Engineering Mathematics - I
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Lecture - 50
Eigenvalues & Eigenvectors: Diagonalization

So, welcome back and today again we will continue our discussion on these eigenvalues and eigenvectors. This is lecture number 50 and we will mainly focus on iagonalization and also it is applications for solving system of linear equations, also for getting the power of the matrices etcetera. So, what is the diagonalization of the matrix? We will discuss here now.

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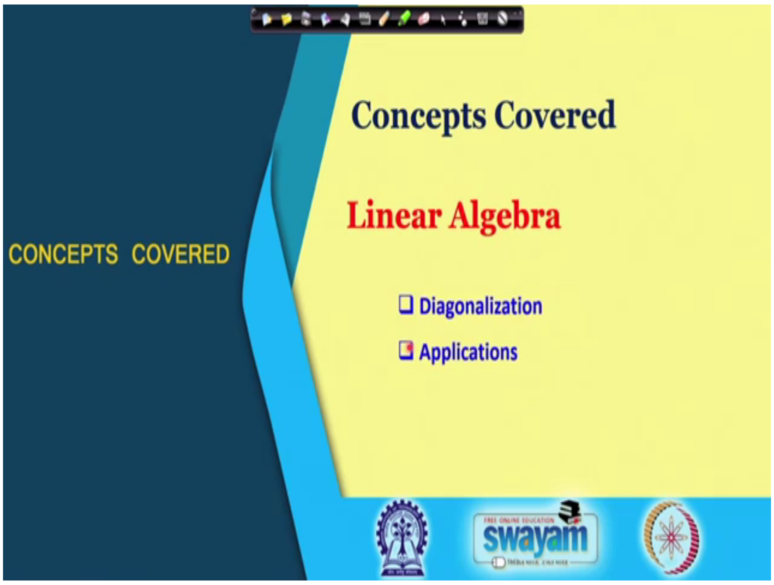

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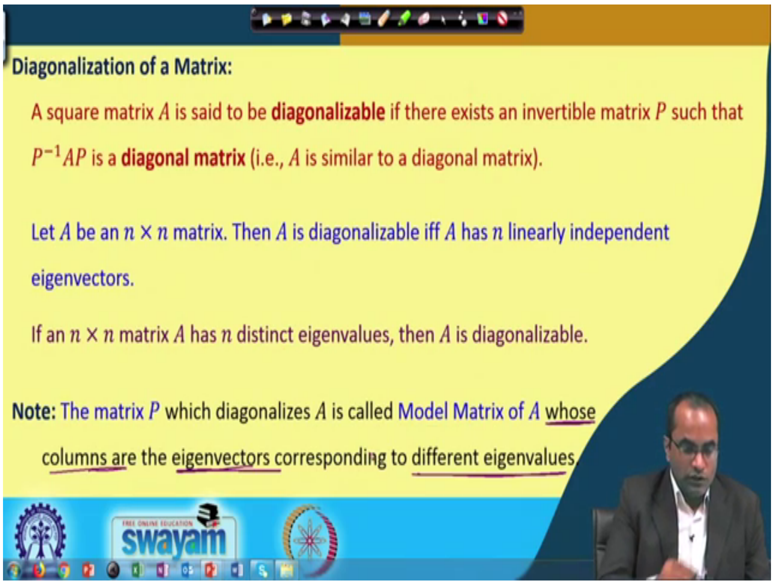

So, A square matrix A is said to be diagonalizable if there exists an invertible matrix P. If there exists an invertible matrix P such that this P inverse AP is a diagonal matrix or in other words, because this concept we have already introduced the similarity of the matrices.

So, in other words if A is similar to a diagonal matrix, because if the point is here A is called diagonalizable. If we can write down this A as this P inverse the P inverse AP is a diagonal matrix then we call that this A is diagonalizable. So, the meaning this A is similar to the diagonal matrix here; AP inverse AP and this P is invertible matrix which we have to find, so that this P inverse AP becomes a diagonal matrix.

So, let A be an n cross n matrix and that is a nice result that A is always diagonalizable; that means, we can find such A P such that this P inverse AP is a diagonal matrix. So, the result is here A is diagonalizable, if A has n linearly independent eigenvector. So, that is a nice result here, nothing to do with this eigenvalues. Actually we have to look for the eigenvector. So, if we get n linearly independent eigenvectors then A can be diagonalized. And if we cannot get these n linearly independent eigenvectors then the A is not diagonalizable. So, that is the main result of this lecture.

Again we will not go through the formal proof here, but we will see with the help of many examples, how this is working. And another subsequence of this result we can we can have here if n is an n cross n matrix here A and it has n distinct eigenvalues, then also A is diagonalizable and then reason is clear, because if we have n distinct eigenvalues, then we will also get n linearly independent eigenvectors; that is a result we have already seen in previous lecture that corresponding to distinct eigenvalues we have a linearly independent eigenvectors. So, eventually this second result here is again the same as this previous one; that is diagonalizable, if and only if it has n linearly independent vectors.

