Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

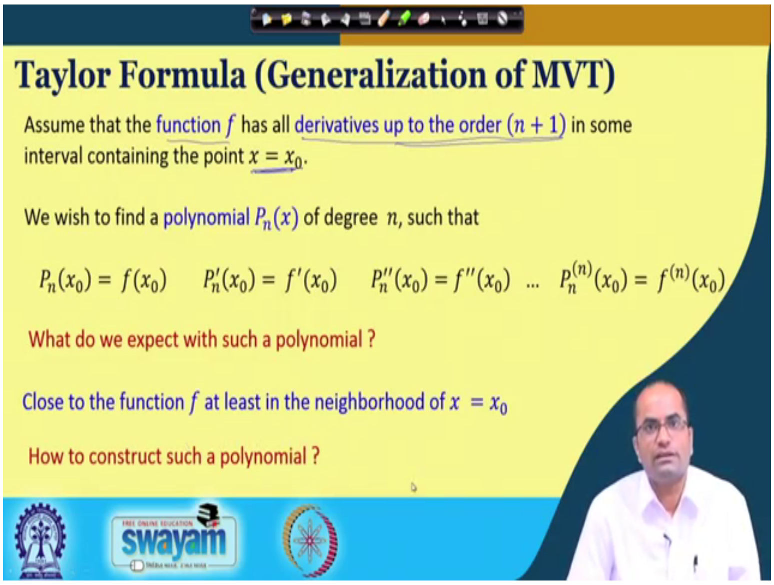
Lecture – 05
Taylor Polynomial and Taylor Series

Welcome back to the lectures on Engineering Mathematics-I and today's we will learn Taylor's Polynomial and Taylor Series.

(Refer Slide Time: 00:22)

So, these are the topics you will cover will start with the Taylor's polynomial and then coming back to this Taylor series from the Taylor's polynomial and some worked out examples.

(Refer Slide Time: 00:36)



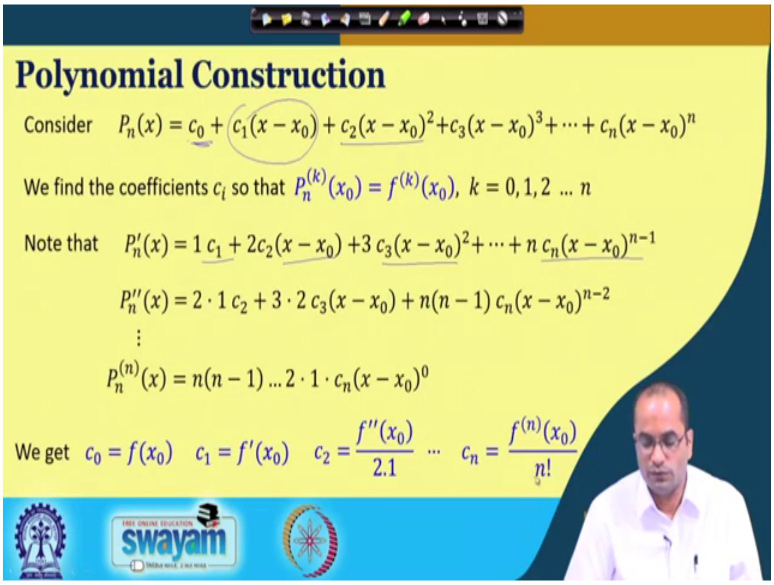
So, this Taylor's formula which is a generalization of the mean value theorem or the Cauchy's mean value theorem which we have learned in the last lecture. So, we assume that the function here f has all the derivatives up to the order n plus 1.

In some interval which contains the point x is equal to x 0. Having this we wish to find this polynomial P n x of degree n, with the conditions that this polynomial satisfies that P n x polynomial at x is equal to 0 is equal to the function value at 0. Second: that the first derivative of this polynomial is equal to the first derivative of the function. The third condition the second derivative of this polynomial at this point x 0 is equal to the second derivative of the function at the x 0 and so on.

So, what we assume basically that all these derivatives, the function value itself the first derivative, the second derivative, and n-th derivative they all are equal to this derivative of the polynomial. So, having these conditions what do we expect from such a polynomial? We expect such a polynomial if we construct then there should be close to the function f naturally at x 0 function value is 0. So, at x 0 several so the derivatives of 2 order n are equal. But in general also we expect that because of these conditions that this polynomial of degree n and somehow in some form will represent the function f.

So, now the question is how to construct such a polynomial.

