Engineering Mathematics – I
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Lecture – 54
Exact Differential Equations (Cond.)

So, welcome back this is lecture number 54. So, we will continue discussing Exact Differential Equations.

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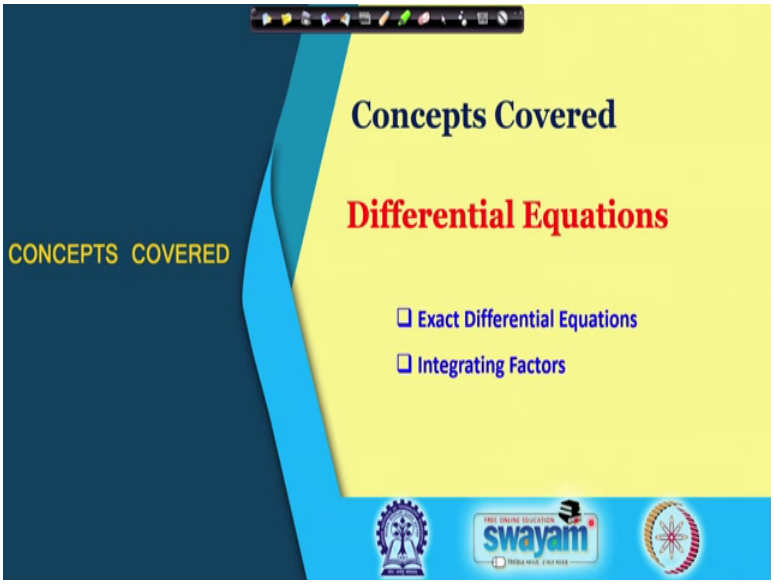

And, in particularly we will be talking about today when the given differential equation is not exact, but it can easily be made exact and that is the introduction of this integrating factor will come into the picture.

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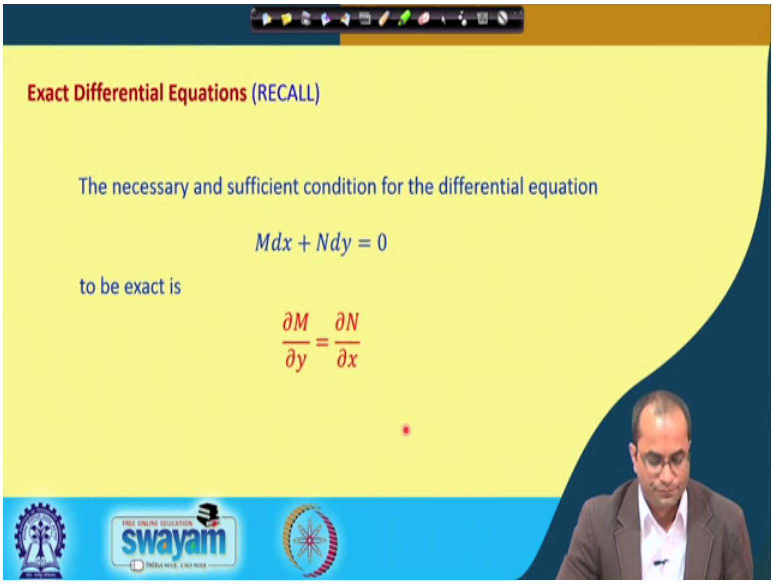

So, here just to recall that we have the necessary and sufficient conditions for the exactness, so, the necessary and sufficient condition for the differential equation to be exact is Mdx plus Ndy is equal to 0 that was the differential equation.

And, this is exact when del M over del y equal to del N over del x, and this was necessary and sufficient meaning that if this condition holds then the differential equation must be exact and if the equation is exact then this condition must hold.

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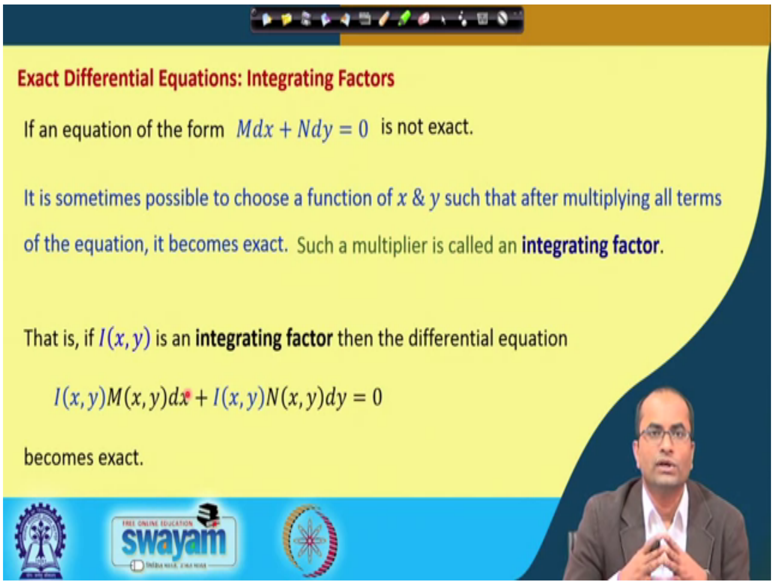

So, now what to be done when the given equation is not accept. So, for instance; so, if an equation of this form Mdx, Ndy is not exact. It is sometimes possible to choose a function here x or function of x and y such that after multiplying all the terms of the equations after multiplying this given equation by that function the equations become the equation becomes exact and such a multiplier here is called the integrating factor. So, integrating factor is nothing but a function of x, y after multiplication to the given differential equation. The given differential equation becomes exact then such a function is called the integrating factor.