So, just a note here that this matrix P which diagonalizes A is called the model matrix of A and whose columns, I mean the point is how to find this A matrix P. So, here this model matrix P has the columns that are nothing, but the eigenvectors corresponding to these different eigenvalues. So, if you have n linearly independent eigenvectors, we will just place them in this matrix P as the columns, and this is our matrix this P. And when we check this P inverse AP that will be nothing, but the diagonal matrix and entries of these diagonal matrix will be just the eigenvalues, corresponding to these eigenvectors we have placed in the sequence as columns of this matrix P.

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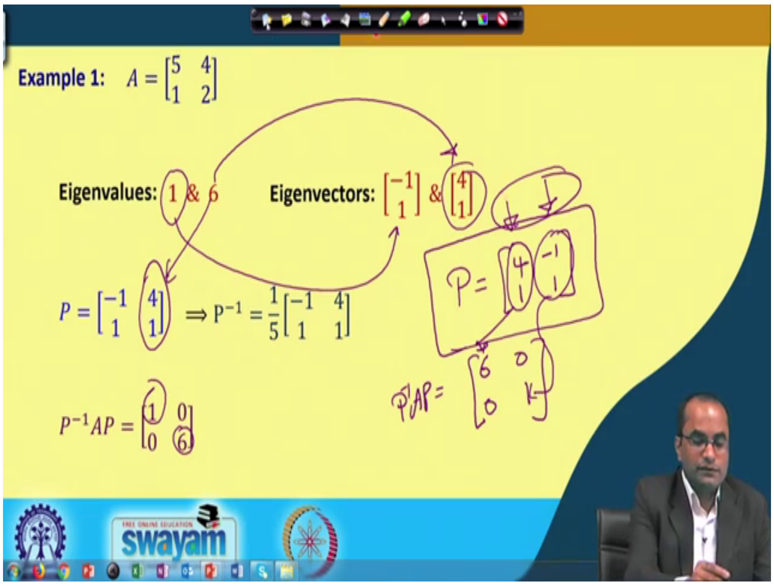

So now here let us consider this example with A 5 4 and 1 2. So, here we will not spend much of our time for computing eigenvalues and eigenvectors, because that we have already seen in previous lectures. So, we can compute the eigenvalues for this example and then it comes to be 1 and 6, because the idea is simple that this 5 minus lambda and then we have 4 and 1 2 minus lambda.

So, this determinant we have to solve which we can make this product here, the 10 and then we will have here minus 2 and minus 5. So, minus 7 lambda and plus this lambda square and minus 4 is equal to 0. So, this is lambda square minus this 7 lambda. And then we have here 6 is equal to 0 and that can factorize to this minus 6 minus 1. So, lambda minus 6 and lambda minus 1 is equal to 0. So, here we get these eigenvalues as 1 and 6.

So, having these eigenvalues now corresponding to each, we have to find the eigenvector. And here we have indeed these distinct eigenvalues. So, we will get two linearly independent eigenvectors and then we can diagonalize this matrix. So, here the eigenvectors corresponding to 1 will be coming as this minus 1 1. So, that is corresponding to this one, and corresponding to 6 we will get here 4 1. And now once we have the eigenvectors we can formulate this model matrix which we call P. So, the P will be we are placing these matrices are these vectors the eigenvectors as columns of this P.

So, here minus 1 1; that is the first column, and then this 4 1 that is the second column. So, our model matrix is ready now and we can verify that how this P inverse AP A P look like. So, here we have to get this P inverse also, that is the inverse. So, for 2 by 2 matrix it is simple, so we have to divide here by this determinant and then we need to change the sign and determined will be again with minus sign.

So, finally, we will get this as the as the P inverse of this matrix P which we can also verify by multiplying these two and we are getting the identity matrix. So, here the P inverse and if we compute this P inverse AP, so that is coming to be 1 6 in the diagonal and it is a diagonal matrix and this is exactly the point here. So, we have kept in our model matrix this, these vector minus 1 as the first column, and that was corresponding to the eigenvalue 1 and that is the reason here this first eigenvalue is coming and in the second case we have kept this second column which was corresponding to the 6 here and therefore, the second element in the diagonal is 6 here. So, this order if we change for instance the order here.

So, if we change it to P like 4 1 and minus 1 1. If we change, if we take this P, if we take this model matrix then here P inverse AP when we compute, this will be 6 0 and 0 1, because here this was corresponding to this 6 and then this is corresponding to this number 1 here. So, accordingly that order will change.