(Refer Slide Time: 02:33)



So, we assume here a general polynomial of degree n. So, we take that c P n x is c 0 and c 1 x minus 0 c 2 x minus x square c 3 x minus 0 cube plus c n x minus x 0 n. So, in this is special form we have taken this polynomial because of the convenience to evaluate these unknown c 0 c 1 and c 2, but one can also assume any general polynomial of degree n. So, now we want to find these coefficients c i's based on the conditions which we have set or we wish to have that this up to n-th order derivative of this polynomial at the point x 0 must be equal to the respective derivative of the function at the same point x 0.

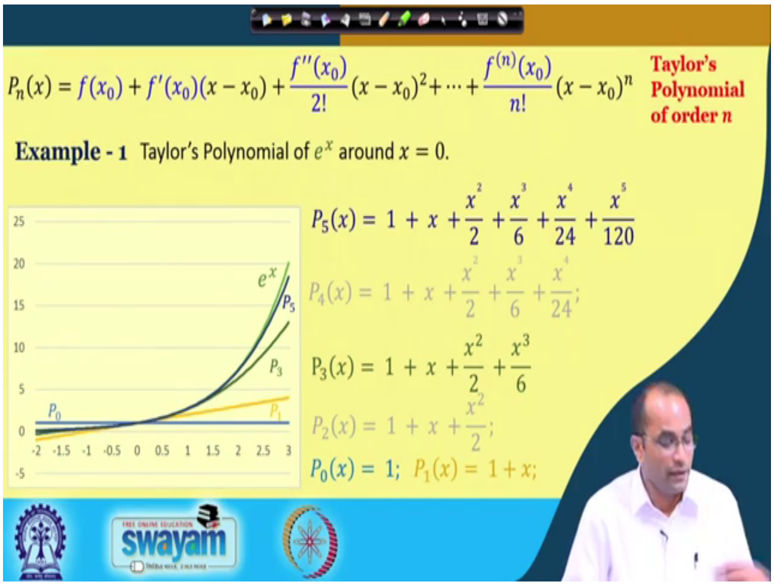
So, having this condition first we know that the first derivative of this polynomial here is equal to. So, once we this is a constant so this will be equal to 0 and then we take the second term which will be having here x there so we will get the c 1 out of this. Similarly from here we will get 2 time c 2 and x minus x 0 then we will get here 3 time c 3 and so on.

Then if we move further, than the second derivative, so out of this first derivative we can again differentiate this you will get 2 c 2 here 3 times tool to c 3 and so on and then we can repeat this process further to get the n-th order derivatives. In n-th order derivative because this was n-th order polynomial we will get only a constant term which will see here the factorial n and the coefficient the c n will come in the expression here and there will be no x term present here because, the polynomial was degree and then we have differentiated end times.

So, having this what we get out of the first condition when we substitute because, our conditions is the polynomial value at x 0 must be equal to the function value at x 0 and further derivatives upto order n. So, from the functional value we will get here P n x 0 and is equal to all these terms will vanish and we will get only c 1. So, P n; sorry from the from the polynomial itself when we substitute x is equal to x 0 we will get c 0 as P n x 0 and P n x 0 is an f n x 0, so we get the c 0 as f x 0 over here.

The second when we substitute in this first derivative so you will get c 1, as the first derivative of the function the c 2 again from this condition so we will get the second derivative at x 0 divided by this factorial; sorry to factorial 2 here and then the c n will be the n-th derivative at x 0 divided by factorial n.

(Refer Slide Time: 06:00)



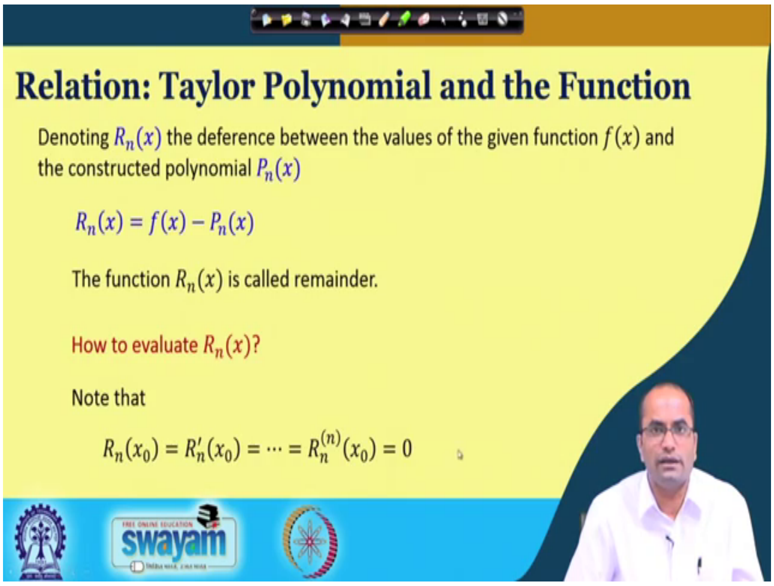
So, having these coefficients now what we have we have the polynomial, the polynomial says that this P n x will be f x 0 which was c 0 there and all these co efferent are substituted here, so this is what we call the Taylor's polynomial of order n.

