrated over y and plus a constant c constant of integration.

So, that we have to also remember that the given differential equation may not be given as a linear equation in y, but if we look take a close look maybe we can write down this as a linear equations in equation in x and then we can use again the same idea having the integrating factor and then writing the solution.

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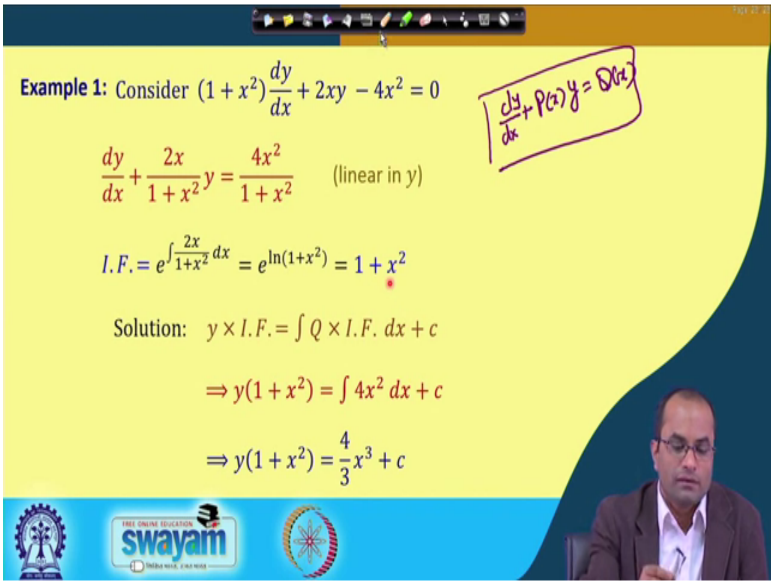

So, going through the example here example number one we consider that 1 plus x square dy over dx plus 2 xy minus 4x square is equal to 0. So, first we have to put into this form which we have discussed that dy over dx. So, the standard form usually we considered as dy over dx plus this P the function of x into y and is equal to this Q x.

So, we will put into this standard form, meaning we just need to divide this by 1 plus x square and this minus 4 x square we can take to the right hand side; that means, the equation here by dividing this is dy over dx plus this 2x divided by 1 plus x square with y and the right hand side 4x square over 1 plus x square. So, this equation is indeed in the in the linear form linear in y because we have y term here and also y term there this is function of x alone. So, this is like P x in the standard form the right hand side corresponds to the Q x.

Now, since this equation is linear in y we can write down its integrating factor. So, the integrating factor will be the exponential of the integral and this P dx. So, P is 2x over 1 plus x square and now what is this here the differentiation of this 1 plus x square term is sitting there as 2x. So, this is nothing, but the logarithmic of 1 plus x square. So, here we have exponential of the ln 1 plus x square which is equal to 1 plus x square because this exponential will cancel out this logarithmic. So, we have this 1 plus x square as the integrating factor.

So, having this integrating factor now 1 plus x square we can use directly the formula for writing down the solution or we can multiply the given equation by this 1 plus x square and then we can see the left hand side is a total differential of some function. So, the solution was y into the integrating factor equal to the right hand side the Q into integrating factor and we need to integrate over this dx plus a constant c.

So, here y into the integrating factor that is 1 plus x square is equal to the Q is therefore, x square divided by one plus x square. So, this one plus x square will be cancelled out with the integrating factor. So, we will get 4x square dx plus c and now here again 1 plus x square into y is equal to this is 4 by 3x cube and plus a constant term c and that is the solution of the given differential equation.

So, with the help of this integrating factor we can compute the solution of a linear differential equation as long as this P and Q are some simple functions where we can integrate here and we can find the integrating factor.

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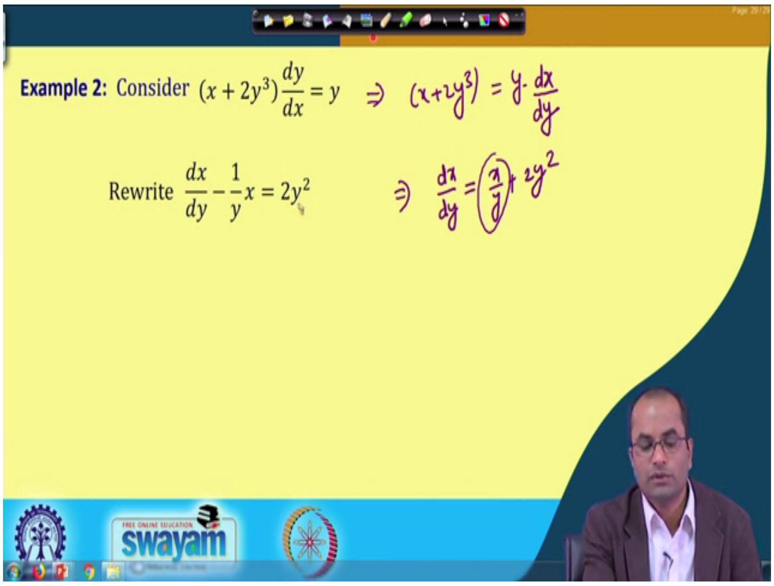