So, the order we place here for the eigenvectors corresponding order will be followed in the diagonal entries as the eigenvalues, so that is important. Second point here which also needs to be mention that this is not the unique eigenvector for instance; so, we can multiply by any number to this minus 1 1 that will be also the eigenvector. Again here also this 4 1 we can multiply by any scalar that will also be the eigenvector, because eigenvectors are not unique and by doing so, also there is no problem we can keep any vector here not only minus 1 and 1, we can also place for example, minus 2 and 2 here in the first column.

And in the second column we can place for instance we multiplied by 2 here, so 8 and 2. So, we can place 8 and 2 in the second column. So, does not matter that will be taken care by this P inverse and still this product will give us the same diagonal matrix 1 0 0 6. So, here it is material that whether we multiply here to these eigenvectors by some scalars; it does not matter with this P inverse AP will lead to the same eigen, same diagonal matrix whose entries will be 1 and 6. The only thing matter again it s the order here we place these eigenvectors.

So, the order we keep here placing as these columns of these eigenvectors in the same order these eigenvalues will appear.

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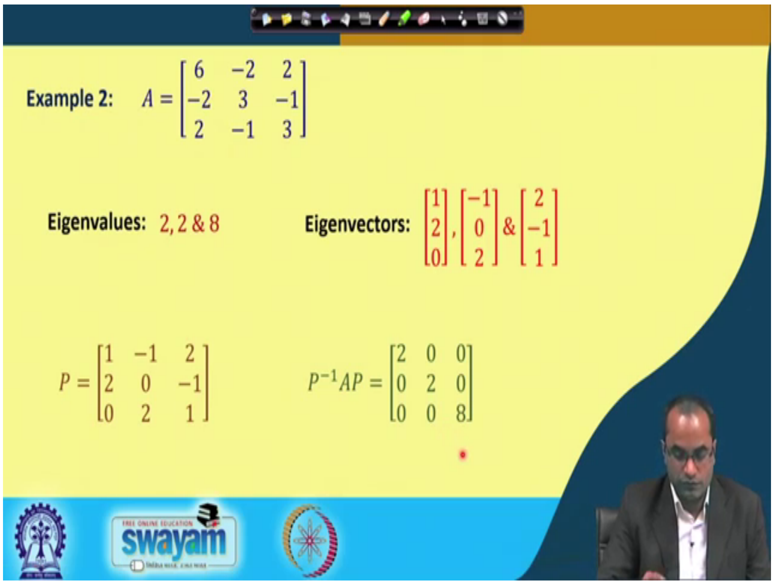

So, here another example of this 3 by 3 matrix 6 minus 2 2 and then we have minus 2 3 3 and minus 1 2 minus 1 3. So, for this matrix also if you want to check whether it can be diagonalized or not and what will be the diagonal matrix? What will be the model matrix? So, for that we need to compute the eigenvalues. So, the eigenvalues for this matrix will be coming 2 2 and 8.

So, again now for each eigenvalue we need to compute the eigenvector to form this model matrix P and the corresponding to this eigenvalues this 2 2 the repeated 1. So, here the algebraic multiplicity of this 2 is 2 and now we compute the eigenvector corresponding to this. So, here this 2 minus 2 and minus 2, so that will be the matrix there which we want to solve as the system of linear equations. So, by doing so what we are actually getting here we are getting 3 linearly independent eigenvector meaning this geometric multiplicity of 2 is 2 and also it is; as the algebraic multiplicity is 2, also the geometric multiplicity in this particular case is coming to be 2.

So, this is corresponding to this 2 this is also corresponding to 2 and here we have this corresponding to this 8. So, we have 3 linearly independent linearly independent eigenvector and that is the reason now we can actually diagonalize this matrix because we need 3 matrices. Remember it is easy to remember here the model P here the model matrix P will be of the same order as A. So, we need this 3 columns to fill the matrix P. So, if we have 3 linearly independent vectors we can form this P otherwise for example, corresponding to 2 if it happens that we have only 1 eigenvector then we cannot form this P in other words the matrix A is not diagonalizable in that case.

So, here the matrix A is diagonalizable because we are getting 3 linearly independent eigenvector and the model matrix P here we will place this 1 to 0 minus 0 2. First two columns and the corresponding to 8; we have this 2 minus 1 as a third column. So, this is corresponding to 2. The first column this is also corresponding to 2 and this corresponds to 1 the third column. So, our order will remain exactly this one and this will become the diagonal entries of the matrix P inverse AP. So, if you compute the P inverse AP now.

So, we need to get this P inverse and then this product we have to make and then we will get this 2 2 and exactly the order we have placed here these eigenvectors. So, we are getting these diagonal entries absolutely the same here 2 2 8. So, that is the diagonal matrix here which is similar to the matrix A and later on we will observe several good properties about this matrix because they share many common properties these similar matrices and some of the applications very important applications one we can once we can diagonalize the matrix we can we can use them in many applications.