Now if you go through the example here and construct the Taylor's polynomial of the exponential function for example, e power x around x is equal to 0. So, the P 0 x the degree 0 polynomial only the first term will be present there and e power 0, so e power 0 will be just 1. So, the polynomial of degree 0 will be simply 1. And if we plot this, so this is the green plot here of the exponential function and this polynomial of degree 0 is just a constant line; so the straight line going through this 0 1 point. And if we compute the polynomial of degree once we will get this f x 0 as 1 and then we have the derivative here e power x the derivative will be e power x and then at x is equal to 0 this will again 1.

So, we have the polynomial of degree 1 as 1 plus x which is plotted here. So, it is again the straight line with slope 1. And now if we continue this process we can evaluate P 2 then P 3, so P 3 again we have plotted here with this curve and then going further so we will get like P 5.

So, for example, if we plot here P 5 then this P 5 if you see it is pretty close to the exponential function in a very wide range of or in a wide interval around this point 0 which was the point of this expansion here. And if we move further for the polynomial of degree 6 or 7, then we will be moving closer to the exponential function in the cid of the Taylor's polynomial. So, by increasing the degree of the polynomial we can approximate our function as good as we like and we will discuss on these further.

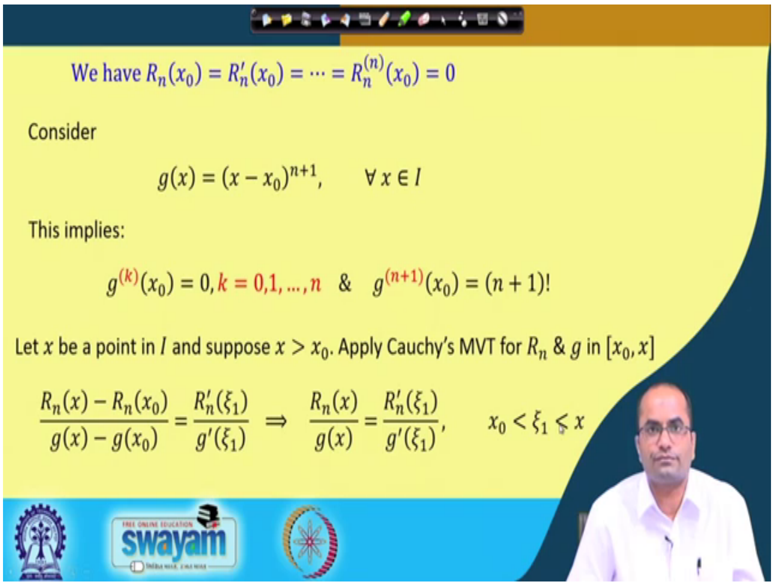
(Refer Slide Time: 08:22)



So, what is the relation of the Taylor's polynomial and the function. That means, because this Taylor's polynomial representing the Taylor functions to some approximation. So now, we will denote this R n x as the difference between the values of the given function and the constructed polynomial P n x. So, in this case what we mean this the R n x we define this difference of the function and the polynomial P n x. And in this case now the R n x is called the remainder, because this is the difference between the actual value of the function minus the polynomial value at the point x.

So, now the question is the how to evaluate this R n x and to go further for the evaluation we first note that the R n at x 0, because f x 0 minus P nx 0 and by construction this polynomial at x 0 is equal to the function value at x 0 so this is 0, and the first derivative the same thing. So, the first derivative of a f at x 0 is equal to the first derivative of the polynomial at x 0 this was the construction of this polynomial. So, all these n-th order derivative are equal, so in this case therefore this R n at point x or the first derivative at point x naught the n-th derivative at point x naught all are equal to 0.

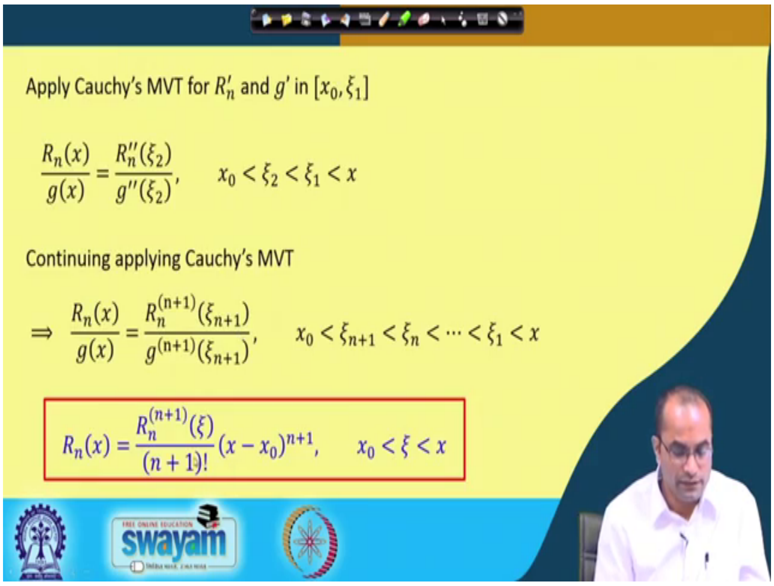
(Refer Slide Time: 09:57)



