Engineering Mathematics – I
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Lecture - 52
First Order Differential Equations

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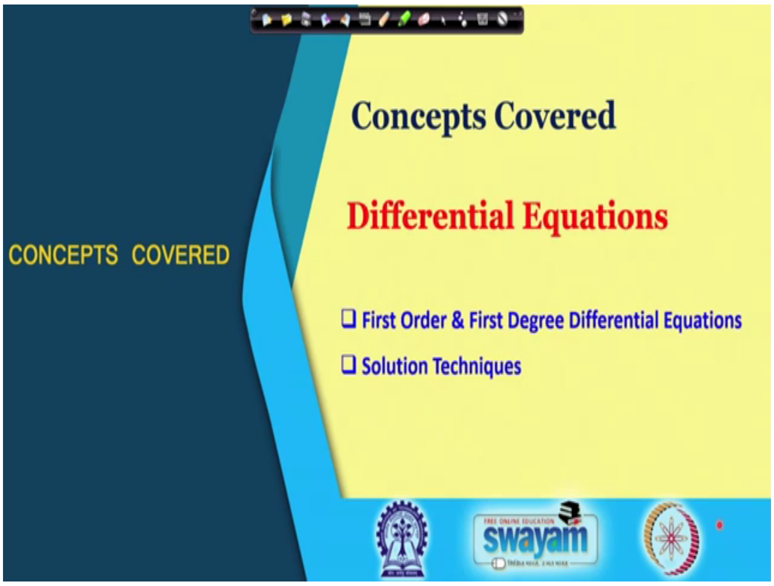

So, welcome back and this is lecture number 52. Today will be talking about this First Order Differential Equations, and mainly we will consider first order and the first degree differential equations and their solution techniques.

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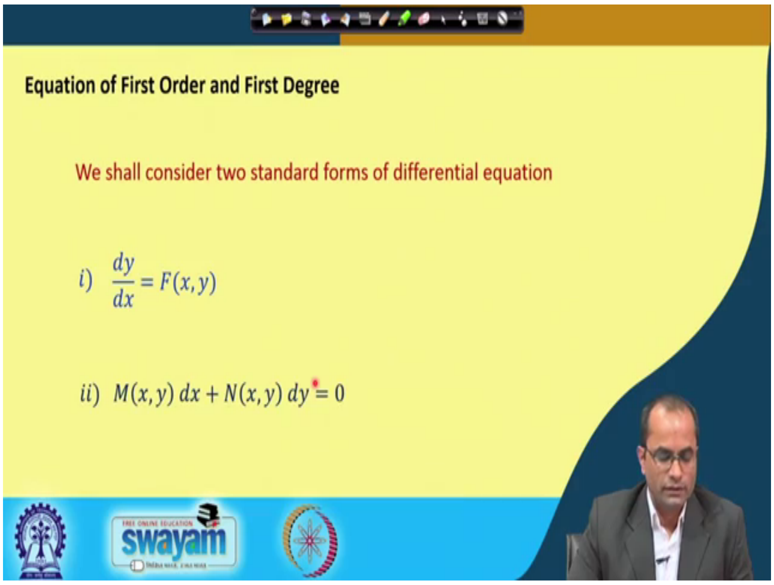

So, today we will focus on these two standard forms; the one we will consider dy over dx is equal to F x y. So, the function of x y right-hand side and the other form we will be discussing in this lecture that will be M x y dx plus N x y dy is equal to 0. So, these are the two types of first order differential equations we will be covering in this and the next few lectures.

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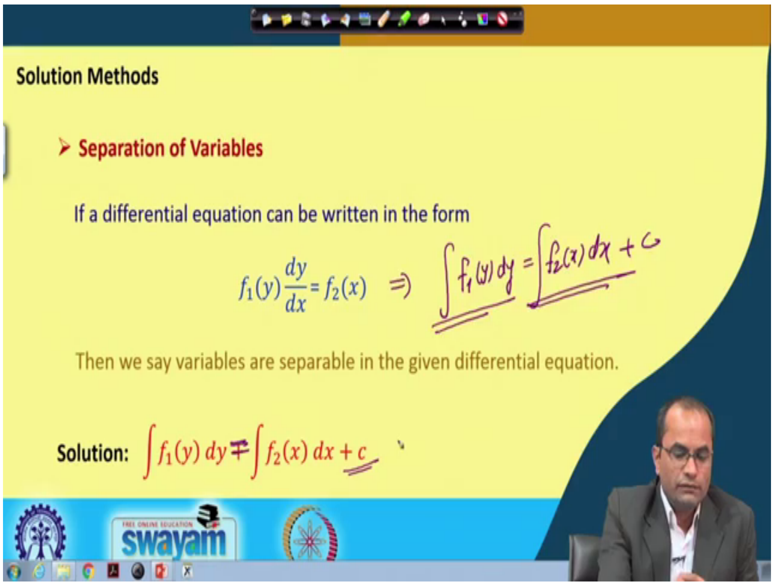

So, the solution techniques. So, what are the solution methods? The first one is the separation of variables. So, we will quickly review some of the techniques which are very fundamental of basics and they are required at a every stage for solving the differential equations. So, the one of them is the separation of variable.

So, here if a differential equation can be written in this form. So, we have f 1 y and dy over dx is equal to f 2 x. So, the point here is that everything the function of y if we can collate to one side and the other side the function of x, in that case we call that this equation is variable separable because now we can easily integrate this because the left-hand side only the function of y and the right-hand side everything of function x.

So, if we can put in such a form then we can easily integrate these equations and then we say that the variable variables are separable in the given differential equation, and the solution for such differential equation once we separate the variables then it can be written as we can simply integrate this here or we can first write down this equation as this f 1 y and dy is equal to this f 2 x and dx and then we can simply integrate this equation with this constant of integration. So, this is equal here. So, f 1 y dy is equal to this f 2 x dx and plus this constant of integration.