So, this I x, y usually the notation we will use for this integrating factor. So, if this is an integrating factor then the differential equation here I x, y times is M x, y dx plus I x, y and x, y dy is equal to 0 becomes exact. And, what is the advantage now? So, the given equation is not exact and then it is difficult to find the solution here. So, what we will do now, we will find the integrating factor and once we multiply this integrating factor to the given differential equation then this equation will become exact and then we know how to solve the exact differential equation. We need to find the function f whose differential is here this left hand side of the given differential equation.

So, here the main topic of this lecture is we will discuss some techniques here how to find integrating factor when a given differential equation is not exact differential equation.

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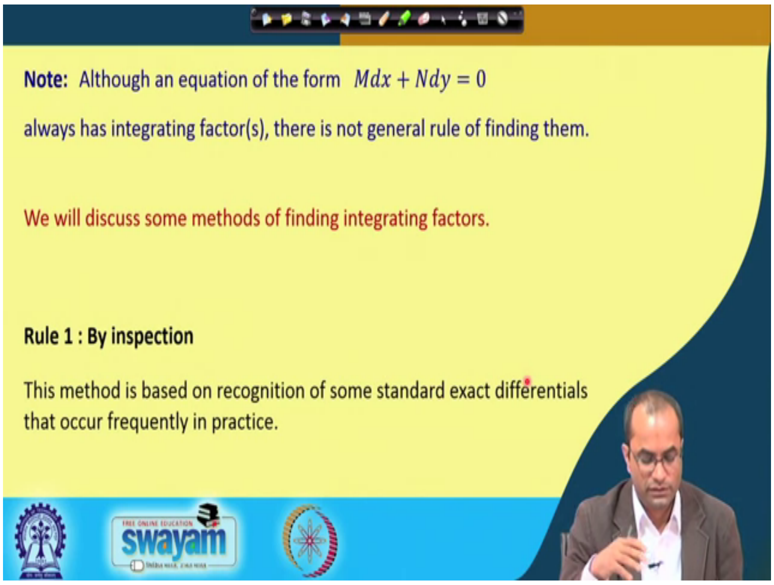

So, although an equation this of the form Mdx plus Ndy always has an integrating factor or factors there could be more factor integrating factors, but there is no a general rule for finding them or rather it is a little bit tedious job to find this integrating factor. It is not in general easy to find the integrating factor of the given differential equation. So, we will consider only some special cases where we can find the integrating factor easily.

So, the rule number 1 is just by inspection. So, this is the most easiest one and also the most toughest approach to find the this integrating factor because it is it is just the gas here looking at the differential equation that, the equation is given in this form and if you multiply by this function then this will become exact, but this is also difficult because it is not a systematic approach to find the integrating factor.

So, this method is based on the recognition of this some standard exact differential. So, if we know some standard differentials here, then looking at the differential equation also we can find that if we rewrite this equation in this form or we multiply or divide by this function. So, we will get exactly the differential of those standard functions and this is what we approach by solving the for solving the differential equation in this approach.

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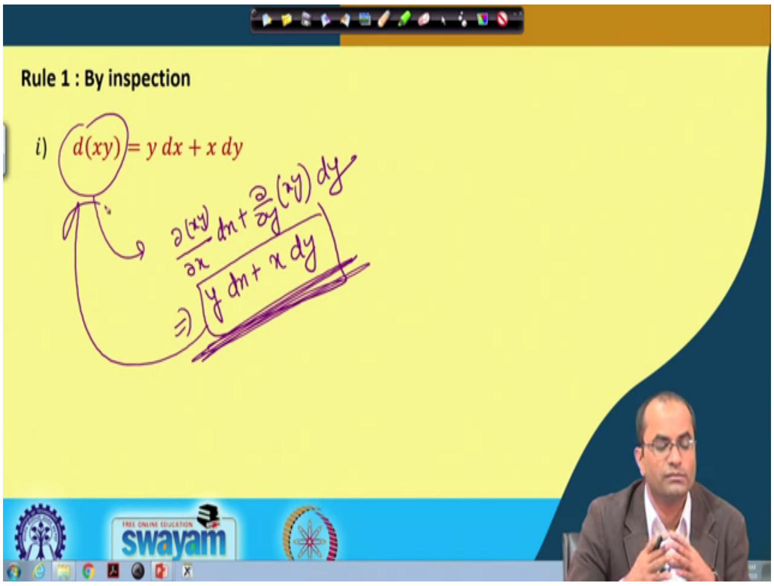

So, here what is what are those stand or some of these standard forms rather we will discuss here and there could be many more which are maybe non standard, but can help if we know them that whose differential is exactly this term. For instance, here y dx plus x dy if we have this term y dx plus x dy then this is nothing, but the differential of xy and we can verify easily.

So, the differential of this xy is again as per the definition, so, del over del x of the function dx and plus this del over del y of this given function and dy. So, here the partial derivative of xy with respect to x will be y dx and plus here will be x dy. So, if we see some such a term in the differential equation or by rewriting the differential equation if we can see that there is a term y dx plus x dy then we know already that this is the differential of xy term. So, that could be useful to remember that the differential of this xy is nothing, but y dx plus xdy.