So, we have these conditions R n x, now we consider another function here which is g x which is g x x minus x 0 power n plus 1. So, this function has also the property if we notice this that g at x 0 will be 0 in fact the first derivative, because n plus 1 and x minus 0 power n that will also become 0 at the point x 0 and so on. So, we can continue this differentiation here up to the n-th order and all these derivatives at point x 0 will be 0. So, what condition we have now that g the case order derivative at x 0 is equal to 0 up to the order n and g n plus 1 is nothing but the factorial n. So, we have 2 functions one is R n x and another one is g x they have similar properties here that all these derivative of to order n they vanish.

So, in this case if it take to x point in the interval and suppose that x is bigger than x 0 we can also assume that x is less than x 0. And now we apply the Cauchy mean value theorem for these 2 functions the rn function and the g function in this interval x 0 to x and now remember what was the Cauchy mean value theorem that you have this function R n x minus the R n x 0 divided by g x minus g x 0 is equal to their Deriratio of their derivatives at some point phi one and g prime at xi 1. Now if he noticed that R n at x 0 is 0 and also the g at x 0 is 0, so we have this resolve that are R n x over g x is equal to the ratio of their derivative at some point xi 1 in this interval x 0 and x we can continue this process.

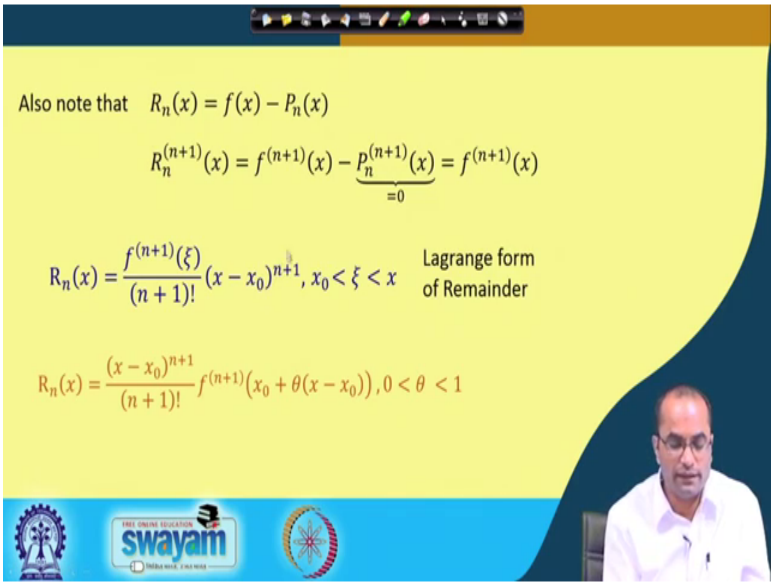
(Refer Slide Time: 12:12)



So, further if we apply again the Caushy mean value theorem for the derivatives rn derivative and g derivative in the interval x 0 and xi 1. So, what we will get we will get because this first derivatives are also 0. So, simply we will get the result that the R n x divided by g x is equal to the R n double derivative over the double derivative of g. Now the sum point in between this interval which we have considered from x 0 to xi 1. And now we can continue this process further for the next derivative supplying this Cauchy's mean value theorem further what you will get we can go up to the n plus 1th derivative because, up to n-th derivative R n and g both are 0.

So, we will end up with this term and then here the xi n plus 1 lies between x 0 and the xi n and there was a continuity up to x there. So, what we get out of this R n x is equal to this R n (Refer Time: 13:25) plus 1th derivative divided by this g n plus 1 the n-th this here the n-th derivative of this g n plus 1th derivatives was the factorial n plus 1. So therefore, this factorial and plus 1 term came here and then we have this gx which was the function x minus x 0 power n plus 1 and the xi which is introduced here xi n plus 1. Now we have replaced by xi it lies somewhere between this x 0 and the point x, so which is written here the xi lies between x 0 and x.