The next example here we have x plus 2y cube and dy over dx is equal to y. So, what do we see here because y cube is reappearing here? So, certainly the idea is that we cannot put into the form as linear in y, but we can try this equation to write down as a linear equation in x. And, then so, we need to rewrite this as a dx over dy, so, that will go to the right hand side. We have dx over dy and minus this x term there and then we need to divide also by y here.

So, there will be 1 over y term x and the right hand side will be 2y square. So, what we have done now. So, the we have x plus 2y cube and then the right hand side is like dy into dx over dy. So, we divided by y, so, we get this x over y and plus 2 y square and then this x over y we have taken to the left hand side. So, our equations dx over dy minus this 1 over y into x and 2y square.

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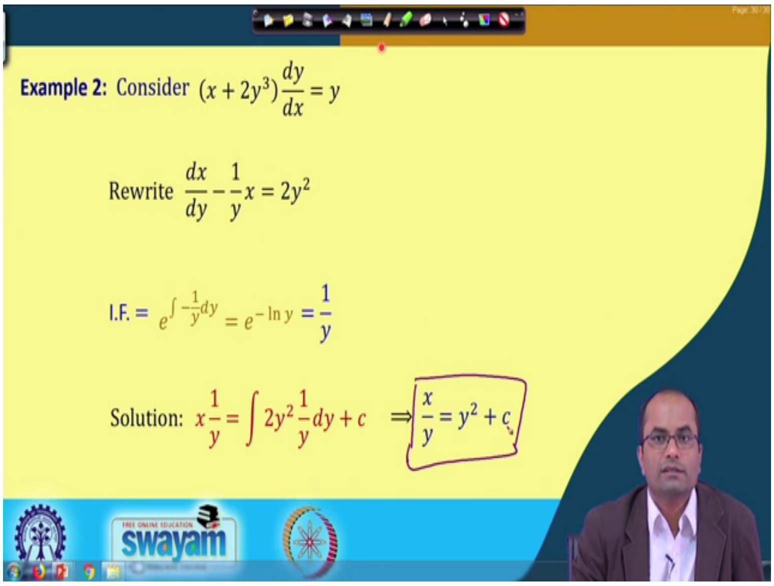

So, with this equation which is linear in x because we have dx over dy term and x here and the right hand side. So, this is corresponds that P and here corresponds to this Q which is which are functions of y alone when we have written this equation linear equation in x. So, we need to compute the integrating factor again using the idea which we developed already.

So, the exponential of integral and this term here minus 1 over y dy. So, which is minus of this ln y the logarithmic y or this minus can be taken here 1 over y and then exponential and this logarithmic will cancel out and we will get just 1 over y as the integrating factor. So, the integrating factor for this equation is 1 over y and then we can use the formula that x into this integrating factor which is 1 over y is equal to the right hand side 2y square and then we have this integrating factor which is a 1 over y and dy and plus this constant of integration.

So, here this y will cancel out. So, we will get to y here dy and then the integration will give us the y square. So, what we have here the x over y is equal to x square plus c and this is the solution of the given differential equation the relation between this x and y.