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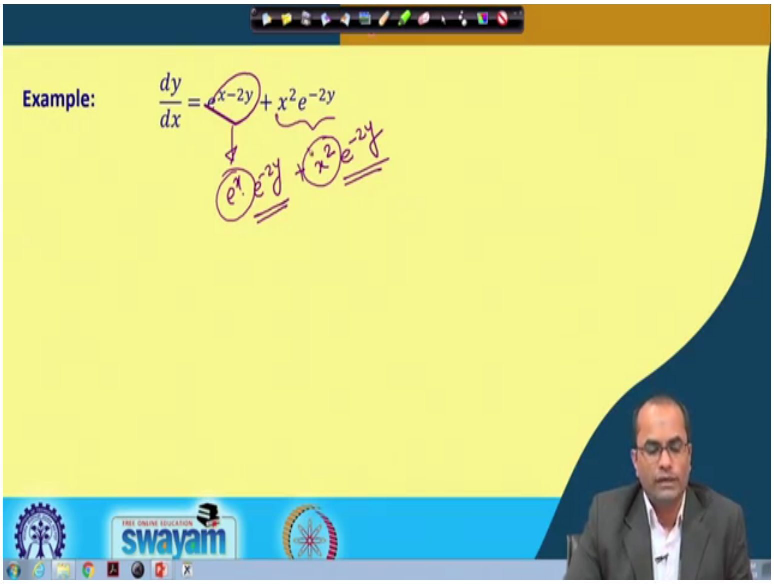

So, see one example of this kind which is given here that dy over dx is equal to e power x minus 2y plus x square and e power minus 2y. So, if we take a close look here at the right-hand side of this function what we observe that e power x because this we can write down this first here as e power x in to e power minus 2y and this term we have already in the separable form, so e power minus this 2y. And what we do now, e power minus 2y we can take as a common and that can go to the other side of the equation and this one side it will remain as e power x plus x square. So, this equation is variable separable which we can easily do and then we can integrate it.

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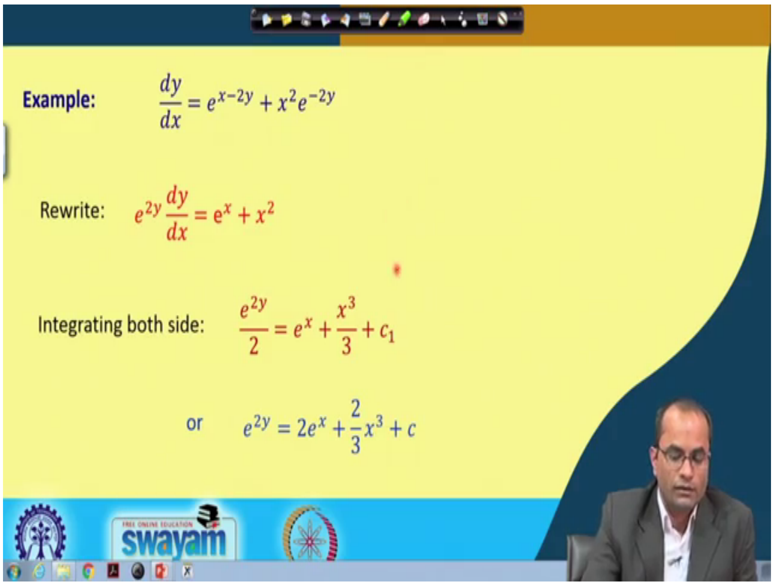

So, here we can rewrite this. So, we multiply it basically e power 2y to this equation and automatically the right-hand side becomes now e power x plus x square, so free from y. So, we have this side everything the function of y and the right-hand side we have the function of x.

So, now, we can easily integrate this equation and integrating we will get because e power 2y, when we integrate we will get e power 2y and divided by 2. The right-hand side here we have e power x which you will give e power x the exponential function will remain as it is and here x square will become x cube by 3, when we integrate this x square and one constant of this integration which we call here c 1.

What we can do here we can multiply by 2 for instance just to again rewrite the solution. So, we have here e power 2y is equal to this 2 will go to the right-hand side, so 2 e power x plus this 2 by 3 and this x cube and plus 2 times c 1, which we have denoted again by a constant here c. So, this is the solution which is given in this implicit form. So, this is the implicit solution of the given differential equation by separating this variable. So, it was easy to integrate and find the solution. So, this is one of the very basic techniques which we use for solving differential equations.

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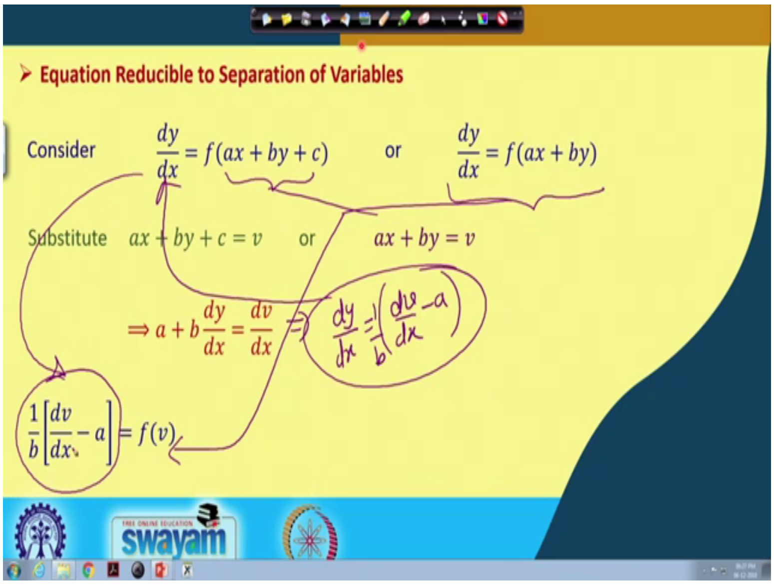

The other one there are equations which can be reduced to the separable of variables. So, as such in the given in the given equation may not be separated with respect to these variables, but it can be made by some substitution to separable form and then we can again repeat the process which we have done for the separable equations.

So, for instance if we consider this dy over dx is equal to f ax plus by plus c. So, if we have here the function of this ax plus by plus c or the form is just ax plus by, so as such this given equation may not be in the separable form, but if we make a substitution here. So, in this case we say it is a we will substitute this ax plus by plus c and we will give a new variable to this expression or we can substitute ax plus by when the equation is given in this form.

So, in either case when we substitute let us say here ax plus by is equal to plus c is equal to just new variable v or in this case ax plus by is equal to v. What will happen? So, if we differentiate now this to get this relation dy over dx. So, what will happen? a times this x which will give just a when we differentiate here b dy over dx and this is a constant, so 0 is equal to dv over dx. And the same expression we will get when we differentiate this ax plus by is equal to v. So, does not matter now, whether we are substituting this one when such a differential equation is given or we are substituting here v when this equation is given. So, in either case we will get this relation that this new variable v and the derivative with respect to x will be a plus b in dy over dx form.

Once we have this we can substitute now back to the equation. So, that we get the new differential equation in v. So, v is now the dependent variable like y was earlier here. And now if you substitute for dy over dx. So, from here dy over dx what we get? The dy over dx will be dv over dx and this minus this a to the right-hand side and then we need to divide also by this b. So, this dy over dx will be substituted here in the equation and then we will get this term now here for dy over dx, and then is equal to f and this we have already substituted as v whether this equation or this equation we will get from both of them exactly this equation which is given here one over b dv over dx minus a is equal to f the function of v.