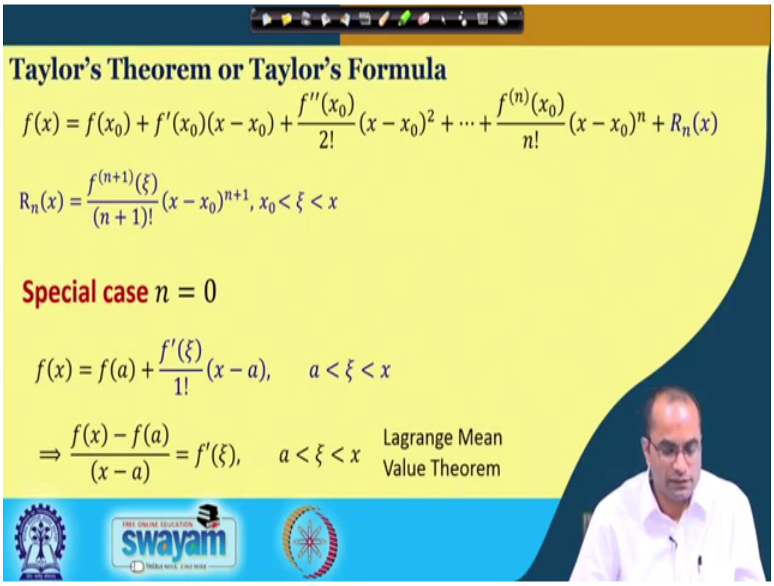
(Refer Slide Time: 14:12)



So, now we note that these R n x which we have define that was the difference between the function f x and P n x. So, if you take the n plus 1 and derivative here of this reminder term the n plus 1th derivative of this a f minus the n plus 1th derivative of this p is equal to the n plus 1th derivative of x because the n plus 1th derivative of n and P n was the polynomial of degree n. So, if we take the n plus 1th derivative this term will become 0 and we have these R n n plus 1 x is equal to f n plus 1 x. And now we can substitute in our formula which was the R n plus 1 here, so we have substituted now this value of this R n plus 1 as f and plus 1 xi.

Well now, we got the polynomial here which is the remainder term the R n x which is the LaGrange form of the remainder, this time we can also we write this remainder form in this form. So, this xi which appear there we have just replaced by this x 0 plus theta x minus x 0 this theta lies from 0 to 1. So, here if theta is close to 0 then this argument here is moving to x naught when theta goes to one this argument here goes to simply x. So, this argument of this f lies between x 0 and x, so it has the same similar meaning what the other form has. So, we can we write this LaGrange form of the remainder in this form as well.

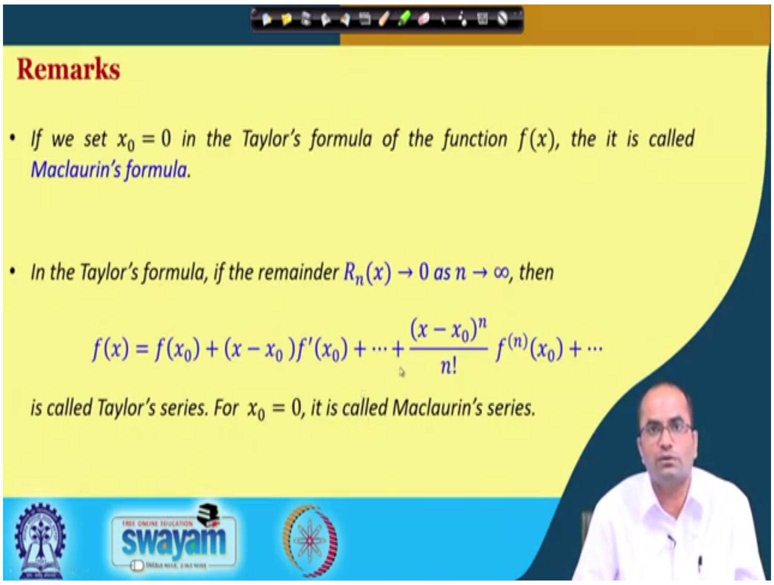
(Refer Slide Time: 16:18)



The Taylor's theorem or the Taylor's formula now if you summarize we have the function f x which can be expanded in this form of f x 0 f prime x 0 x minus x 0 up to the this n-th order term and plus this reminder term which we have just derived as this form which is called the LaGrange form of the reminder.