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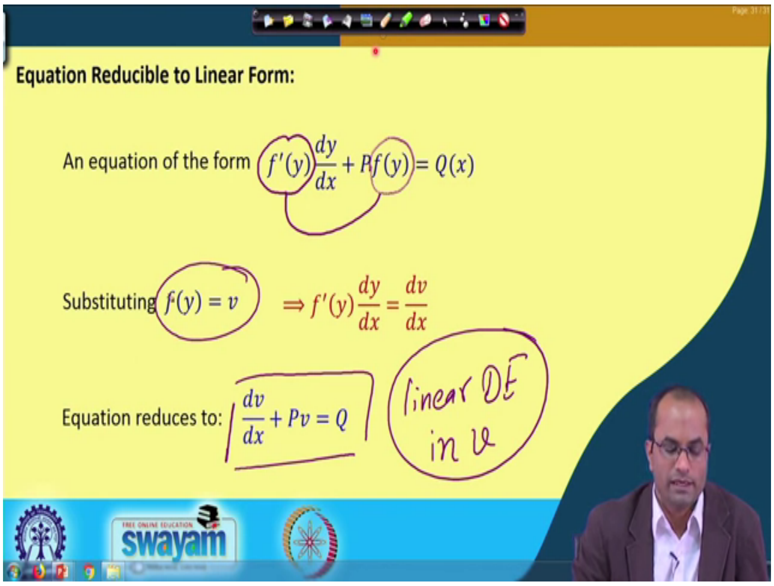

Next what we will consider about these equations which are reducible to the linear forms. So, an equation of this form f prime y and dy over dx plus P and f y is equal to the Q y. So, this equation is not linear in y because this f y sitting here and also the f prime y is sitting here. The linear equation was dy over dx P y is equal to Q x, but what do we see here that if the derivative of this term which is sitting with the P here the f and its derivative is just in front of this dy over dx of which we can pull our equation in this form we can write our equation in this form then we will see now that this can be reduced to the linear form by appropriate substitution and what is the substitution you are going to have it here.

So, if we substitute here f y this function here is equal to v some variable name we have introduced here that this f y is equal to v. And, then what we can get if you differentiate this the f prime v, so, the derivative of f with respect to y. So, f prime y and then dy over dx is equal to the right hand side will be dv over dx and now we can substitute and that was the point here that this f prime dy over dx. So, f prime dy over dx is becoming now dv over dx. So, this term is dv over dx then we have P and this f y is v and is equal to Q x.

So, the new equation, which after this substitution, this equation reduces to a dv over dx. So, this term with this derivative a dv over dx plus the P into v. So, we have the P there and f y is substituted as v and the right hand side equal to the Q which is given there. So, what we observe now so, now, this equation here this equation is a linear equation linear differential equation and that linear in v. So, this is linear differential equation in the variable v which we can solve for v and then later on we can substitute this v as f y and we can write down the solution in the form of y and x.

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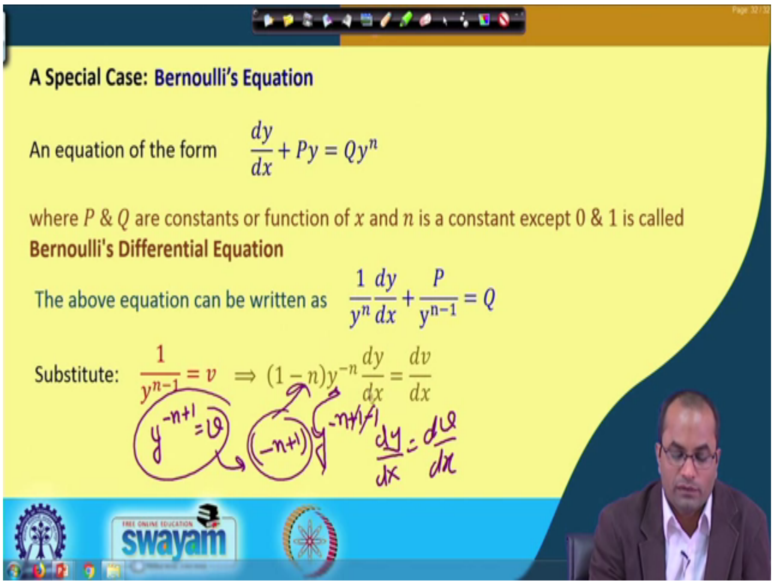

As a special case here we have the Bernoulli equation and this is an equation this of the form the dy over dx plus Py is equal to Qy power n. This such a form of the equation is called Bernoulli equation, where this P and Q are constants or functions of x. So, they can be functions of x and this n is a constant except 0 and 1. So, it can take any other value other than this 0 and 1 and such equation is called Bernoulli's differential equation.

So, this Bernoulli's differential equation is a special case of the idea we have discussed earlier that it can be reduced to linear differential equation by an appropriate substitution. How we can do that here? So, if we divide this y power n then this equation can be written as this 1 over y power n. So, this y power N goes to the left hand side in the denominator. So, 1 over y power N then we have dy over dx then P and here y power N minus 1. So, this was y you already. So, we got here y power n minus 1 and it is equal to this Q, the right hand side.