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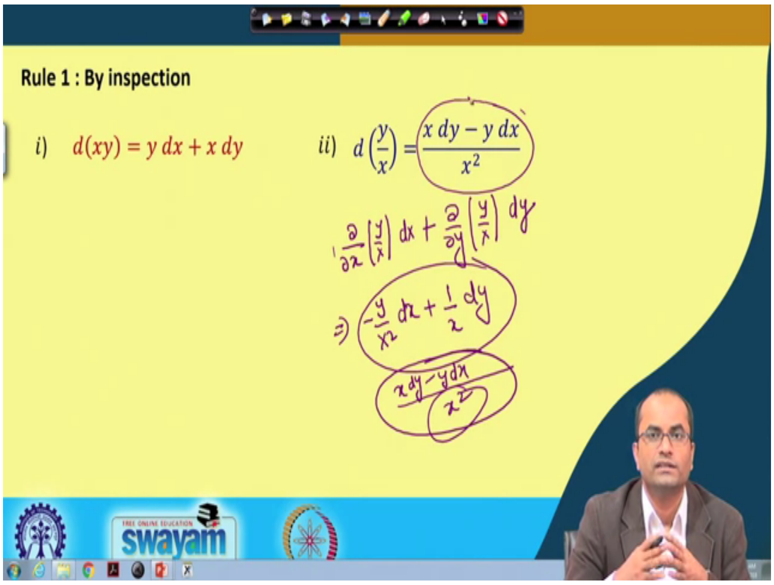

Another a standard form could be the differential of this y over x. So, again by using the formula which we have just used we will realize that the differential here of this y over x is nothing, but x dy minus y dx over x square and this just to again the approach is same that we take the derivative with respect to x of this function y over x and then this dx plus this partial derivative of the function y over x with respect to y and then the dy.

So, here with respect to x we will get minus y over x square dx and here we will get with respect to y only this 1 over x and dy. And, this is exactly the given differential equation given expression there. So, we have x square. So, here x dy and then the minus d y dx. So, if this term such term appears or just by dividing for instance by x square we do see such a term in the differential equation. So, it will immediately click to us that this is the differential of the y over x and then we can again proceed easily for the integration.

So, this is one of the standard differential which we can use a in guessing the differential equation.

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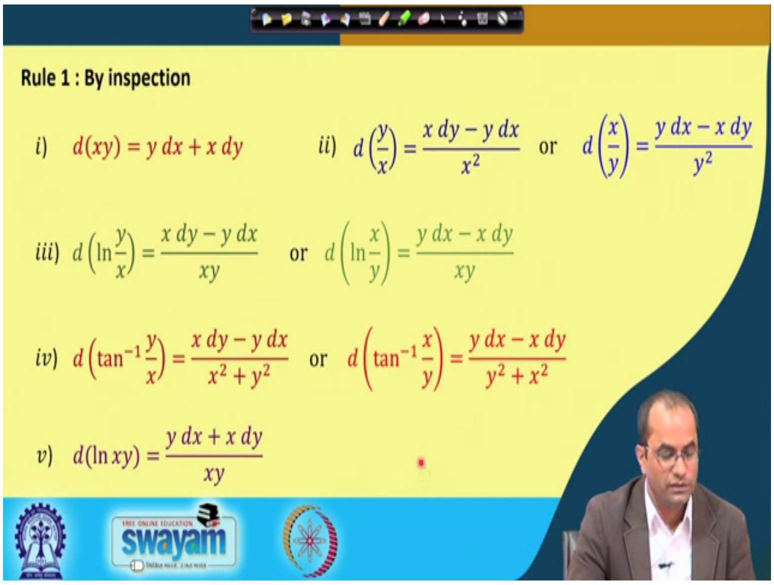

So, here or this can be also like x over y set of y over x. So, the formula will be just y dx minus x dy over y square. There is another standard form differential here or. So, instead of this x dy minus y dx. So, here this x square or here y square, but instead of this x square and y square. For example, this xy is given. So, here again the same expression x dy minus y dx over xy. So, for this one this is the differential of the logarithmic of y over x which we can again prove like we have done for this case just using the definition of the differential, getting the partial derivative of this with respect to x, partial derivative of ln y over x with respect to y we can get this expression here or the differential of the logarithmic x over y as well we can discuss.

So, here this will be similar to what we had here for x over y. So, y dx minus xy and again this xy here. So, they both looks almost the same, but the differences here it is a xy and there x square and here y square, but here we have the xy. So, once we see such terms we can immediately recognize that these are the differential of the logarithmic functions of y over x or x over y.

Another standard form could be this x dy minus y dx again we have this common term there x dy minus y dx, but here we have x square plus y square now. So, if we have the possibility to rewrite our equations in this form then we know that this is nothing, but the differential of the tan inverse y over x or tan inverse x over y. So, we have then y dx minus x dy over this y square plus x square. So, this is the standard differential of this standard function tan inverse x over y.

And, the another one which we have discussed here the logarithmic xy. So, if we have this plus term like we had their y dx plus x dy and, but by dividing this number this expression xy we get this differential of the logarithmic of xy.