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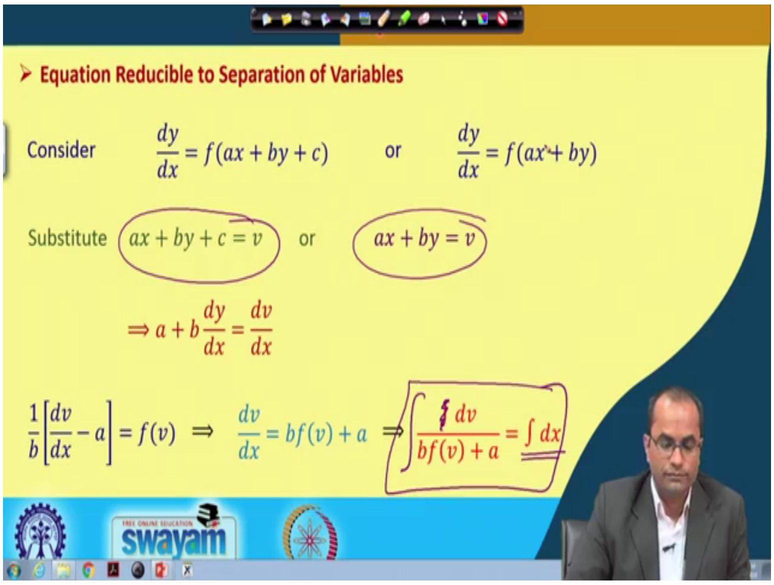

So, having this equation now we can just rewrite this. So, dv over dx will be equal to, so we multiply by b first, so that will be b F v and then this a will also go to the right-hand side. So, we have dv over dx is equal to b times this function of v plus this a. And now this equation is in separable form because this right-hand side is having only v, there is no x here anymore, so we can take everything to the left-hand side and then dx this differential of x can go to the right-hand side. So, we have this separable form which we can integrate as this integral of this one over b v plus a and this with respect to v and then the right-hand side will be the integral of x.

So, we have this the solution after getting the solution in terms of v we can again substitute whether this v is equal to ax plus by plus c depending on the situation of the giving differential equation. So, we can write down final solution again in terms of y.

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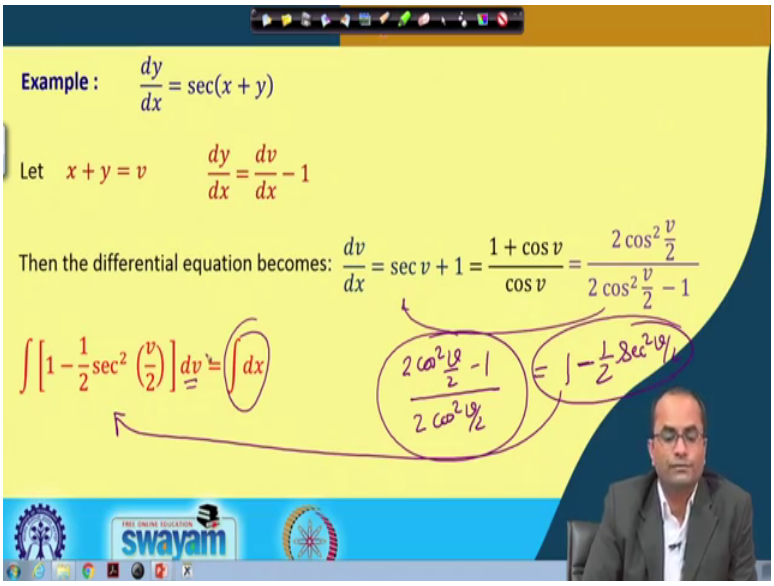

Now, we take this example of this kind. So, dy over dx is equal to this sec x plus y. So, this equation as such given in this form is not separable because we cannot bring all this x to one side and y to other side. So, we have to use the idea which is described just before. That means, this x plus y we will set as y v and having; so here we can now get this dy over dx. So, this is 1 plus dy over dx is equal to dv over dx and then this dy over dx will be dv over dx minus 1. So, we have this change enough from y to v which we can substitute in the given equation. So, the given equation will be considered now as dv over dx minus 1, dv over dx this minus 1 is equal to the sec of this v.

Now, this equation is in separable form and we can integrate this easily. So, before we do. So, let us just simplify this term. So, this is sec v means 1 over cos v, so we can write 1 plus cos v over cos v. And then we can also simplify little most in terms of this v by 2 we can write down as the 2 times cos v means 2 cos square v by 2 that is for 1 plus cos v and here cos v will be 2 cos square v by 2 minus 1.

We have done this because now the integration will be easier. So, all this terms will go to the left-hand side and the dx this differential will go to the right-hand side and then we will integrate it. So, what I will getting the left-hand side when we, so this will be reverse the order then we have 2 cos square v minus 1 and divided by 2 cos square v by 2 and that will be, so let us just check here. So, this when go to the left-hand side then this will become 2 cos square v by 2 minus 1 over this 2 cos square v by 2. And that is exactly what we have there. So, 1 because 2 cos square v by 2 divided by 2 cos square v by 2 that will be 1 and minus this 1 by 2 and 1 over cos square v by 2 will be sec square v by 2. So, this is now the integrant here 1 minus half sec square v by 2 and this we will integrate over this v, in the right-hand side we have integral of this dx.

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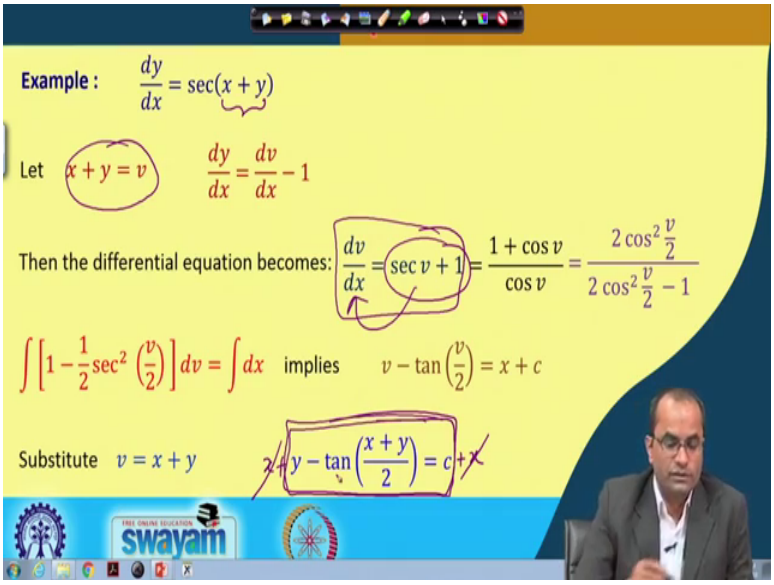

So, now this we can integrate. So, this v, 1 when we integrate over v this will give us v and then here the sec square v by 2, so this half is already here. So, the sec square this v which will give us tan of this v by 2. So, we have the integral of the left side v minus for this sec square we have tan v by 2 and the right-hand side it just the x plus this constant of integration.