And, now this is exactly the same which we have discussed earlier that the function which is sitting with P its derivative is somehow coming here as a factor of this dy over dx; that means, if we substitute this 1 over y power this n minus 1, 1 over y power n minus 1 as a function here as a variable v and then what we see? So, if we differentiate here what will happen? So, we have y power n minus n and plus 1. So, when we differentiate minus n power this n minus. So, here we have n minus 1. So, again this was like y power minus n plus 1 is equal to v.

So, when we differentiate this what we will get minus n plus 1 and y power minus n plus 1 and minus 1 and dy over dx is equal to dv over dx. This is a differentiation here. So, this 1 minus n term is sitting here and y power minus n is coming here and dy over dx is equal to dv over dx. So, we have exactly the derivative of this written here and now the idea was this y power minus n which is here y power minus n with dy over dx. So, this term will become this dv over dx. This factor we can adjust by dividing this.

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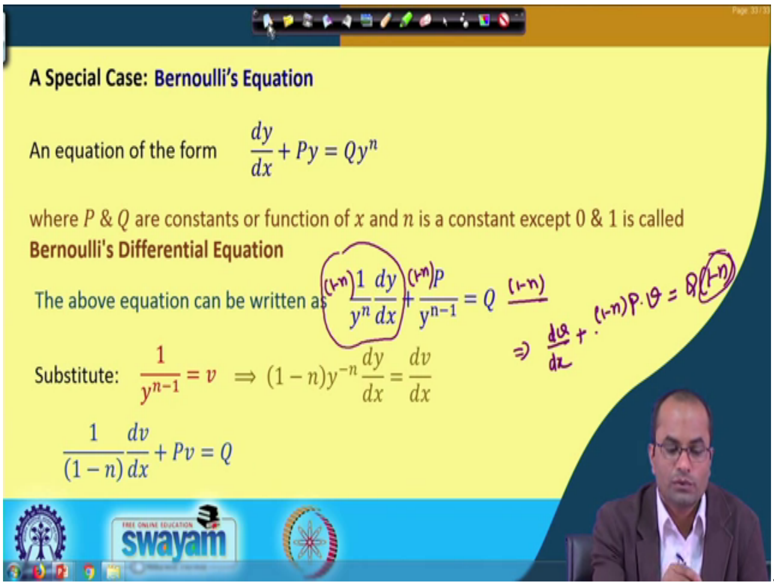

So, once we substitute this 1 over y power n minus 1 is equal to v into this equation what we will get? We will get 1 over 1 minus n and then just dv over dx. So, because we have to now we can think in this way. So, we multiply this equation we multiply this whole equation by 1 minus n and also so, we can just 1 minus n. So, we have multiplied here 1 minus n, here also 1 minus n and they are also 1 minus n.

So, this 1 minus n and y power this n minus 1 with dy over dx will become as dv over dx plus this 1 over. So, here 1 minus n will be there and this P 1 over y n minus 1 will be v and the right hand side the cube with 1 minus n. And, then we can divide again with this 1 minus n factor. So, we will get here 1 over 1 minus n dv over dx plus this Pv is equal to this Q.

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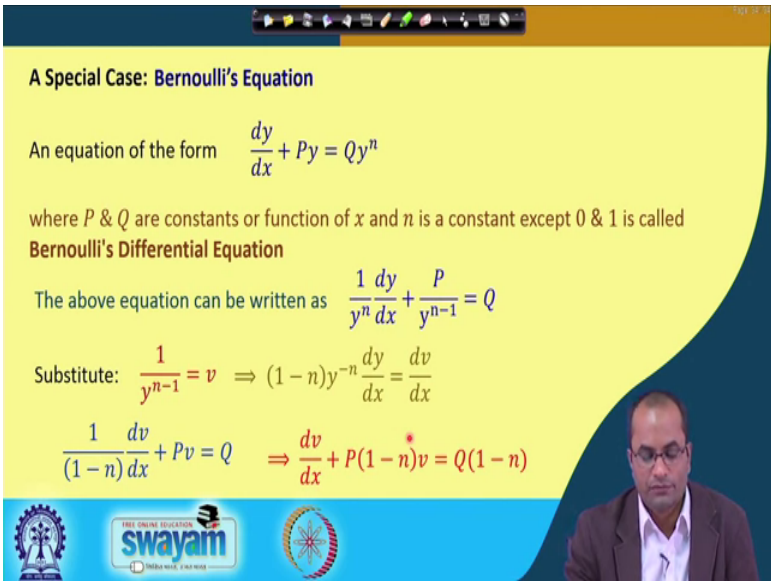

Or we can write down in this form itself. So, because this is the standard form of the of the linear equation in v because we have dv over dx term here and we have v here together with this 1 minus n which is a constant and this P could be a function of x and then the right hand side this is function of x and multiplied by this 1 minus n.

