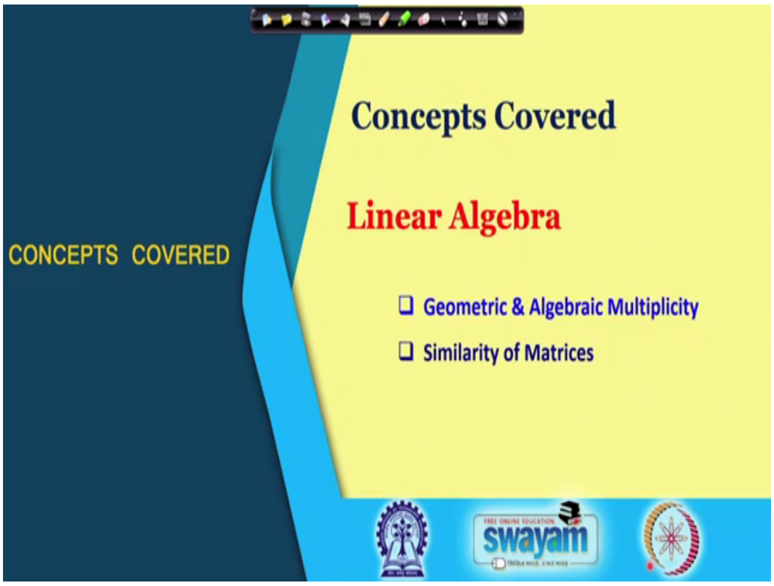
Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

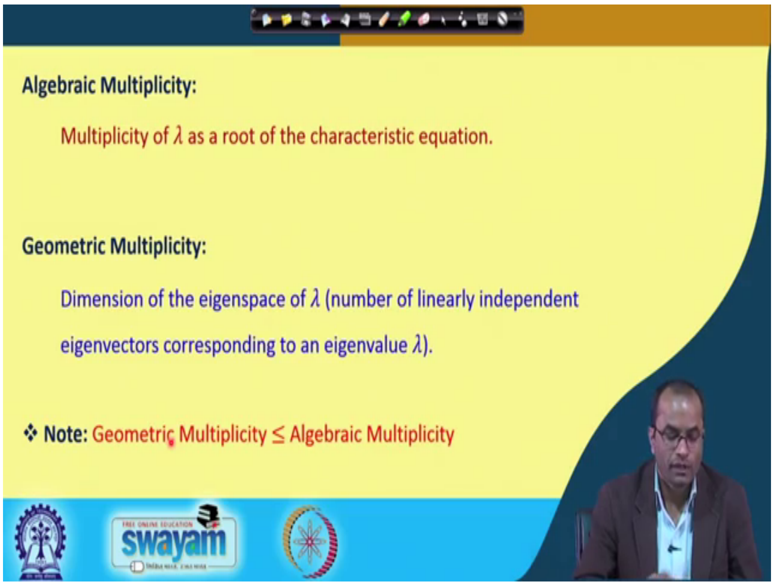
Lecture - 49
Eigenvalues & Eigenvectors (Contd.)

(Refer Slide Time: 00:22)



Welcome back, so this is lecture number 49 and we will be talking about today the geometric and algebraic multiplicity and the similarity of matrices.

(Refer Slide Time: 00:29)



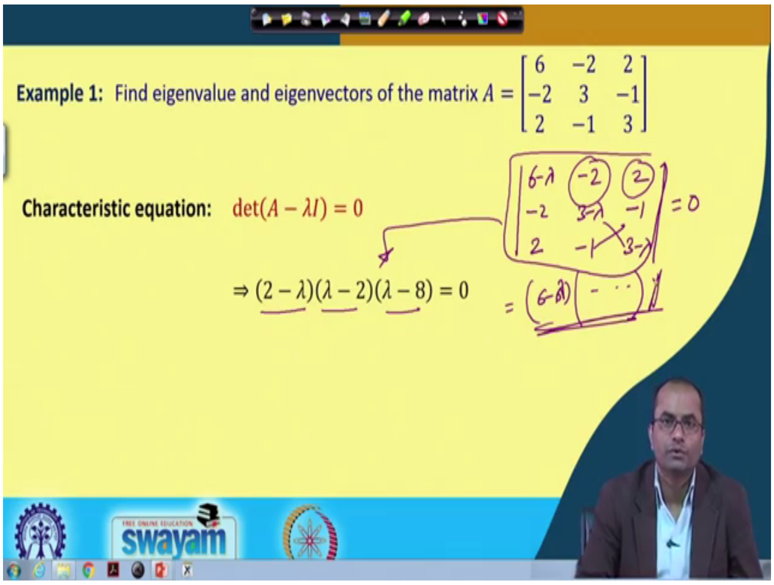
So, what is the algebraic multiplicity? So, algebraic multiplicity of lambda as a root of the its a multiplicity of lambda as a root of the characteristic equation. And the geometric multiplicity is nothing but, the dimension of the eigenspace of lambda; that means, the number of linearly independent eigenvectors corresponding to an eigenvalue lambda.