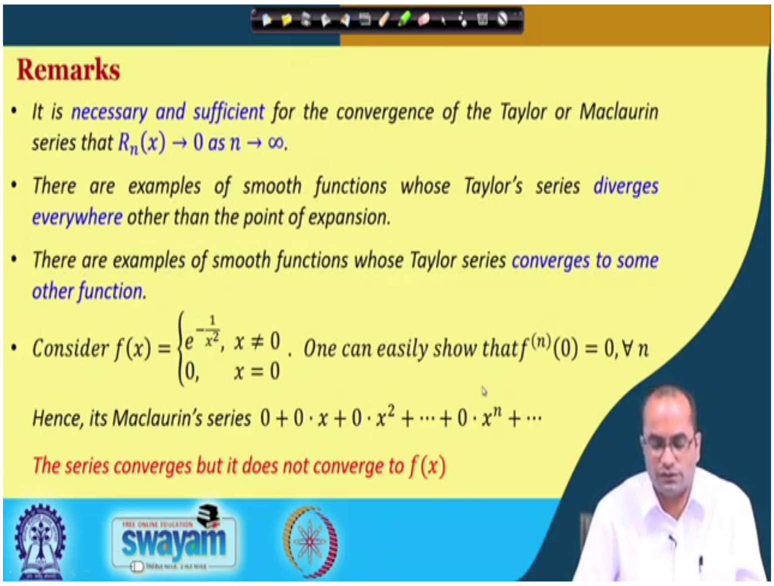
So, in the special case when we take n is equal to 0 so that means, this up to the order one we have to write this reminder term. So, we get this f prime xi divided by factorial 1 and then we have this x minus either way or x 0 whatever we consider. So, then this x the xi lies between this a and x. And in this case we get this form of the Taylor's theorem which was which was the LaGrange form mean value theorem. So, this is a special case of the mean value theorem which we have seen just by putting n is equal to 0 in this case.

(Refer Slide Time: 17:20)



Some remarks so if we set x 0 is equal to 0 point of expansion in this Taylor's formula then it is called the Maclaurin's formula and in the Taylor's formula if it is reminder goes to 0 as n goes to infinity, so this is an important remark here. If this reminder goes to 0 as n goes to infinity then we can write down that Taylor's formula in the form of the series so f x 0 and so on you can continue for infinite term and this is called the Taylor series. So, for x is equal 0 if we said this x 0 is equal to 0, then this is called the maclaurins series.

(Refer Slide Time: 18:03)

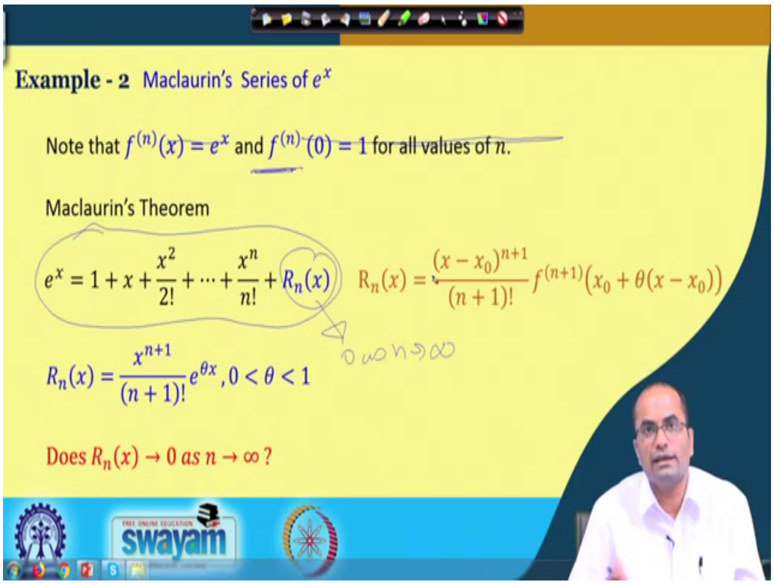


So, what we have seen this last remark again, so it is necessary and sufficient for the convergence of the Taylor's or the maclaurinseries that R n x goes to 0 as n goes to infinity. Because they are examples of smooth function series. So, smooth means we have the derivatives of whatever or we lie, but the Taylor series diverges everywhere rather than the point of expansion, because a point of a expansion the series will have the value same as the function as per the construction so. And there are the example (Refer Time: 18:35) of this smooth functions whose Taylor series converges to some other function and for instance you take this example fx is equal to e power 1 over x square and the 0 here.

So, x not equal to 0 and x is equal to 0 so in this case one can easily show that (Refer Time: 19:01) at 0. So, if we take the derivative of this function here e power minus one over x square which I am not doing this calculations, but one can easily compute at all the derivations of any order of this function at 0 will be 0 and then we if we write the Taylor series or other maclaurin series around x is equal to 0 then we will get because all the derivative are 0.

So, you will get 0 plus 0 into x 0 into x sqaure and so on. And then what we see here whatever x we keep the maclaurin series is giving 0. That means, the series converges whatever x the series is just 0, but it does not convert to the function of the function was for x not equal to 0 e power minus 1 over x square. So, it is very simple example where we can see that these maclaurin or Taylor series they do not converge to the function.