So, this is the linear equation in v. So, or this Bernoulli's equation we can convert by this such a substitution to the linear equation in this another variable v and then we know how to solve a linear equation.

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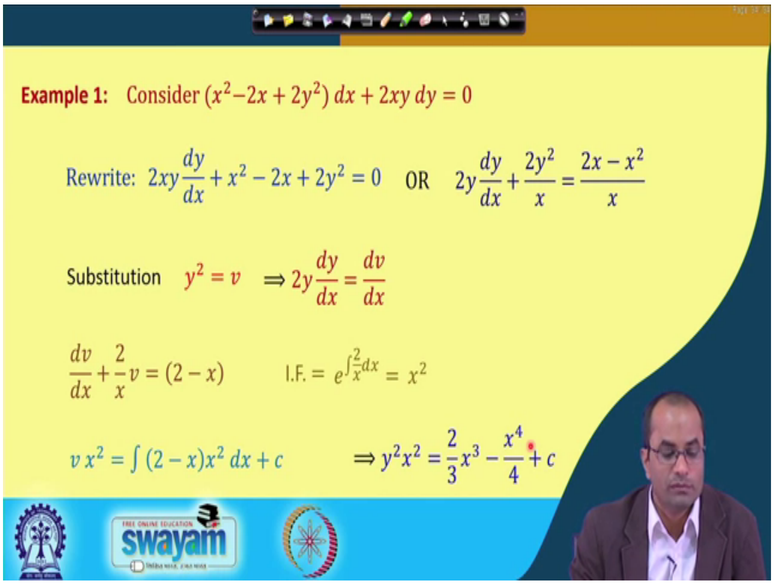

Going through the example; so, first example we will consider as x square minus 2x plus 2y square dx and 2xy dy. So, this is written as M dx plus N dy form, but we can write down as dy over dx form. So, writing this into dy over dx form. So, we have 2xy and dy over dx plus this x square minus 2x plus this 2y square that is the term sitting here. So, we have divided just by this dx, so, we got this equation.

And, now we can take everything of this x to the right hand side. So, we have actually 2y. So, we divide by x also. So, 2y and this dy over dx plus this 2y square over this x which we are dividing. The right hand side goes to now 2x minus x square and we have divided by x. So, we have rewritten this equation into this form and now if we substitute here this y square as a new variable then it is differential and derivative is sitting here next to this dy over dx. So, we can convert this to the linear equation.

So, with this substitution y square is equal to v we can get this 2y and dy over dx as this dv over dx and then we can substitute here. So, this term will become dv over dx, then we have 2 over x term together with this y square which is v and the right hand side will be 2 minus x. So, this x when we divide we will get 2 minus x there. So, this is the linear equation in v and we know how to solve a linear equation in v. So, we can write down the integrating factor and this is our P here. So, e power this P dx which is x square now. So, 2 ln x or ln x square exponential ln x square, so, here we get this x square.

So, v into x square the solution we can write down now. So, v are multiplied by this x square the right hand side here 2 minus x into this x square dx and plus a constant of integration here c. So, we get this v as y square, we substitute again. So, y square x square is equal to this 2 and this x square dx. So, 2 into x square dx means when we integrate x cube by 3 this factor will come and this minus this x cube which will become here x 4 divided by 4 and plus this a constant of integration.

So, this is the solution of this given differential equation written in this xy as in implicit form.