So, that will be the also topic of discussion of this lecture.

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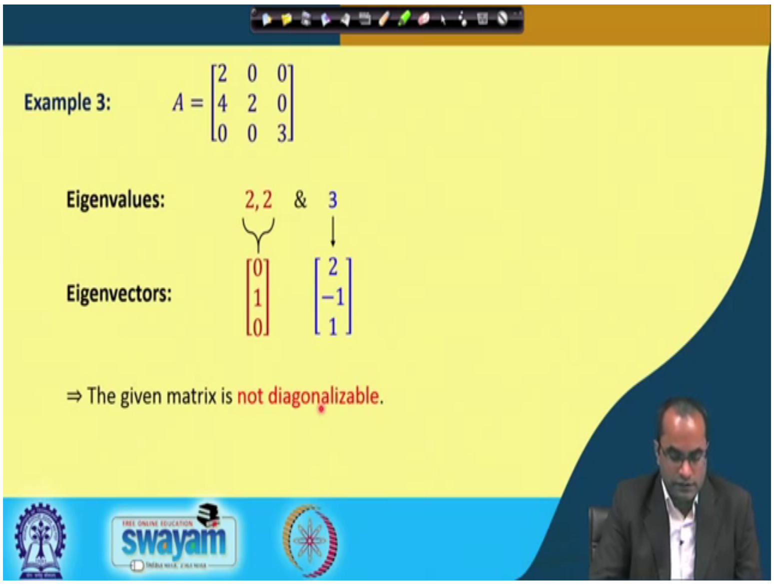

So, the last example here we will take another one where we do see this is the lower triangular matrix with entries 2 2 3 in the diagonals and then we have this 4 in the off diagonal rest everything is 0. So, in this case if we compute the eigenvalues we know for the triangular matrices. So, this is 2 2 3. So, the eigenvalues will be 2 2 and 3 and we have to compute again the eigenvectors corresponding to the 2 and also corresponding to this 3. What happens in this case that here this algebraic multiplicity of 2 is 2 and it comes to be that the geometric multiplicity of this 2 is 1 and that is the point where actually we cannot diagnolize the system because geometric multiplicity is not equal to the algebraic multiplicity for this eigenvalue 2 then we will get less number of eigenvectors and we cannot form this model matrix P.

So, here the corresponding to these 2 eigenvalues the repeated eigenvalue we are getting only 1 eigenvector and the reason is clear because if we formulate this equation A minus lambda I x is equal to 0. So, what will happen here? We get this 0 0 0 and then we have here 4 this again 0 0 and 0 0 1. This is the situation of this system of equation for the eigenvector here and the right hand side 0. So, for this system of equation what do observer what do we observe here that we have the 2 pivots element here 4 and also this 1. These are the pivot elements and the free variable we have only one that is here x 2.

So, x 2 is free variable free variable; that means, we can choose this x 2 whatever we like. So, let us take this alpha and then directly from these equations we have observe that the x 1 is 0 from this second equation and from this third equation we observe that x 3 is equal to 0. So, the eigenvectors x 1, x, 2, x 3 in this case corresponding to this repeated eigenvalue is coming to be 0 1 0 and any multiple of this 0 1 0.

So, that is here we have taken just alpha 1. So, this is one of the eigenvectors here and then corresponding to 3 also we can compute the eigenvector and that is naturally it will come 1 only. So, we have 2 minus 1 and 1. So, with these 2 vectors because we need 3 vectors to fill the positions of this model matrix P. So, we cannot do in this case because we are getting 2 linearly independent eigenvectors and therefore, this matrix A is not diagonalizable. So, the given matrix is not diagonalizable.

So, what we have seen here that every matrix we cannot diagonalize. We can diagonalize only those matrices when the eigen vectors the set of this eigen vectors is full means if it is a n by n matrix then if we get n linearly independent Eigen vectors then we can diagonalize the matrix, otherwise we cannot diagonalize the matrix.

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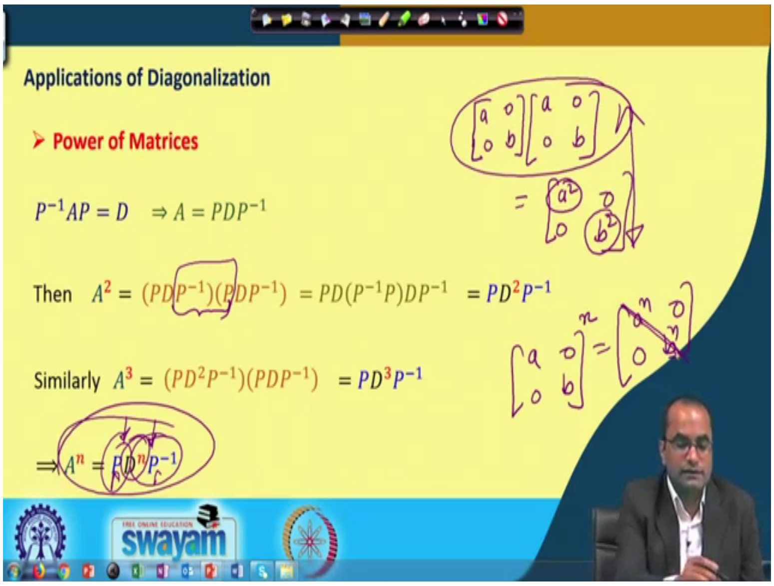