So, these are the few some standard differentials, but there can be many more if we can recognize these standard differentials. It is a very useful for solving the equations which are not exact, but can be made exacts by multiplying or dividing by some functions. So, that we can see those standard forms and then we can write down the differential.

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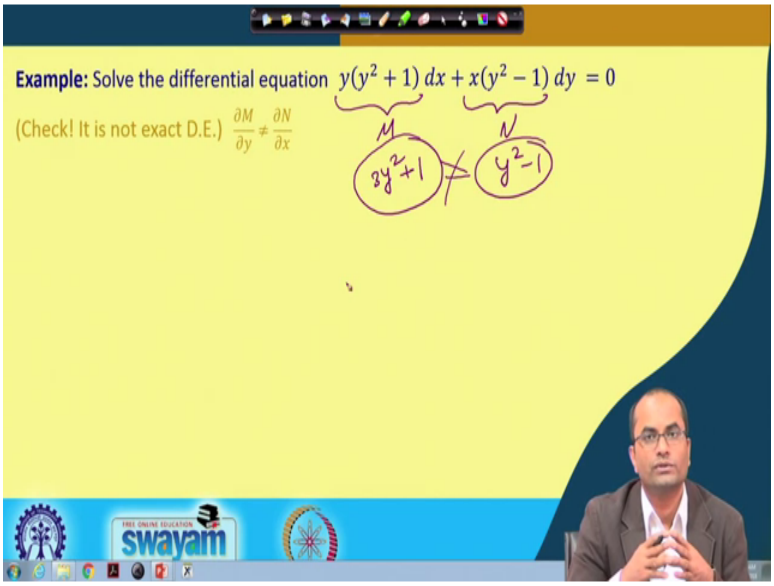

So, we will just go through at least one example where we can we have such a possibility to recognize those standard differentials. So, again we can check that this is not an exact differential equation because del M over del y is not equal to del N over del x. So, what is del M over del y here? So, here we have M and this is N; M dx, N dy. So, del M over del y we will be this is y cube plus y. So, 3 y square and plus 1 that will be the derivative of this M with respect to y and derivative of N with respect to x will be y square minus 1. So, they are not equal.

Of course, so, this equation given in this form is not an exact differential equation. So, this approach would by inspection we will do now. So, here we have y square here also we have y square term, y square with this dx here also y square with the dy. So, if you take y square common out of these two terms, so, what we will get? y dx x dy that is the known term which we have seen before as y dx plus x dy and then the rest will be here y dx and minus x dy, that term also we have seen before.

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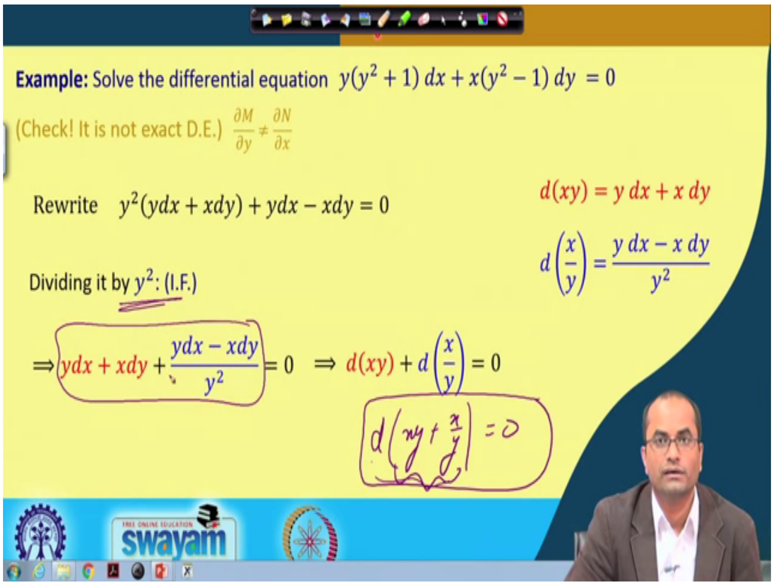

So, rewriting this equation first in this form taking this y square common from these two terms we get y dx and we also get here x dy and then plus this y times this dx and minus this x times minus this x times dy. That is equal to 0 and now this dividing this equation by this y square. So, if we divide this equation by y square because all those standard expressions we have seen that there was something in the division. So, if we divide this y square this is very standard form here y dx x dy and this y dx minus x dy divided by y square.

And, just to recall, from the previous slide we have that the differential of this xy is y dx plus x dy and the differential of x over y is y dx minus x dy over y square. So, we can use these two differentials, but we should have in mind that this is the differential of the xy and this is the differential of x over y. Once we know this. So, we can just rewrite as the differential of xy plus the differential of x over y; meaning that we have the differential of this x y plus this x over y is equal to 0.

So, we have this function f whose differential is now the given differential equation. So, though the equation was not exact, but by just dividing by this y square that equation becomes exact and whose differential we know now that this is a differential of xy plus x over y, the differential of this is the given modified equation after this division. So, with this now we can now write down the solution as well because we have the function f directly here. So, f is equal to constant will be the will be the solution.