So, these are the two numbers which we will now use for telling about the multiplicity of this lambda, because we have seen in several examples the characteristic, roots all the eigenvalues were repeated. So, that we can now quantify with the help of this algebraic multiplicity; so, algebraic multiplicity if for instance one root is repeated 3 times. So, then it is algebraic multiplicity of that particular root is 3 and the geometric multiplicity will be the dimension of the eigenspace or the number of linearly independent vectors we have corresponding to that particular eigenvalue lambda.

So, with these two classification we will move further, but before that there is a note here, that this geometric multiplicity is always less than or equal to the algebraic multiplicity, that is a important result which one can formally prove. But, it is it requires little more knowledge of a diagonalization etcetera.

So, we will not prove this result now, but we will keep in mind that this geometric multiplicity is always less than or equal to the algebraic multiplicity. Meaning that for example, one a particular root; one particular eigenvalue is repeated 3 times than the corresponding geometric multiplicity meaning they are number of linearly independent eigenvectors cannot be more than 3, they have to be less than or equal to 3.

(Refer Slide Time: 02:31)

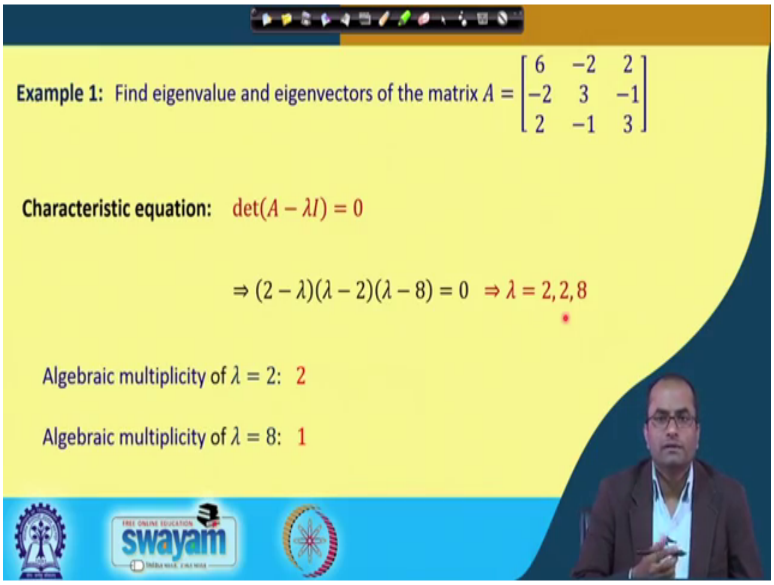


So, first we will see with the help of many examples that what are the situations arises here. So, in this case we find the eigenvalues and the eigenvectors of this matrix A which is given as a 6 minus 2 and 2 here of minus 2 3 1 and 2 minus 1 3. So, for this eigen, for this matrix we will compute the eigenvalue and eigenvectors we have already computed for several matrices in the last lecture. So, we are on now the familiar with the computation of the eigenvalues.

So, here first we need to write down the characteristic equation for this given matrix which is the determinant of this A minus lambda I, determinant of this matrix A minus lambda I is equal to 0 and for this matrix we can compute this determinant here. So, the determinant would be like the 6 minus lambda minus 2 2 and then minus 2 here 3 minus lambda and minus 1 2 minus 1 3 minus lambda. So, this lambda would be subtracted from the diagonal entries.

And with this now we can expand this, so here the 6 minus lambda and then we have this product minus this, then we will take this 2 then, minus 2. So, with this value of this determinant here which will be coming as; when we do the factorization of this polynomial that will become 2 minus lambda lambda minus 2 and the lambda minus 8. So, I skip this portion here because in this lecture that is not important, we have already seen for several examples in the last lecture.

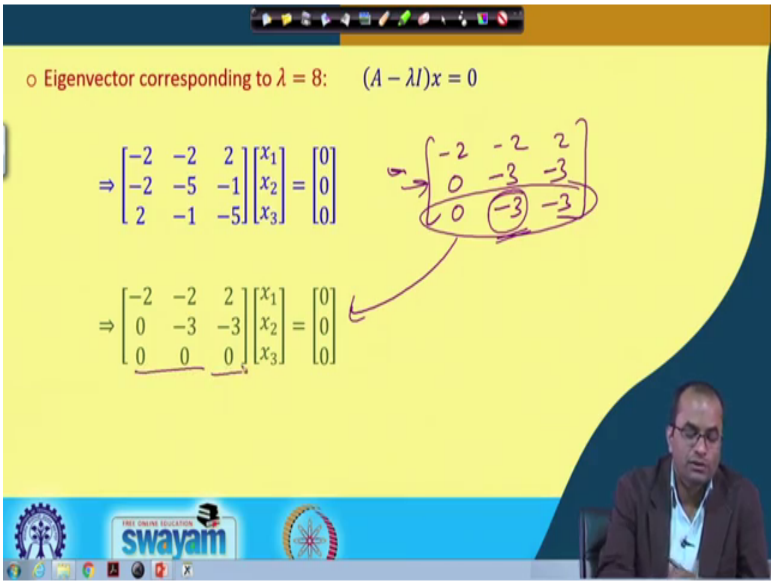
(Refer Slide Time: 04:18)