Now, coming to the applications of the diagonalization, the first application we will consider that we can easily compute the power of the matrices once we can diagonalize the matrix. So, why so? What is the connection here or to the to the eigenvalues, eigenvectors that we will see now. So, this P inverse AP what we have seen that A can be diagonalized then we have this relation P inverse AP is equal to D or we can rewrite it that A is equal to so we multiply by P here first there will be PD and then the right side we will multiply by P inverse.

So, we will get out of this relation here A is equal to PD and P inverse. So, having this relation then if you want to multiply or we want to get this A power 2 or A square in that case. So, we need to multiply this PDP inverse with the PDP inverse and then with this associativity property of this product we will realize here that this P inverse, P is there which we can put as the identity matrix and then we will get here P D and D P inverse meaning this P and the power of this diagonal matrix. So, here the A square the power of this matrix is 2 power of this matrix; it is coming to be that this power is exactly translated to the power of this diagonal matrix. What is the use here that this diagonal matrix we can easily get this power here because this power will directly go to this diagonal entry. So, we do not have to actually multiply these matrices D here for this power, but only the diagonal entries will be squared and that is a reason here.

So, the A square is very simple now the PD square P inverse. So, we have to only do these multiplications. Naturally when we have a high power here and not just for the two because in any case now we have to do this multiplication PD square and also this with the P inverse. So, here naturally the work is more if we just want to find out this power 2, but in case for example, we want to power to get the power 1000 then definitely this will be very very useful because D power higher power would be easier to compute. So, why this A square is coming D square? We can continue this idea for example, A 3 also the same similar structure will happen.

So, this PD square P inverse that is for A square and then again multiplied by A and this P inverse P will again become the identity matrix and we will have PDQ P inverse. So, what is the point here that we can see out of these calculations that A power n will be also just D power n here and this PD power and P inverse. So, this will continue this power here on translating to this matrix T. So, in general also we can prove that this A power n is nothing, but the P D power and n P inverse. So, we can use as I said before when we want to compute this very high power of this matrix a large number here and then this is very very useful because D power n the computation here is very simple because if we take 2 diagonal matrix for example, this here and we want to multiply with the same.

So, what will happen now? So, if you multiply this a square will come and then this product will be 0 here. Also this will be 0 and b square will come. So, what happens when we do the product of the diagonal matrix just simply we will this power; this power will go to the diagonal entries. So, we do not have to do this multiplication as the matrix multiplication just simply when we have the power n. So, for instance we have this a 0 0 b and we want to get this power n there.

So, this will be nothing, but the a power n zero and b power n. So, that is the point here. So, having this relation that A power n is nothing, but the D power n here. So, that is very simple to compute and then finally, we need to make this product with the P and the P inverse. So, that is the only computation load here for this matrix multiplication, but for D power n only that linear relations here. So, these diagonal entries will be powered and nothing else.

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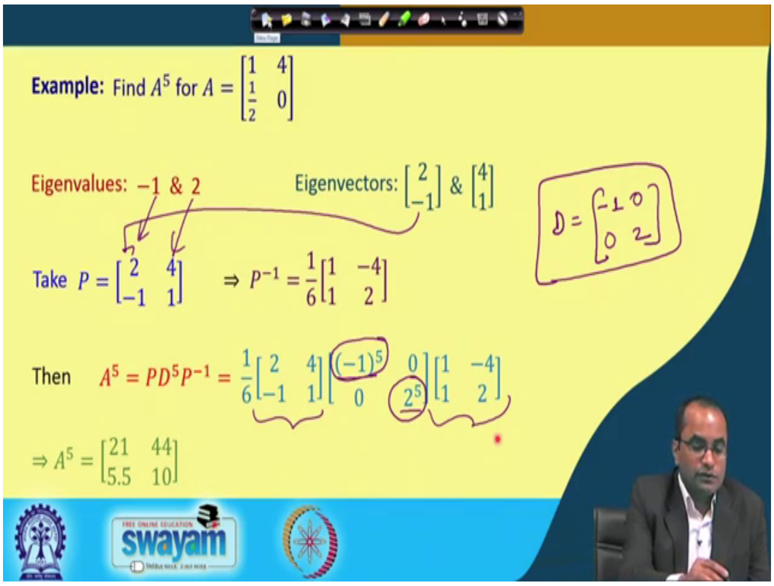