And now we need to substitute back this v in terms of the x plus y. So, that will be now v is from here x plus y. So, we can write down the x plus y here minus this tan v is again x plus y by 2 is equal to x plus c, so this x and the left-hand side also we have this x plus y. So, when we substitute for v this was x plus y and the right-hand side was also x. So, this x gets cancelled and then we have this as the solution that y minus this tan x plus y by 2 is equal to this constant c.

So, here the given equation again was not in separable form, but by doing this substitution this here x plus y is equal to v this one, then the new equation which is written now in v here dv over dx is equal to sec v plus 1 this is in separable form and therefore, we could integrate this easily and finally, we got this again implicit solution. So, y minus tan x plus y by 2 is equal to arbitrary constant.

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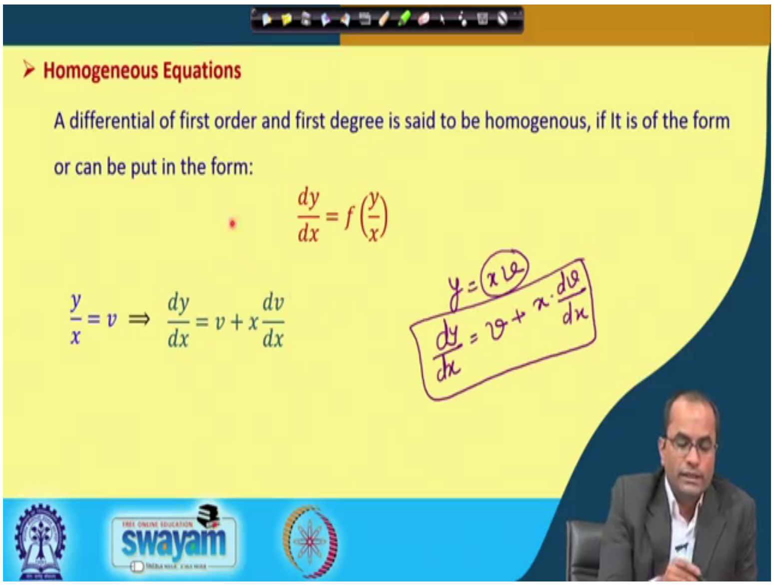

So, another type of equations we will consider now these are called the homogeneous equations. So, these homogeneous equations differential equation of first order and first degree is said to be homogeneous, if it is of the form or can be put in the form here of this. So, dy over dx is equal to this function of y over x. So, if we can bring every the function here right-hand side in the form of dy over x as a function of y over x then we call this differential equation homogeneous differential equation.

And now we have the solution technique which we can use for solving this such a homogeneous equation. And the trick here is that we take this y over x as a new variable, so here we have taken this v and then we can now get this dy over dx from this term, so having this relation we have basically y is equal to x v. And then dy over dx then we differentiate here, so the product rule will be applicable and we have v here the derivative of x and plus this x and the derivative of v. So, this is what we will get out of this relation.

So, dy over dx is equal to v plus x dv over dx. So, having this relation now we can convert our equation which was given in this form dy over dx is equal to function of y over x. So, this will become now dy over dx will be replaced by v plus x dv over dx.

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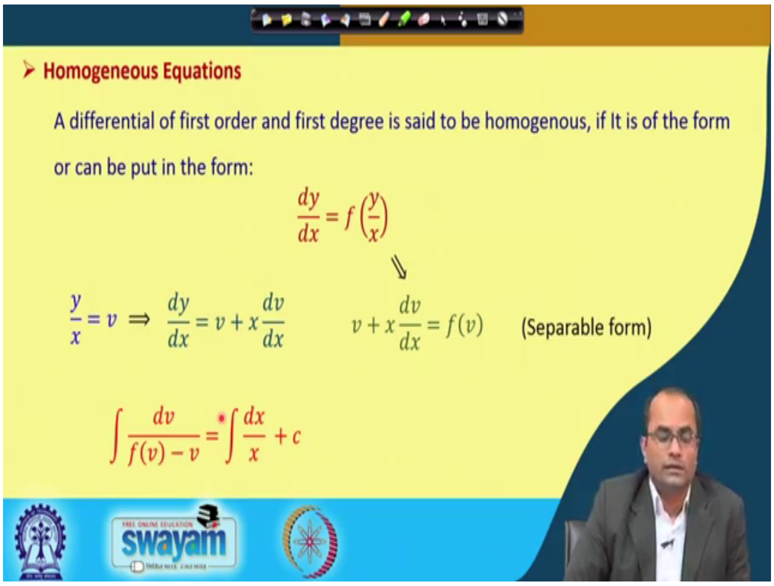

Here v plus x dv over dx is equal to the function of v because y over x we have assumed as v here. So, this is new equation which is again in separable form because this v we can take to the right-hand side, so we have f v minus v. So, the right-hand side contains everything with v and the left-hand side with this x. So, here this is also a separable equation now in v and now we can write down the solution, so dv and the right-hand side when we take this v there this will be now, f v minus y is equal to dx over x plus c. So, that is the separable form which we can integrate and find out the solution. After that we can replace this v by y by x to get the solution in terms of y and x.

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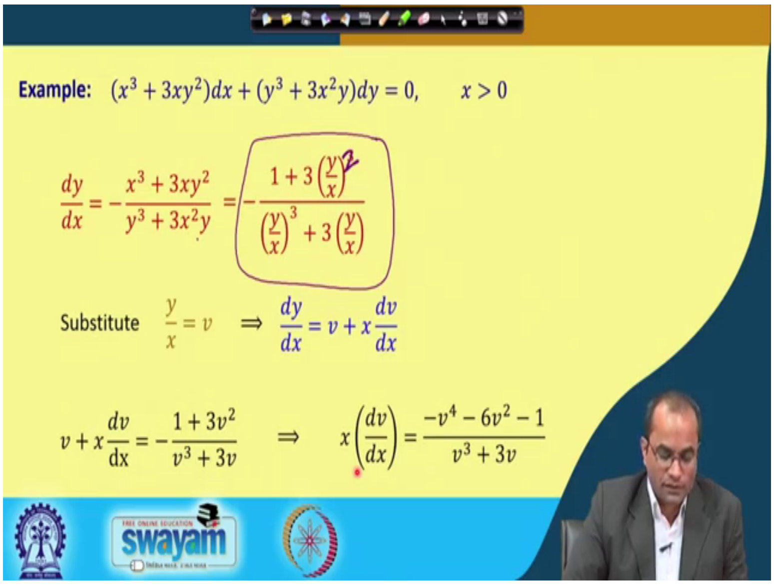

So, we go through this example which is of this homogeneous differential equation. So, if we rewrite this equation this dy over dx form by dividing this differential here. So, dy over dx will be minus because this term will go to the right-hand side so with minus sign and this x cube plus 3 x y square that is a numerator term here and divided by this y cube plus 3 x square y. So, here now we can realize that we can put this in to this y over x form because all these degree here is 3 here also its adding to 3, 2 plus 1, here 3 and here 3.