So, what we have? We have this characteristic equation of this matrix here as this 2 minus lambda lambda minus 2 and lambda minus 8 equal to 0. So, what we observe now that there are two distinct eigenvalues and one is repeated 2 times so; that means, the lambdas are the 2, 2 and 8. So, this eigenvalue 2 is repeated 2 times and this 8 is repeated 1 times, and exactly that is what we have discussed about this algebraic multiplicity.

So, the algebraic multiplicity of this lambda is equal to 2 this eigenvalue 2 is because it is repeated 2 times here. So, that I the algebraic multiplicity of this 2 is 2 and the algebraic multiplicity of 8 because this is repeated only once. So, here the algebraic multiplicity of this lambda is equal to 8 is 1. So, this is how the algebraic multiplicity and the geometric algebraic multiplicity is defined to define the geometric multiplicity corresponding; to lambda is equal to 2 or lambda is equal to 8, we need to get the eigenspace of these vectors of these eigenvalues.

(Refer Slide Time: 05:32)



So, the eigenvector corresponding to this lambda is equal to 5. So, remember this lambda is equal to 8 was repeated once. Though, we have already the result that the eigen the geometric multiplicity cannot be more than 1 now in this case, because the algebraic multiplicity of this lambda is equal to 8 is 1. So, without calculation of the eigenvectors as well we can claim that the geometric multiplicity will be 1, because definitely there will be one linearly independent eigenvector corresponding to this lambda is equal to 8.

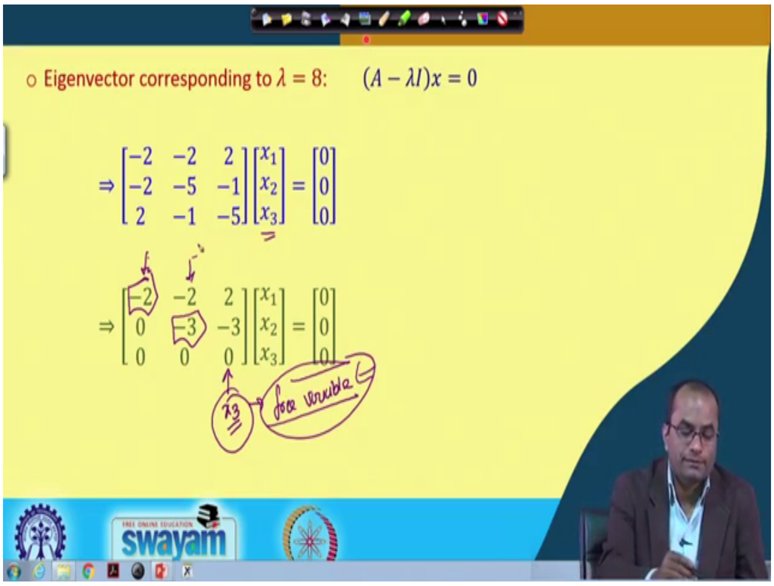
So, there cannot be two linearly independent eigenvectors, that was that result says where we have that these algebraic multiplicity is always bigger than the geometric multiplicity. So, here we know though beforehand that this there will be there cannot be two linearly independent vectors, it has to be only one because the algebraic multiplicity of this lambda is equal to 8 is 1.

And it had it just must to have at least 1 eigenvectors because that is what the foundation says so, we have already this a minus lambda I and the determinant is equal to 0 we have non-trivial solution always. For this equation here A minus lambda x is equal to 0. So, there will be definitely one linearly independent eigenvector, but there cannot be two this is what we will see in this case as well.

So, here this is a minus lambda I; so, this lambda means 8 here was subtracted from the diagonal entries of A and then we have x 1 x 2 x 3 and the right hand side this 0 vector. So, we can reduce to this echelon form this matrix and so, 2 minus 2 2 first row as it is and then here we can subtract this. So this will be 0, when you subtract will be minus 3 and this is minus 3 and here also we can add in the first step. So, this will be 0, this will be minus 3 and this will be minus 3 and then in the second step again with the help of the second column.