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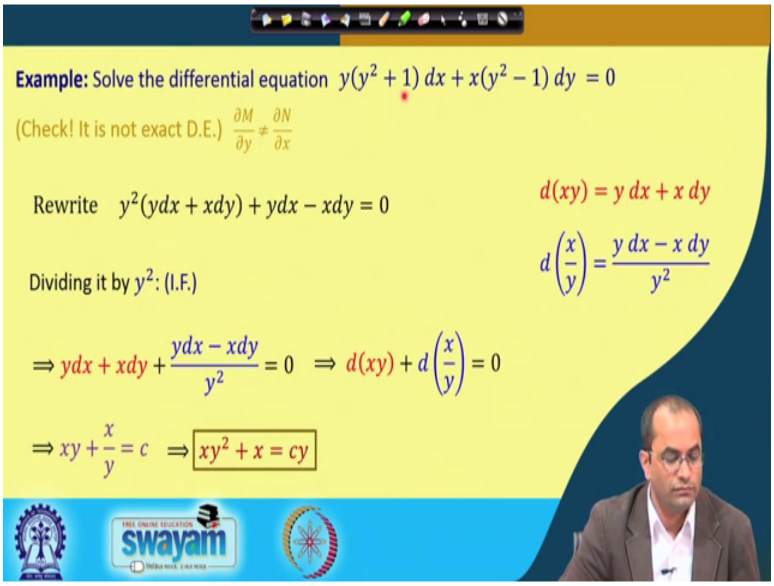

So, here xy plus x over y that is our f now whose differential is this one and is equal to some constant. So, this is the solution of the differential equation again we can multiply by y to see this form here xy square plus x and cy. So, this is the differential this is the solution of the given differential equation.

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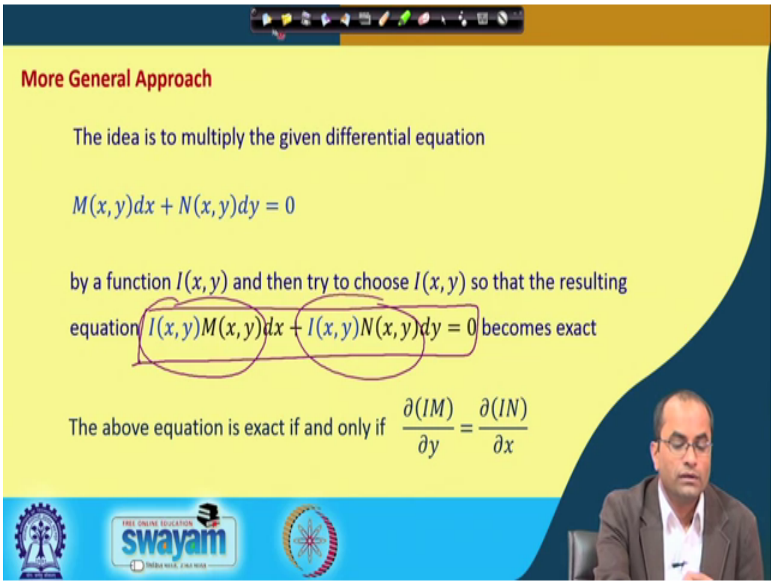

Slightly more general approach we will discuss because here by inspection it is not always easy to see the those standard differentials in the equation after multiplying or dividing by some functions. So, here we have a more or rather systematic approach which we can follow to get the integrating factors.

So, the idea here is to multiply the given differential equation which is not exact. So, this is suppose this is not an exact differential equation, but if you multiply by this function I x, y which we call as the integrating factor and then try to choose this I x so that this function is this equation here after multiplying the given differential equation by this I this becomes exactly, that is the idea of the integrating factor that.

We are looking for such a function here I x, y when we multiply this I x, y to this given differential equation this should become exact difference in equation. So, this is the point where we will now proceed that suppose that I mean we know that the if and only if condition or the sufficient necessary condition, this equation to be exact now would be that del of this function with respect to y must be equal to partial derivative of this with respect to x.

So, we know that now these equation is exact if this del I M over del y is equal to del I N over del x that is a necessary and sufficient condition for the exactness of a differential equation. So, here if this is exact then this condition must hold and out of this condition we will try to derive some in some special cases this I here which we need as this multiplier or the integrating factor.

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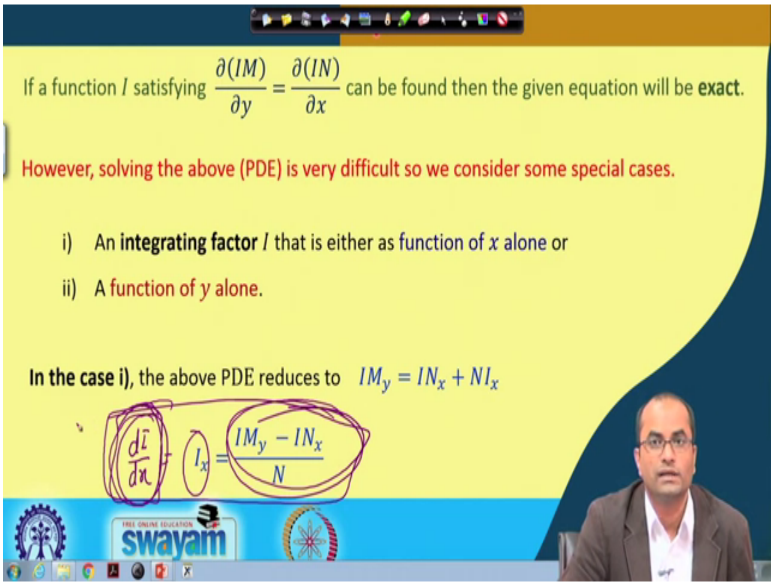

So, if a function I satisfying this equation del over del y IM is equal to del over del x of IN can be found then the given equation will be exact.