So, once we have this equation we can easily rewrite in this form because we can divide by this x cube here and also x cube in the denominator. So, by doing this we have 1 plus 3 y over x cube because we have divided this x cube. So, this x will get cancelled we have this square here. So, this will be indeed square term not the cube term. So, we have here the square y over x square and then here y over x this cube and 3 when we divide x cube here, so y over x. So, now, everything here is in terms of this y over x as a function of y over x. So, this equation is in separable form and we can now substitute this y over x as v which we have explained this before and as a result here this dy over dx will be v plus x dv over dx. So, we can substitute now in our equation here. So, v plus this x dv over dx and then we have with this minus sign and then 1 plus 3 this y over x is v, so v square and divided by this v cube and plus this 3 v.

So, now, this equation we can take this v to the right-hand side and the equation is separable in terms of this v and x. So, while taking this to the right-hand side we will get now this minus v 4 and minus 6 v square minus 1 and divided by v square plus 3 v, when we subtract this v from this right-hand side. So, now, this equation is in separable form we can take this term to the left-hand side and that the right-hand side will go this dx over x and then we can integrate easily.

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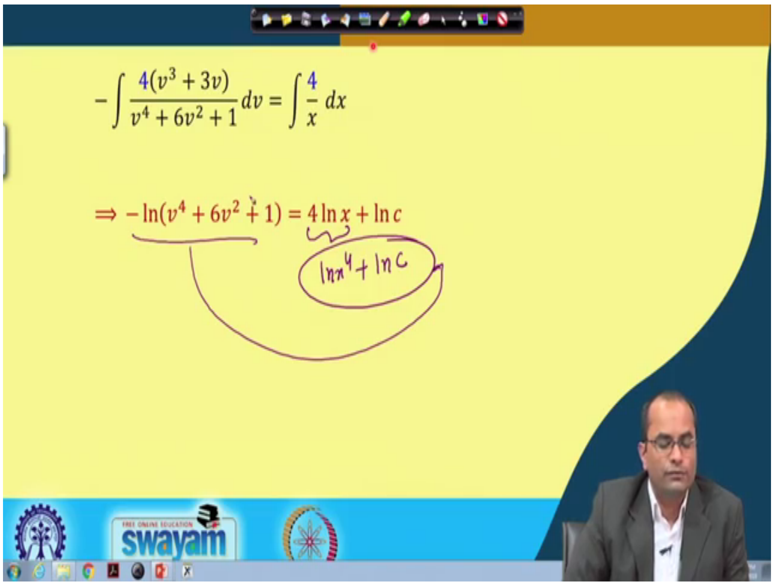

So, we have done, so 1 over x dx and this goes in another other round here. So, v cube plus the 3 v over this v 4 plus v square plus 1 and we have multiplied both the side with 4 because the differentiation of this when we get this v 4 plus 6 v square, so that will be coming 4 times v 3 and then 12 times v, so again here 12. So, exactly this is the differential of this sitting there. So, we can easily integrate now and here we have 4 over x, so right-hand side also easy to integrate.

So, now, we can integrate. So, this will imply minus. So, again the logarithmic because this derivative of this is exactly this sitting in the numerator. So, we have the logarithmic of this one v 4 plus 6 v square plus 1 and is equal to the 4 times and again the logarithmic of x we have assumed x to be positive. So, plus this constant of integration which we have taken in that logarithmic form. So, any arbitrary constant here, so, logarithmic of c. And now we can combine these two. So, this we can write as this ln x 4 and then here we have this ln c. So, this is basically cx 4 and this also we can take to the right-hand side, so we have everything this log x 4 plus log c and plus this log v 4 plus 6 v square plus 1. So, we can combine all these terms of all these slopes. And what we will get here?

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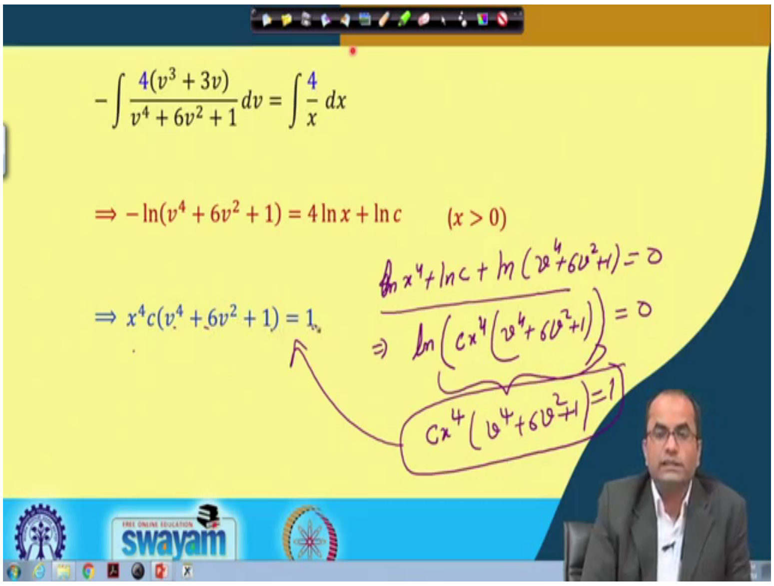

So, just one step before here what we will get the the ln here x 4 plus this ln c and also, we will get this ln v 4 plus 6 v square plus 1 is equal to 0. So, here if we combine this everything now, so this will be ln and the cx 4 and also this v 4 plus 6 v square plus 1 and the right-hand side 0 and then taking the exponential both the sides we will get v 4 plus this 6 v square plus 1 and is equal to the exponential 0 that is one here. So, we got exactly this one here cx 4 and v 4 plus 6 v square plus 1 is equal to 1.