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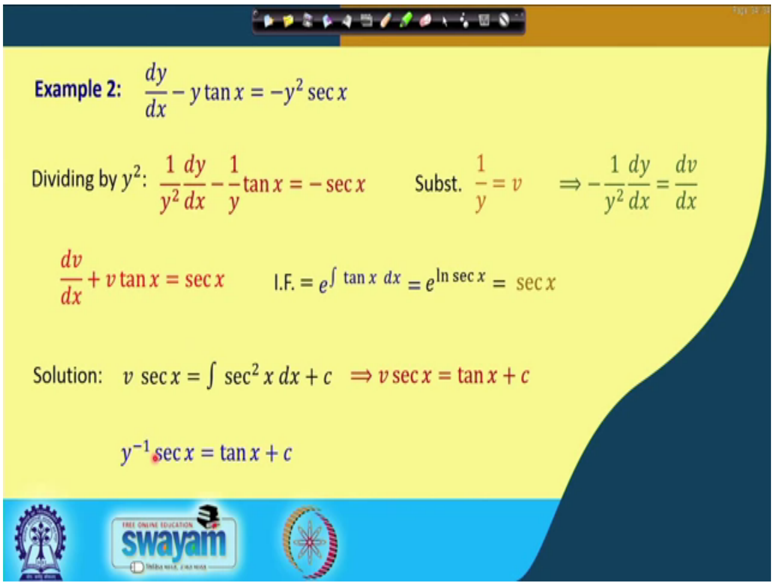

So, just one more example we will discuss here. So, dy over dx minus this y tan x is equal to minus y square sec x for instance. So, if we divide by this y square what we are getting 1 over y square dy over dx minus this 1 over y and this tan x and then we have minus sec x there. So, it is a kind Bernoulli's type of equation.

So, if we substitute here this 1 over y as v and then we differentiate here, so, we will get minus this 1 over y square dy over dx is equal to dv over dx; that is coming from this 1 over y is equal to v substitution. And, then when we substitute back to this equation here 1 over y square dy over dx that is dv over dx and this 1 over y; 1 over y is v this tan x and this minus sec. So, minus also we have just multiplied. So, we got this equation dv over dx plus v tan x is equal to this sec x.

And, now this is a linear equation in v which we can solve by calculating this integrating factor. So, the integrating factor will be exponential of this tan x. So, exponential this tan x and tan x will be this after integral will become this ln sec x. So, the integrating factor is nothing, but this sec x which we can use now in the solution. So, the solution will be v into this sec x term is equal to the integral. Here this sec and then one this sec x will come, so, square dx and plus a constant of integration.

So, we get this v into the sec x is equal to the integral of this sec square x that is tan x plus this constant of integration and this v was nothing, but 1 over y. So, this 1 over y sec x is equal to tan x plus c and that is the solution of the given differential equation written in this explicit implicit form of the y and x. So, in fact, this we can write down as an explicit form also because y appears here. So, we can bring everything to the another side of this y. So, this solution can be written in explicit form as well.

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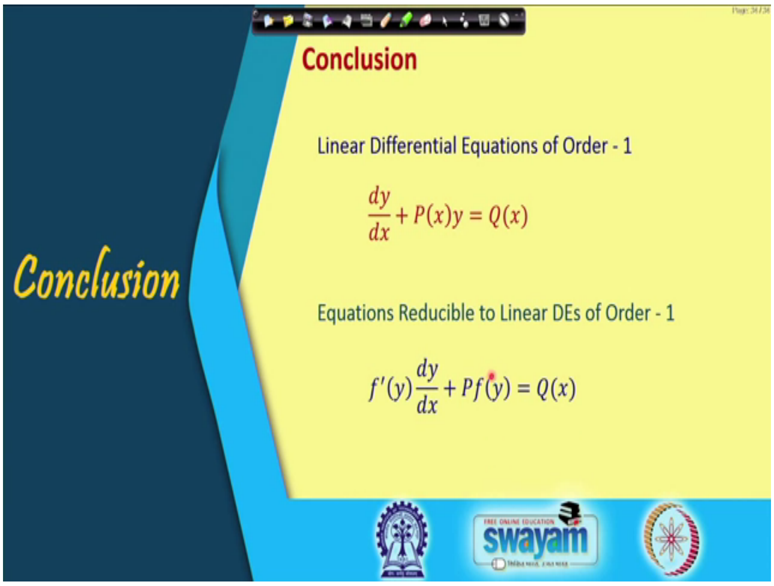

So, coming to the conclusion we have gone through the linear equations of order 1 and the standard form which we have discussed was dy over dx plus this P x y is equal to Q x and also we have discussed the equations which are reducible to linear differential equations of order 1 again. And, there we have considered this again very general equation where this f y is sitting with P and its derivative if we see it is lying here with this dy over dx.

Then, by this substitution of this f y is equal to v we can convert this equation into the equation into the linear equation which is in v and then we can solve the linear differential equation which we have explained in today's lecture.

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So, these are the references used for preparing the lectures and.

Thank you for your attention.