So, let us just go through one example which says this find A power 5 for A is equal to 1 4 and half 0. So, but to do so, we have to compute the eigenvalues and also the eigenvectors, because we need that P and we also need that diagonal matrix. So, in this case the eigenvalues are coming to be minus 1 and 2 and then we compute the eigenvectors corresponding to minus 1 it is 2 1 and corresponding to 2 it is coming as 4 1. So, having this we can now form this P the model matrix P. So, placing these eigenvectors here as the columns.

So, the first column we have 2 minus 1 and the second column we have this 4 1. So, having this model matrix now we can compute this P inverse again easily and then this A power P as per our discussion in the earlier slide. So, we have this P and the D power P and P inverse. So, we have to get this power of the diagonal matrix and then we have to multiply by this P and P inverse. So, looking at this one, so we have this P inverse and then this D; D was; so, what will be the D? We have the eigenvalues minus 1 and 1.

So, this D will become simply we have to place these eigenvalues in the order we have placed these eigenvectors in P. So, this was the corresponding to 2 minus 1, the second was corresponding to 2 2. So, therefore, this order will be maintained here in the eigenvalue. So, that is the matrix this D here. So, the D power 5 is just minus 1 power 5 and this 2 power 5 which is here.

So, we can actually compute a very high power. Also the computational load will remain the same. Only thing we have to we have to just do this power here of the diagonal entries. So, and then later on we have to in any case just multiply by this P and the P inverse. So, it is easy now to find having this relation any power of the matrix A. So, here the A power 5 when we do this multiplication that is coming this 21 44 and this 5.5 and this 10. So, that is A power 5, but it is usually used when we really want to have we want to compute a high power of this matrix A then this is computationally very very efficient as compared to doing the product of matrices A.

So, here was the one of the applications where we use this eigenvalues, eigenvectors or in particular this idea of the diagonalization of the matrix.

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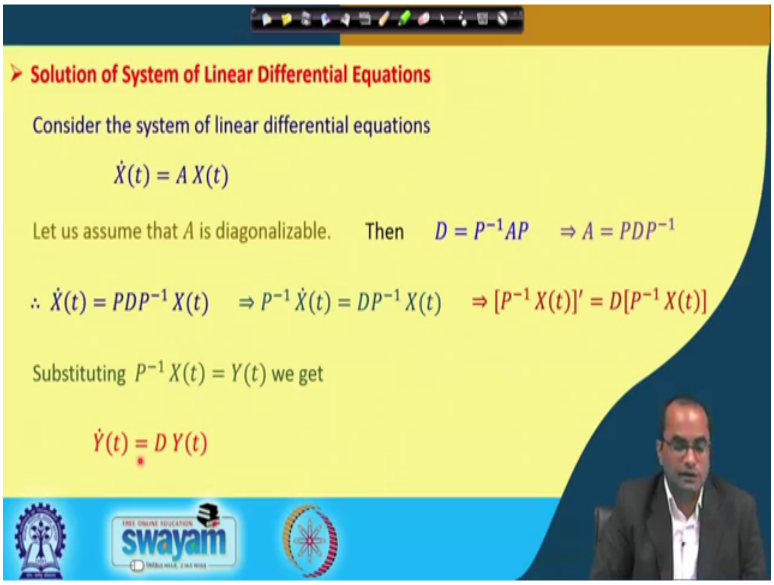

Now, coming to the next application which is the solution of the system of linear differential equations though the differential equations will be the topic of the next few lectures, but here just to introduce the idea of this or the application of this diagonalization. We will be doing very simple example also. So, here we consider the linear differential equations. So, here it is a system, system means we have a more than one differential equations and their variables are coupled. So, here we have this; the left hand side for example, this is the derivative term.

So, meaning that we have like this dx 1 dt and suppose we have two equations dx 2 over dt and then right hand side some matrix is given here. So, a 1 1, a 1 2, a 2 1, a 2 2 and then the variable here X 1 and X 2. So, we have these two unknowns for example, in the system X 1 and X 2 and this is the system because the first equation of this system dx 1 over dt is having this X 1 and X 2. So, a 1 1 X 1 plus a 1 2 X 2 and the second equation here is dx 2 over dt a 1 1 a 2 1 X 1 and plus a 2 2 X 2. So, we have basically these two equations. These two are the differential equations we have the ordinary derivative.

So, these are the system of ordinary differential equations and these are coupled because this X 1 and X 2 is percent in this equation 1 as well as in the equation 2. So, these are the coupled equations and they are not very easy to solve, but with the idea of this diagonalization it becomes very very simple as we will observe now here. So, what we have to assume? We have to assume that this matrix A is diagonalizable; this coefficient matrix here is diagonalizable. So, once we know that.