(Refer Slide Time: 20:02)



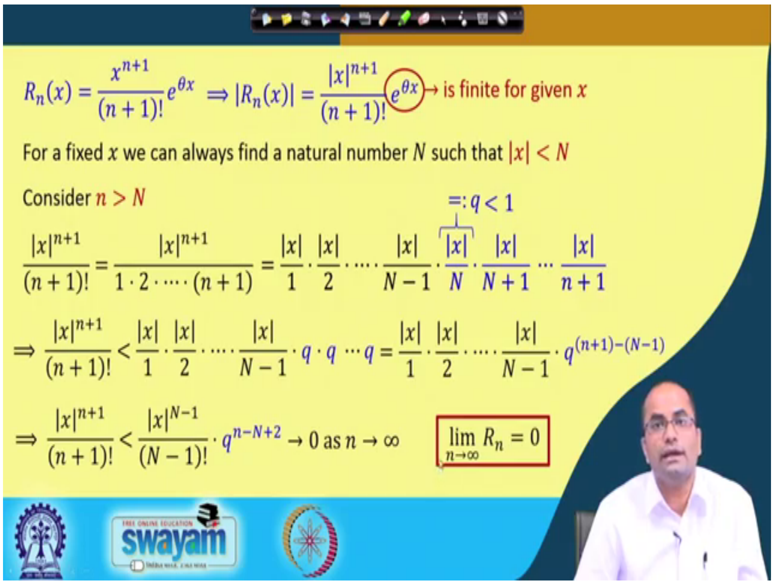
Now, let us just take this example of the maclaurin series of e power x now again not that. So, whatever derivative we take for this function exponential it is just the exponential x and at 0 we have the value 1. So, for all values of n all the derivatives of this function exponential function is 1. So, we can easily write down this maclaurins series theorem that exponential x is equal to 1 plus x plus x square by factorial 2 all the derivatives are 1.

And x n power factorial n plus R n x; so the R n x is the remainder term which we have just seen the remainder term is x minus x 0 power n plus 1 divided by the factorial n plus 1 and the n plus 1th derivative at some point which is x 0 plus theta x minus x 0. So, here if we substitute this x 0 is equal to 0 then we get this reminder x power n divided by factorial n and e power theta x and theta is between 0 and. So, one now you will see what happens to this R n x as n goes to infinity.

If this is the case if R n x goes to 0 as n goes to infinity, then we can write down the Maclaurin series of exponential function x. So, here again we note that this is the Maclaurin's theorem. So, if this term goes to 0 as n goes to infinity, in this case we can write down this exponential function as a series. So, we can remove this R n x and then we can continue with the further terms as a series.

So, let us just check.

(Refer Slide Time: 22:04)



So, this is the R n x the remainder term x power n plus 1 divided by this factorial n plus 1 the exponential theta x and if you take the absolute value of this remainder term as x power n plus 1 over n plus 1 factorial e power theta x, then we notice that this e power theta x for whatever given x this will be a finite quantity. So, this will not disturb because there is no n term here. So, if we can now focus on this term that what will happen when n goes to infinity.

We should notice that when x when n goes to infinity. So, this becomes infinity and here whenever this x is for example, large number greater than 1, then this term is also going to infinity. So, we cannot simply say that what will happen to this term when n goes to infinity. So, we have to carefully check this limit that what will happen to this term when n goes to infinity. So, what we consider for a fixed value of x, we can always whatever x as could be very large number, but we can find a natural number n such that the absolute value of this x is greater than n.

So, whatever x we take here, then this n the big N we take greater than the absolute value of this x. Having this we will also consider one more n the small n term which is appearing there in the formula. So, which is a bigger than this number n as well. Now we consider this term modulus x power n plus 1 over factorial n plus 1 factorial n plus 1; this factorial n plus 1 is the product of 1 plus 1 and upto one and the we have here modulus x power n plus 1.