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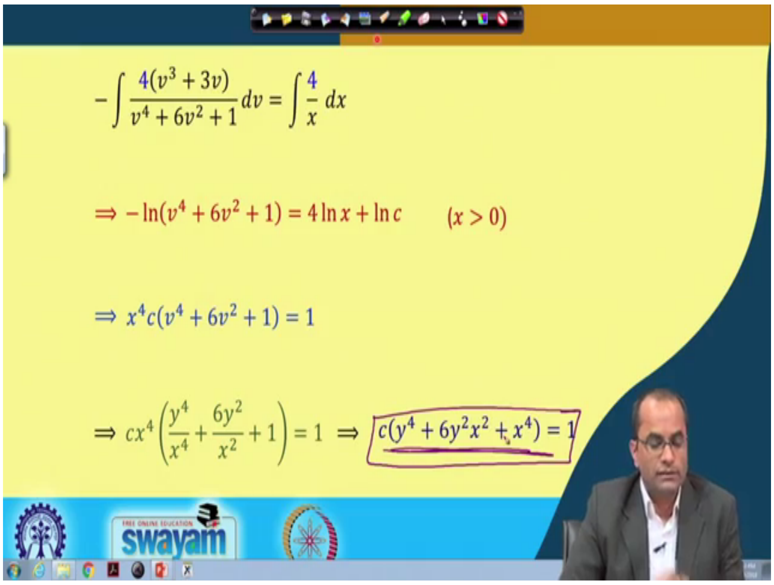

So, now this one, so we need to substitute back this v which was y over x. So, here we have cx 4 and this v is y over x. So, we have here y 4 over x 4, here also we have 6 y square over x square plus 1 and is equal to 1. And now this x 4 also we can bring to the inside this bracket. So, what we will get? c times this y 4 plus this 6 x square y square plus this x 4 and is equal to 1, and that is the solution again given in implicit form. So, we have y and x given in implicit form as the solution of the given differential equation.

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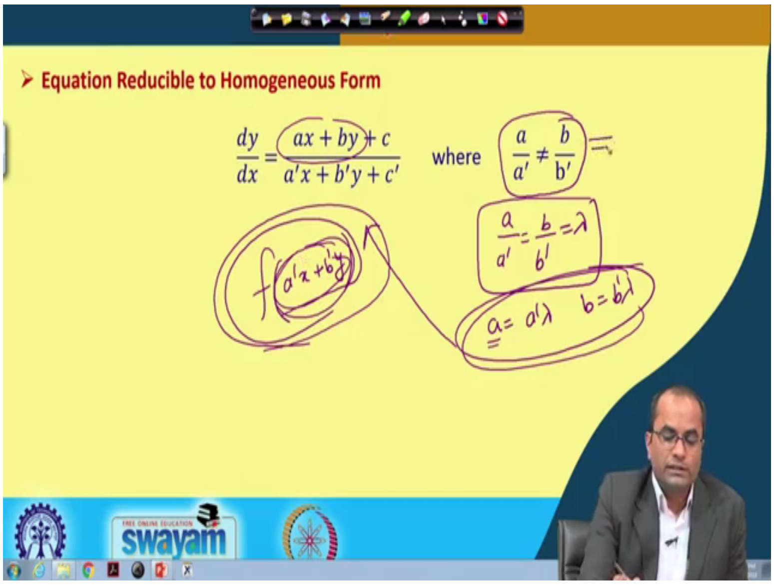

Now, the last one for this lecture we have the equations which can be reduced to this homogeneous form. So, again the given equation may not be in the homogeneous form directly, but by doing some substitution we can reduce it to the homogeneous form and this is exactly the equation we are talking about the dy over dx is equal to ax plus by plus c and divided by this a prime here. So, it is just the new name of this constant. It is not the derivative naturally. It is just the another constant a prime here b prime and another constant plus this c prime the another name for the constants.

So, when the equation is given in this form and also together with this condition that this a over this a prime is not equal to b over b prime because if this is the case here if they are equal this ratio is equal that b over b prime and is equal to then we can assume let us say the lambda and then we have the a is equal to the a prime lambda and also the b is equal to this b prime lambda. Then what will happen? When we have this relation and when we substitute this one then this will become as a function of, so either we replace this a or we replace a prime. So, what we will get here in this case when we replace a. So, this everything will become as function of this a prime x plus b prime y and then we can solve we do not have to exactly do any further substitution by substituting this to the new variable we can solve this equation. But here the condition is given that this a over a prime is not equal to this b over b prime.

So, in that case otherwise this is just the by substituting we can um reduce to the separable form and then we can solve as we have done before for the equations reducible to the separable form. But here this condition is given; that means, we have to do little more here.

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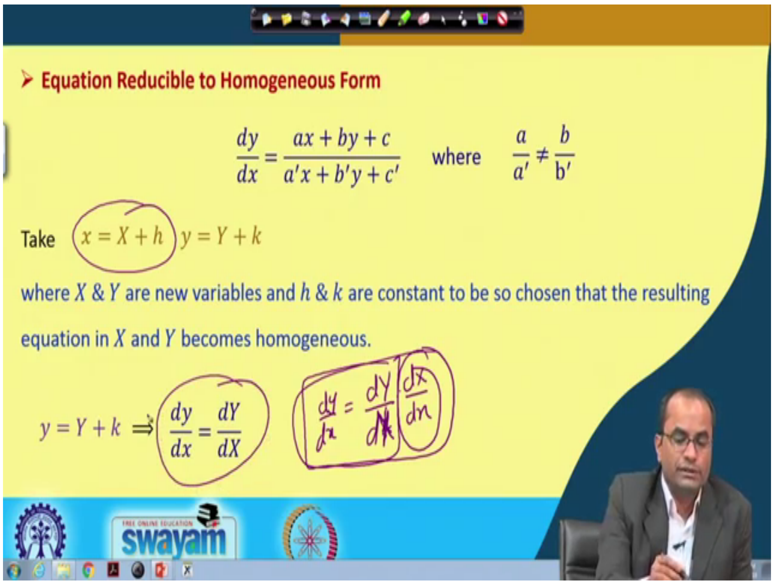

So, what we do? We take here the substitution that x is equal to X plus h and we take y is equal to Y plus k as the new variable here big X and this big Y. So, now, our independent variables we are changing here the x is X plus h and this y the dependent variable also as Y plus k.

Now, by having this substitutions now, this h and k are the constants and we have to choose these constants such that this equation become homogeneous because at present this equation is not homogeneous because of these constant terms here. So, these equation we will choose now this h and k so that this equation becomes homogeneous. Now, if we substitute this first here let us see what will happen. So, we have this relation y is equal to Y plus k from there we can get this dy over dx, we can easily get this dy over dx is equal to d big Y over big Y as a function of x and then here big X and then big X over the function of x. So, we can do this and dx over x from this will be one. So, this relation dy over dx is equal to dY over this big X remain as we have written here.