The problem here is that solving this equation because this is not an ordinary equation. This is a partial differential equation; we have the partial derivatives in this equation. So, this is a partial differential equation which is not very easy to solve to get this I out of this equations. So, we will have to consider some special cases where we can solve such a equation. So, we will consider here at least two of them and there could be many more special cases can be derived from this partial differential equations so that we can get this unknown here I.

So, these are the special cases an integrating factor I that is either a function of x alone or a function of y alone, that is our first assumption that we are looking for integrating factor which is a function of x alone or it is a function of y alone. So, these two special cases we will consider here to get the solution of this differential equation or this partial differential equation here.

So, in the case 1, when we assume that this is a function of x alone then this PDE, the condition here can be reduced because we have assumed that this function here I is a function of x alone. So, here this partial derivative with respect to y of this IM, I we can take out because this is I is a function of x alone. So, we have some simplification at least we have I the partial derivative of M with respect to y that is a short notation for that is equal to here this derivative we have to because I is a function of x. So, we have to use the product rule. So, it is I x with this N and I with this N x. So, here the product rule the right hand side.

So, we have this in a rather simplified form of the given differential equation which tells us that I x is nothing, but I M y minus I N x and divided by this N, but again the purpose is not solved because this the here dI over dx I is a function of x alone. So, this is the derivative here dI over dx. The right hand side, if this is not a function of x then we cannot solve because we will get this inconsistency here the left hand side because we have assumed that I is a function of x alone. So, here this is nothing, but the ordinary derivative here dI over dx and this is a function of x.

So, the right hand side should be the function of x at least to proceed from here to solve this differential equation for I. So, this is a differential equation the ordinary differential equation for I. So, if this one here is a function of x again; so, another assumption, that if this is a function of x alone then the we can solve perhaps this ordinary differential equation to get this I the integrating factor.

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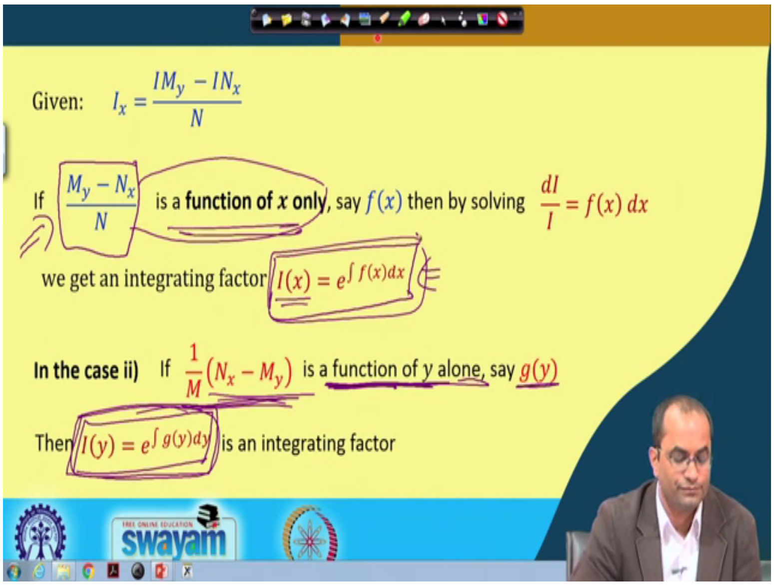

So, our next assumption would be that this M y minus this N x over N because I is anyway a function of x. So, if this one is a function of x only. So, again further assumption here which we take to get this I as a function of x alone. So, here if this is a function that is the main condition because it depends on this M and N and the divided by N. So, if this is a function of x alone, then we can solve this ordinary simple differential equation if this function is not very complicated here.

So, we say let us this is the function f x because we assume that if this is a function of x alone then only we can proceed for solving this I x is equal to I M y minus I N x over N equation. So, if this is a function of x only then we can solve this equation dI over this I from the right hand side is equal to this f x which we have used for this function here and this dx. So, this is a simple differential equation if this f x is a simple function, then we can get this integrating factor from here as because this is a logarithmic of I is equal to this integral of f x dx. So, this I x is equal to exponential of this f x dx. So, that will be the integrating factor.

So, what is the condition? We have to check this M y minus nx over and if this is a function of x alone, then we can write down that this will be the integrating factor e power integral f x dx because as per the construction we have seen through the necessary and sufficient condition that this I will now if you multiply the given differential equation by this I x in this case when this is a function of x only, then this will be serving as the integrating factor.

Now, the other case also we can deal which we have just stated before in the case 2, when we assume that this I the integrating factor is a is a function of y alone in that case again such a differential equation will be formed and we will get again for the consistency of this differential equation. This condition, that 1 over M N x minus M y is a function of y alone, yeah. So, here now if it is a function of y alone this one and we say let us see this is g y, then we can solve this differential equation again this ordinary such a ordinary differential equation and we can get as this I y is equal to e power g y dy as the integrating factor.