So, we know this relation that D is equal to P inverse AP. So, this D then this system here which was the derivative term of this X is equal to A is replaced now by this PDP inverse. So, PDP inverse and the X the right hand side and what we do? We multiply this to P inverse now. This equation to where P inverse is derivative the vector of the derivative terms then we have DP inverse and this X term. Now this is just the coefficient it is just the matrix here, which we have the model matrix.

So, these are just the constant they increase your P, they are the constant. So, we can actually rewrite this P inverse and the derivative terms here are coming in this vector. So, we can simply take the derivative of this P inverse X because these are the constant terms. So, this will not matter for the derivative. So, we have collectively taken this as P inverse X and the derivative right hand side PD and this P inverse this X and now what we do we just substitute a new variable here we take for this P inverse X.

So, substituting this P inverse X is equal to Y a new variable. We have named the new variable when we multiply this P inverse to X. We are setting this new variable Y and our differential our system here which was this X dot is equal to AX t, which is now reduced to this Y dot is equal to DY t. So, what is the benefit now; having from this system to this system. Now do you notice that this t is the diagonal entries here now in the D and they are the eigenvalues here are placed in this t.

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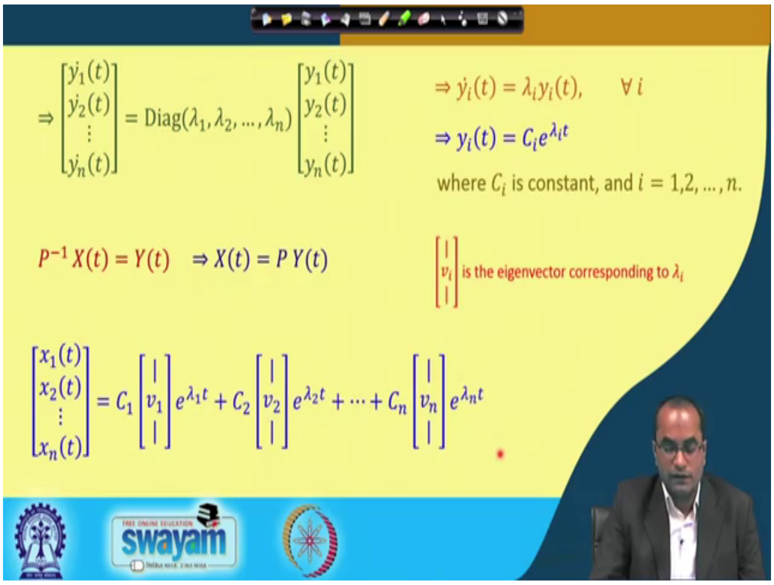

So, the benefit here that our system the reduced system is that here we have the derivatives from the, here we have this diagonal matrix and then we have the y the unknown. Now if we just look at this multiplication what we are getting? We are getting these n equations and they are actually decoupled equations now because once we multiply this with the diagonal entries what we are getting lambda 1 y 1, the second entry here lambda 2 y 2 and lambda n y n and here also we have these; the derivatives of y 1, y 2, y 3, y n.

So, we got here the i equation I mean n equations, but they are decouple now because the for instance the first equation has only y 1, the second equation has only y 2, the third equation is having only y 3 and the single equation this dy over dt is equal to lambda y type equations we know how to solve because all these equations are of this type dy over dt is equal to yt and we know that the solution here that this is y as some constant exponential t. So, that is the solution of this equation y is equal to c exponential time t and here we have all these equations which are no more coupled equation.

So, by this idea of the diagonalization we got this from the coupled system of equations just these uncouple system of equations, which are easy to solve now. So, we will solve these each question here the solution will be y i t is equal to C i and e power lambda i t. So, the C isare this constant of integration. So, what do we get? We have the P inverse X t that was the our substitution which we have made for Y and now we got the vector Y.

So, from there we can indeed get this X again back because our main variable in the given system was X or we can write down this in this expanded form.

So, this X was having these n components x 1, x 2, x 3, x n. Here this is P the vector P; the vector P here is having the columns as the eigenvectors right. So, here the d was the diagonal matrix whose entries were lambda 1, lambda 2, lambda n and here we will get the corresponding eigenvector. So, v 1, v 2 and this v 3 and so on and then we will get this v n. So, these are the corresponding eigenvectors of these eigenvalues lambdas. So, this was the model matrix here.

So, then we can do this product as well. So, this product with the Y t. So, here we have basically y 1, y 2, and y n t whose value also we know that C 1 e power lambda i t. So, this matrix vector product we can take as the first column multiplied by this first element, the second column multiplied by the second element and so on. This is what we have written here for Y we have substituted this C 1 e power lambda 1 t, here C 2 e power lambda 2 t, C n e power lambda n t.