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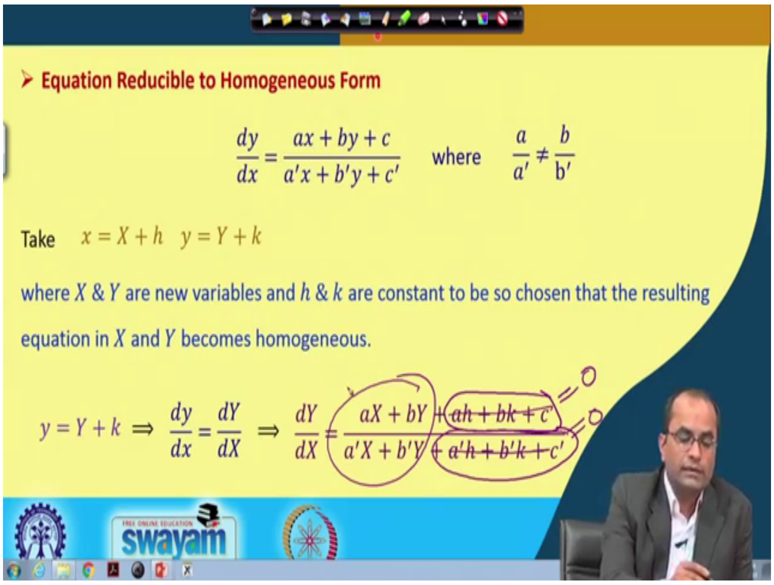

So, now by this here dy over dx is equal to dY over d big X and we can now substitute in this equation, so this dy over dx will be replaced by this d big Y over d big X. And then again, this x and y we have to substitute here also x and y. So, the new expression here will be again in terms of x and y, but then we have also this here a h plus b k plus c and here a prime h plus b prime k plus c prime.

And now the idea is that we will set this here we will set because h and k we need to choose. So, we will choose this so that this becomes 0 and this also becomes 0. So, once our h and k we have chosen, so that these two expressions vanishes here our equation will become as the homogeneous equation and that is the that is the aim here.

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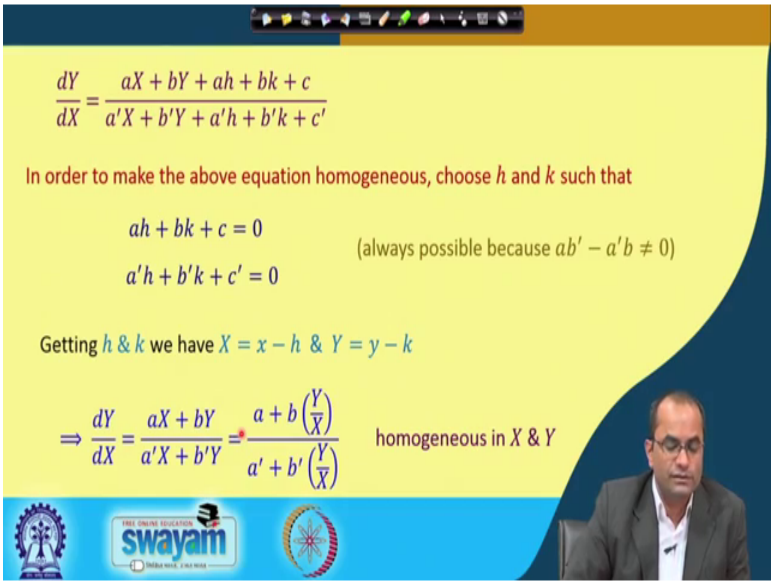

So, by setting these two terms equal to 0 what will happen now? a h plus b k plus c is equal to 0 and also this a prime h plus b prime k plus c prime is equal to 0 which is always possible to so to find such h and k because these equations are always solvable if we look at the at the determinant of this a and this coefficient matrix here a b and a prime b prime for this h and k are unknowns. So, this is the determinant which is non-zero that condition is given already in the equation. That means, that we can always solve this equation, these are solvable equation and will get the unique solution indeed.

So, here we will get now the values of h and k such that these terms will vanish and then we have these relations, already which we have substituted. So, our equation will convert now dY over dX in to aX plus bY, a prime X plus by b prime Y and this equation is homogeneous equation which we can write in this Y over X form, homogeneous in X and Y which we can solve using the technique which we have discussed earlier.

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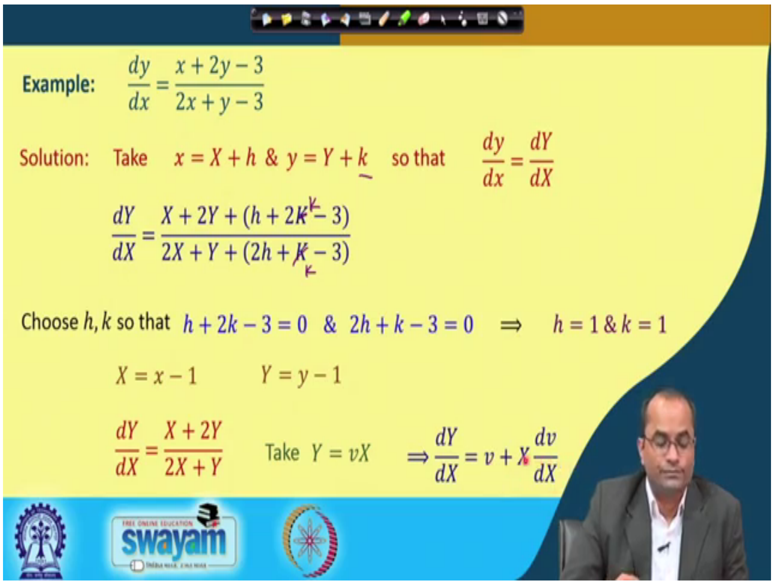

So, let us do this one example in the last one for today's lecture. So, here we assumed this x is equal to X plus h and y is equal to Y plus k and substitute in the equation as discussed before. So, then we have this h plus 2k minus 3 and 2h plus k minus 3. So, here this k is indeed a small k as given here. So, now, we will set this to 0 and also this to 0 and solve for h and k. So, by doing so, we have this h plus 2k minus 3 is equal to 0 and this 2h plus k minus 3 is equal to 0, from here we will get h is equal to 1 and k is equal to 1. So, this satisfy this equation also h is equal to 1, k is equal to 1 satisfies the other equation.

That means, our substitution is this big X is equal to x minus 1 and big Y is equal to y minus 1 and now we can substitute in this equation naturally this term is 0, this term is 0 and we have the homogeneous equation which we know how to solve this equation. So, we need to again substitute this Y is equal to vX as explained before for the homogeneous equation and then that will give us this dY over dX in terms of this big Y and big X, v plus X dv over dX.

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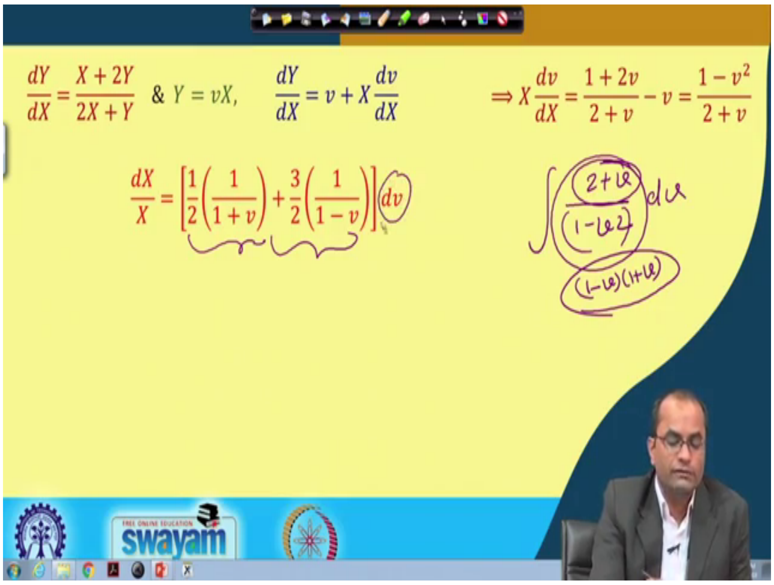

And now we have already this that dy over dx is this was the equation the homogeneous equation we have substituted this which leads to this dY over dX is equal to v plus X dv over dX.