So, we got these two simplified cases there as I said there can be many more cases where we can solve this differential equation this condition necessary and sufficient condition to get this I, but we have considered at least these two which is special cases. The first one was when this M y minus N x over N is a function of x alone; the second condition we have taken here when this 1 over M N x minus M y, y is a function of y alone. So, in that case this is the integrating factor. In the other case, this was the integrating factor.

So, with this we will now go through some of the examples.

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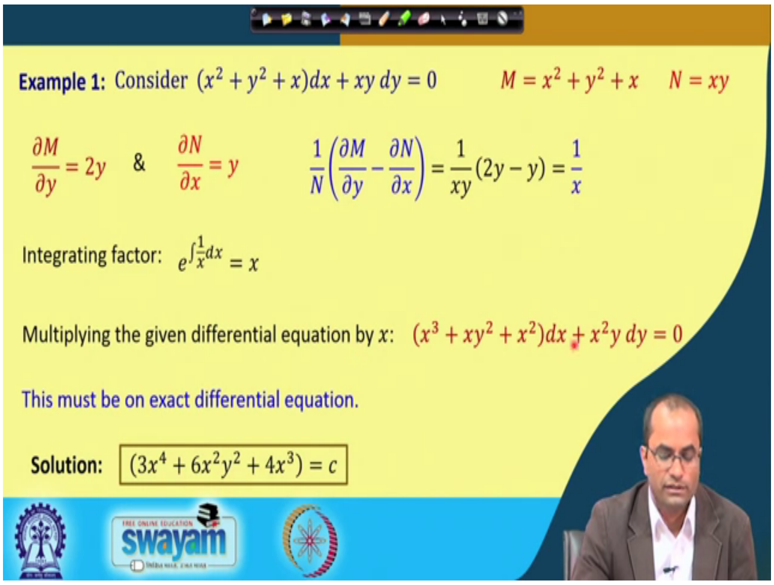

Let us consider this first one: x square plus y square plus x dx and plus this xy dy is equal to 0. So, we have M here we have N M is x square plus y square plus x and N is xy. So, we can check that this equation is not exact as this del M over del y here is 2y and del N over del x is just again here y del N over del x is just y. So, naturally these two are not equal.

But, what we observe in this case these are the from the previous discussion which we have derived those conditions what do we see that 1 over N of this del M over del y minus this del N over del x that expression there when we divide by this N what are we getting is the N was 1 over xy and here we have this 2y and this minus y. So, this is y here and this y, y get cancelled and we have 1 over x. So, this here is a function of x alone that is 1 over x and that was one of the conditions we have just studied that if this is a function of x alone we can easily integrate it.

So, the integrating factor is e power 1 over x dx which is ln x and e power ln x will give us x only. So, this is the integrating factor of this differential equation meaning if you multiply by this x to this differential equation the equation will become exact. That means this is the exact differential equation which we can also verify. So, del M over del y here will be 2xy and here del N over del x will be also 2xy. So, this equation is now exact and we know how to solve the exact equations. So, we will not go through all the solution process, but it was explained already in the previous lecture.

So, we will now find f whose differential is given here. So, this is an exact differential equation and the solution after the process we follow for getting such a f whose differential is this we can get this as these solution of this given differential equation.

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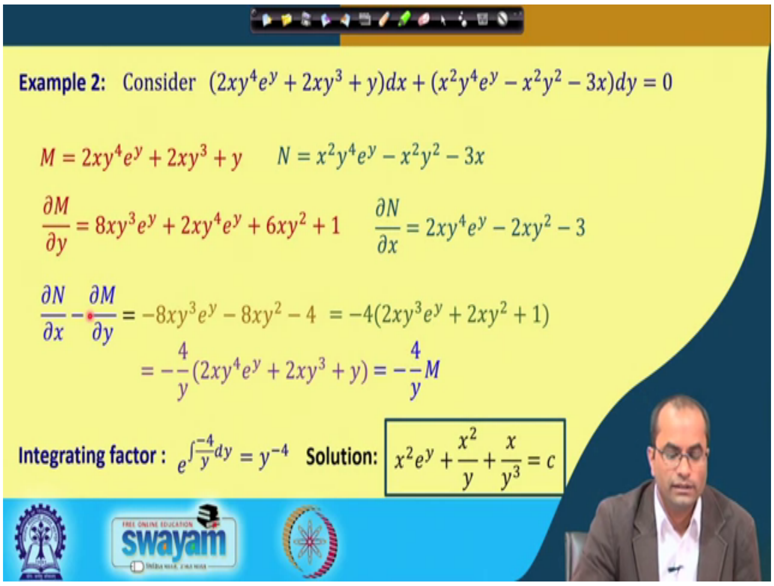

Another example we take with what this M and this N here. So, M is given by this, N is given by this and again we can verify that del M over del y is coming to be this complicated expression here. Also, the del N over del x is coming here of out of this N this complicated expression again. But, if we take this difference then N over del x minus del M over del y naturally it is not 0 otherwise the equation will be exact.