So, what is the final remark here that we need to compute the eigenvalues of the given matrix A and also the eigenvectors and then we can write down finally, the solution directly that the constant terms the first eigenvector a corresponding to this lambda 1. So, e power lambda 1 t, this eigenvector corresponding to lambda 2, eigenvector corresponding to lambda n. So, what we have to do? We have to compute the; this original coefficient matrix A which was given for the system of equations. We need to just compute the lambdas and the eigenvalues eigenvectors and then we can find a solution.

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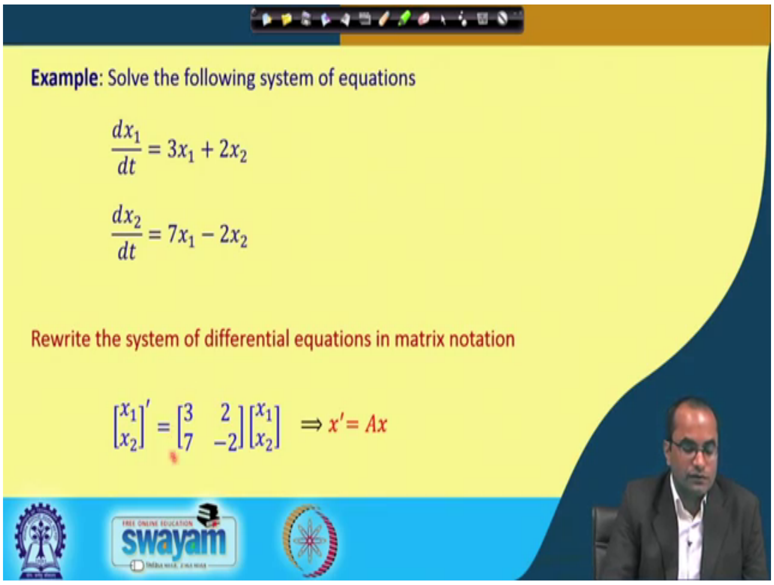

So, just to demonstrate this we have taken the simple example here. We can rewrite in the system form here. So, dx 1 dt, dx 2 dt we can write down in this vector form this coefficient matrix 3 2 7 and minus 2 and this x 1 x 2. So, what we have to do? We have to just compute the eigenvalues and eigenvector of this matrix here.

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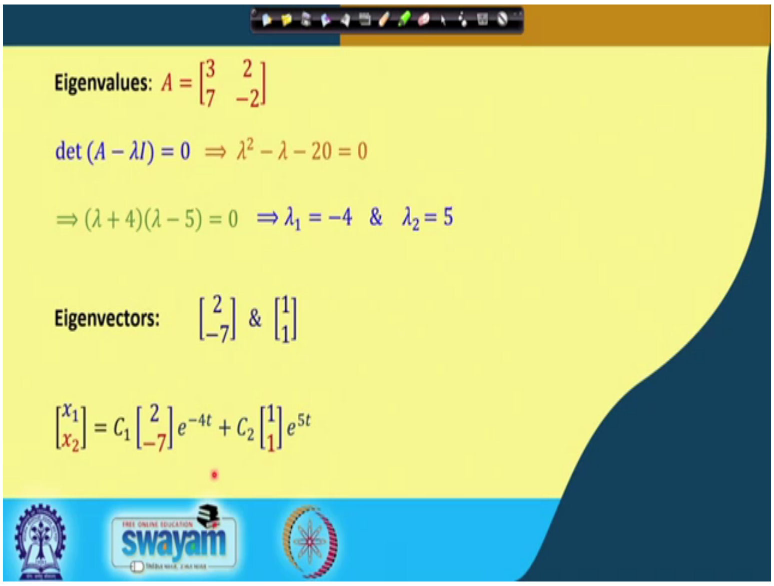

So, the eigenvalues of this matrix are coming as when we write down this characteristic polynomial we are getting this minus 4 and 5 as the eigenvalues and the corresponding eigenvectors we are getting here 2 7 and 1 1.

So, 1 1 is corresponds to 5 and minus 4 and corresponds to this 2 and minus 7. So, now, we can write down the solution directly in terms of the eigenvalues, eigenvectors. We have the constant term we need to put this eigenvector and e exponential power this lambda t, again the second constant, the second vector and exponential 5 t.

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So, what we have seen here that with the help of this diagonalization; this was very easy to solve such a system, and the conclusion here is that this diagonalization of the matrix we have learnt, and in particular we have seen these two applications, the power of the matrices, we can easily compute with the help of this idea and also the solution of the system of linear differential equations we can compute with this diagonalization.

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So, these are the references we have used and thank you for your attention.