So, we substitute now back to the equation here and we will get this equation which is separable equation. And now we can bring this big X to the right-hand side and these v to this left-hand side; that means, we have dX over X. And this 2 plus v over 1 minus v square here, so this term here two 2 v over 1 minus v square which will come with this dv; this with the partial fractions because this 1 minus v square we can write down 1 minus v and 1 plus v and then we can do this partial fractions easily. So, that is the result here of the partial fractions and then we will integrate with respect to v.

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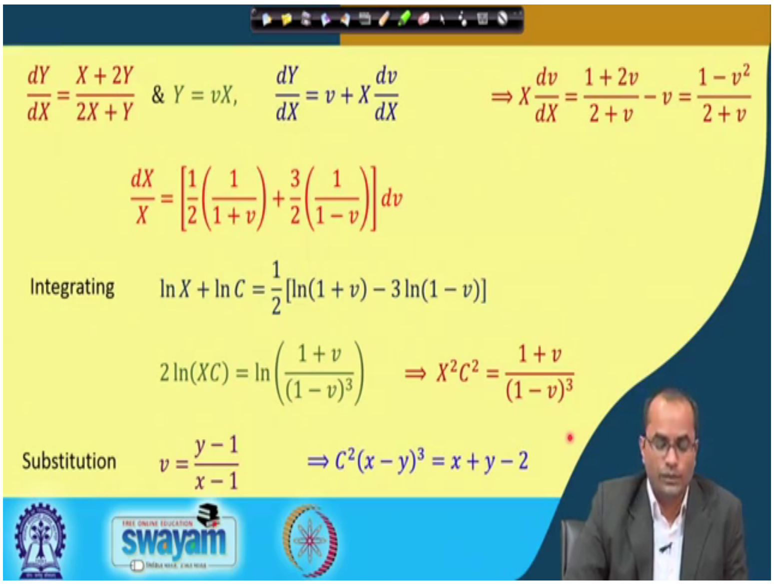

So, here by integrating this we get ln x and this constant here ln c and is equal to this half again here ln 1 plus v minus 3 times this half is outside, so this is just minus 3 here because of this minus sign and ln one minus v assuming that these are the constants, otherwise we have to take the these are the positive numbers otherwise we have to take the modulus as well.

So, then we have here 2 times, this 2 will go to their and this ln cx is equal to this ln 1 pus v over this 1 minus v cube, and from here we will get this x square c square is equal to 1 plus v over 1 minus v whole cube. And then we need to substitute this v as y minus 1 over x minus 1 because this h and k were this was k and this was h which was 1 there. So, v in terms of this y and x that was given here as big Y over big X and then big Y is y minus 1 and big X is x minus 1. So, finally, this v we have to substitute here to get the relation of x and y. So, by substituting this we get this relation c square x minus y cube is equal to x plus y minus 2 and that is the solution written in x implicit form of a given differential equation.

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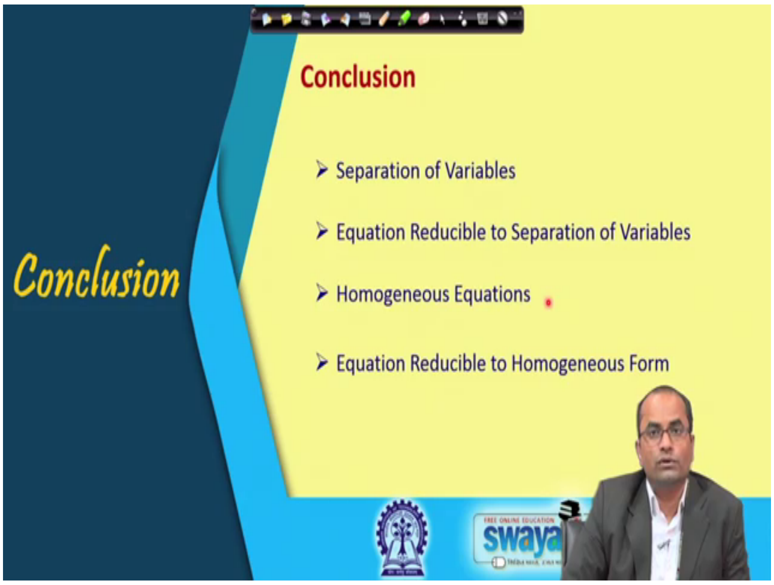

Well, so we come to the conclusion now. So, this is what we have learnt today, the separable of variables. So, it was kind of review because many of you are already aware with all these techniques. And we have also discussed about the equation reducible to the separable of variable form again, we have also discussed the homogeneous equation and the equation reducible to the homogeneous form.

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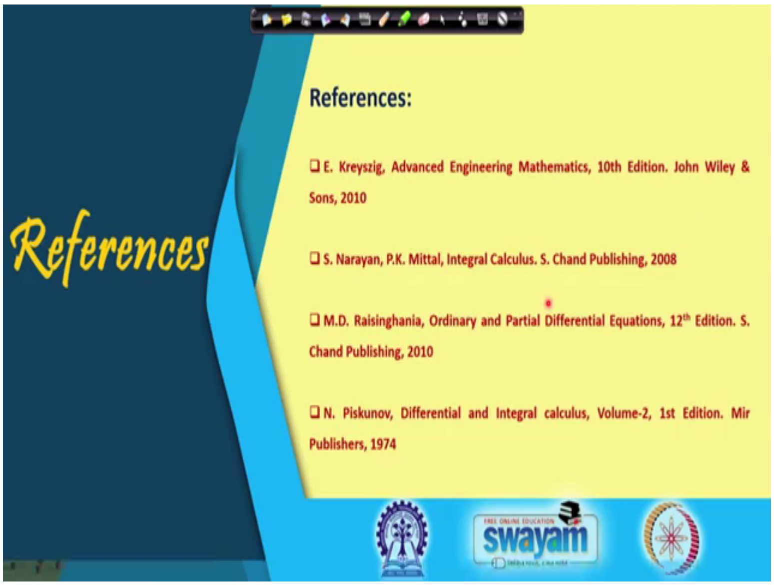

These are the references used, and thank you for your attention.