So, here when we take this difference we do see this little simple simpler version here when we take this minus 4 as common we will get this one which again by taking this y from here. So, minus 4 over y and because we do see that this is matching precisely with this M here, 2 xy cube plus y when we just multiply by y and divide by y. So, what are we getting here? Minus 4 over y and this is M sitting there; so, minus 4 over y M. So, this one when we divide by M is a function of y alone. So, that was exactly the second case where we have seen that this N x minus this M partial derivatives with respect to y divided by M if it is a function of y alone then we can find the integrating factor easily and the integrating factor will be exponential of this minus 4 over y with this dy, that is an integrating factor which comes to be this y power minus 4.

So, the conclusion here again that if you multiply this divide by this equation by y power 4 or multiplied by y power minus 4. This equation then this equation will become exact differential equation and then we can follow the approach to solve this differential equation to find f whose differential is exactly they are after this multiplication.

So, one can solve this and the solution will be coming this x square e power y plus x square by y plus x over y cube is equal to some constant. So, here again that approach which we have discussed which was rather general approach at least by the derivation, but we have considered only few simple cases where we can get this integrating factor y.

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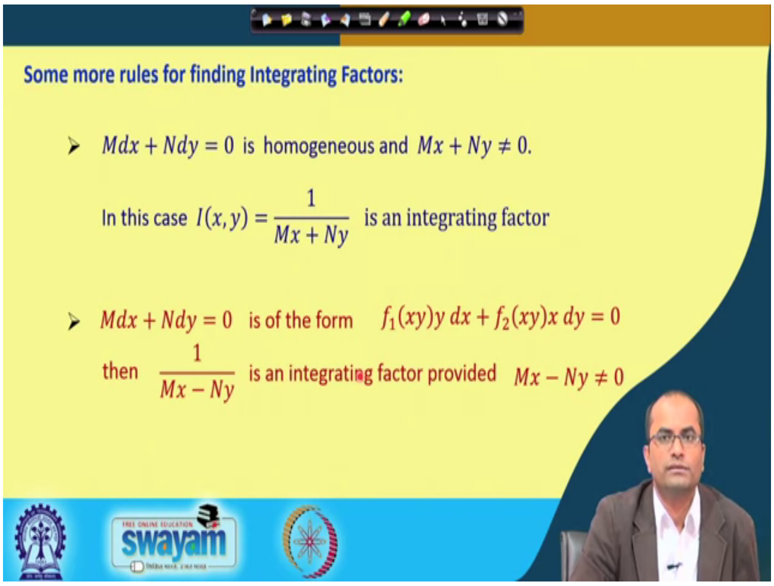

There are some more findings from that that those that condition where we can find the integrating factor. So, we are not going into the details of finding them, but for instance this is one of the possibilities that if this equation is homogeneous and this M x plus N y is not equal to 0, then this comes to be an integrating factor. So, again directly one can find out this I x, y as 1 over M x plus N y this will be the integrating factor of this differential equation. But, here the condition is this should be homogeneous differential equation and we already know how to solve the homogeneous differential equation. So, we may not follow exactly this approach here finding this integrating factor and then solving the exact equation.

Another possibility here is M dx plus N dy equal to 0, is of this form. If this equation is given off this form means the sum function xy and y dx; another function x dy, then this 1 over Mx minus Ny if this term is not equal to 0 will be an integrating factor and if we multiply this to this equation this equation will become exact. So, there are many more such rules can be derived from there, but what we have discussed a very general approach and from there also we have at least seen how to get the integrating factor in those special cases.

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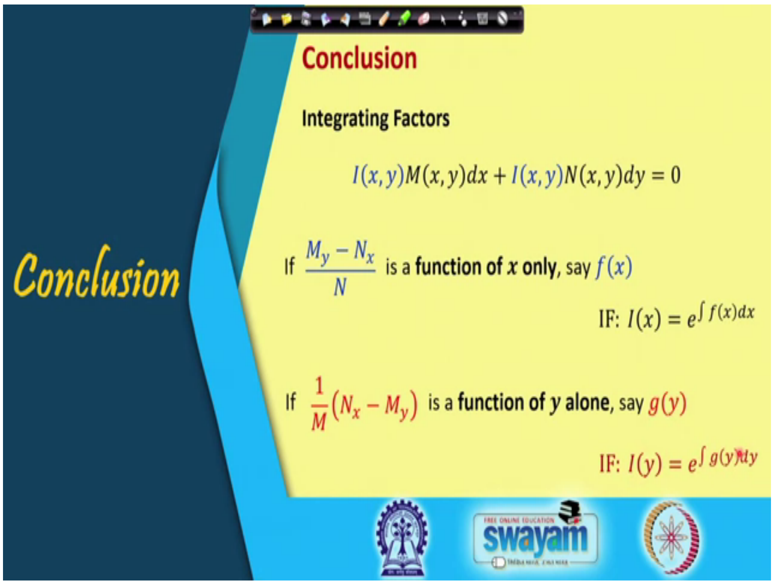

So, coming to the conclusion the integrating factor is nothing, but a function here. If you multiply to the given differential equation which is not exact, but after multiplication the equation becomes exact we call such a function as the integrating factor and we have seen this special case when M y the partial derivative of M with respect to y, partial derivative of N with respect to x. So, this difference divided by this N is a function of x alone, then we have derived that this I x is equal to e power this integral f x dx will be the integrating factor and this is a function of y alone say this g y then this will be the integrating factor I y is equal to exponential integral g y dy.

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These are the references used here. And.

Thank you for your attention.