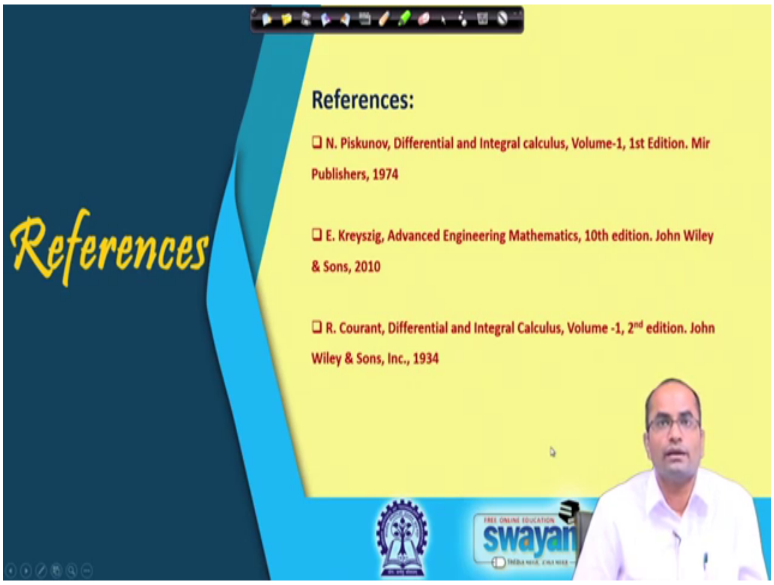
So, we can write down in the form of the product as. So, factorial x divided by 1 factorial x again divided by 2 and so, on we can continue now just look at this term here which have appear after this N minus 1 term. So, modulus x divided by n here also this absolute value x divided by N plus 1 and so, on. So, this term here absolute value of x divided by n. So, out of this expression what we see? The absolute value of x divided by this N is less than 1 and now if we assume if we assume this as a number q. So, we have a q here and this is in fact, divided by N plus 1. So, this is less than q and all other terms hear less than q.

So, again you note that this n was bigger than N. So, we have gone up to this N plus 1 term. So, this with this N will be somewhere in the middle and now this q what we have notice because the absolute value of x divided by N is less than 1. So, what we see now we can replace this equality by the inequality. So, less than equal to because here we have replace by q and this one is also replace by q though it is a less than 1 q this is also less than q all these terms are less than q.

So, now how many q's we have here. So, N minus 1 terms are already there and then the total terms were n plus 1. So, if we can re write now the total term n plus 1 and already these terms are n minus 1; so n plus 1 minus N minus 1. So, this number here is n minus n and minus 2 and modulus as n minus 1 over factorial n minus 1 and now we can take a limit here in this case and q is less than 1 and when n goes to infinity. So, this n goes to infinity this n here it goes to infinity and q is less than 1 then this goes to 0 and here some fixed number is appearing. So, this everything goes to 0.

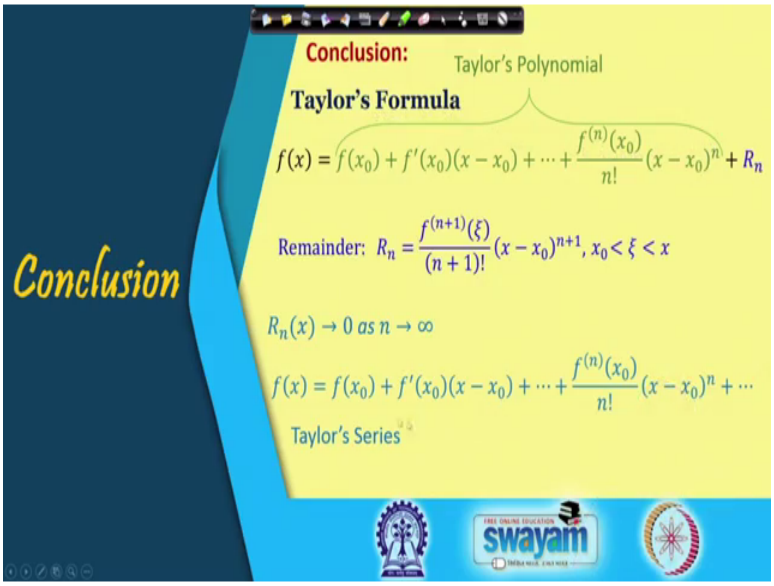
So, this absolute value of this remainder term which goes to 0; so what we have seen that this reminder term which was a part of the remainder term is less than which term which goes to 0 as n goes to infinity and we can conclude that the remainder goes to 0.

(Refer Slide Time: 27:04)



So, these are the references which were used for preparing these lectures and the conclusion.

(Refer Slide Time: 27:12)



So, what we have learnt today is the Taylor's formula and very important topic in the differential calculus. So, a function which is smooth enough we can write down as f x 0 f derivative x 0, x minus x 0. So, on plus this R n x term which is called the remainder term, and this is one form of the remainder term f power n-th plus 1 derivative divided by 1 plus 1th factorial x minus 0 n plus 1. And what we call this the polynomial term we call the Taylor's polynomial, the whole resolve this is called the Taylor's formula. And then we have also observed that when R n the remainder term goes to 0 as n goes to infinity, then we can write down this Taylor's formula as in the terms of a series which is called the Taylor series. So, that is all.

Thank you for your attention.