So, in the first step what do we get, it is like minus 2 minus 2 2 and then here we have 0 and then minus 3 and minus 3 here we get when we add row 1 and row 3. So, we will get 0 well get minus 3 and minus 3. So, again with the help of the second row, we can now get actually get rid of this number here minus 3, but this will also become 0 together. So, this is the situation, this is the row reduced echelon form for the system of equations for this matrix.

(Refer Slide Time: 08:21)

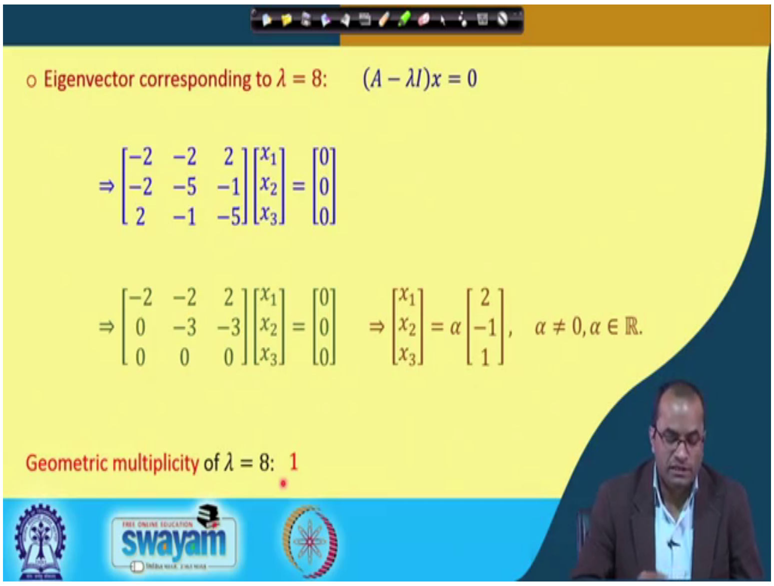


And now, we observe here that this is the pivot element and here also we have the pivot element. So, the first two columns have pivot element that third column does not have pivot element. So, that corresponds to this x 3 component of this vector and which we can take as the free variable. So, there will be only one free variable which was clear from there also because the algebraic multiplicity was one and corresponding to that we cannot get two free variables. So, the number of free variables tells about the number of linearly independent eigenvectors.

So, here we cannot have two linearly independent eigenvector. So, we know in advance that there will be only one free variable in this case, there cannot be two free variables.

So, this x 3 is the free variable which we can choose again as as alpha; having that alpha we can compute the x 1 and x 2 in terms of of alpha.

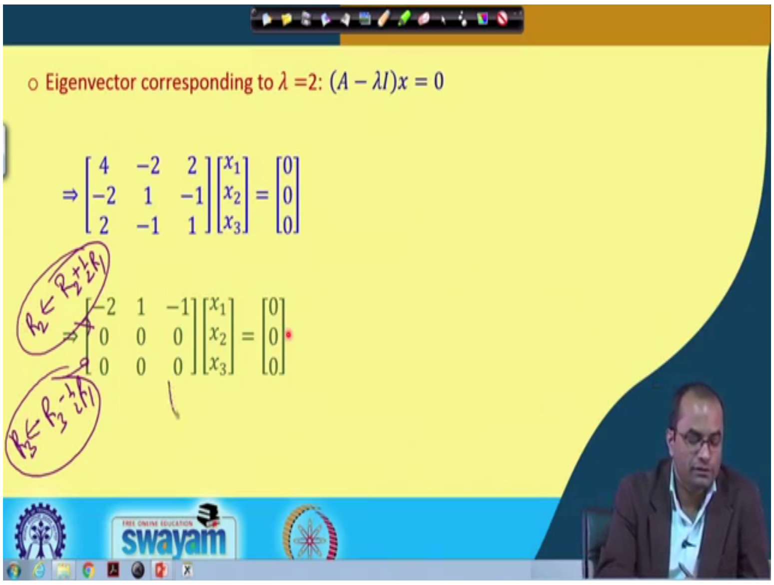
(Refer Slide Time: 09:20)



So, then we can write down this solution of this equation x 1 x 2 x 3. So, here this x 3 was taken as alpha and x 2 comes to be from here the minus of x 3. So, we got this minus alpha and from this equation number 1, we will got this two time; we will get this 2 times alpha, this vector here x 1. So, we have this alpha not equal to 0 and alpha belongs to this real number we can take any real number here. So, this is for any alpha not equal to 0 these are the eigenvectors. And basically the dimension of this eigenspace is one or in other words we got only one linearly independent eigenvector which we can take for instance this is 2 minus 1 1.

So, that is the only one eigenvector which is linearly independent, any other eigenvector which we get out of taking this value alpha where they are the dependent eigenvectors on this 2 minus 1 1. So, in this case we got only one linearly independent eigenvector and therefore, we say that the geometric multiplicity. The geometric multiplicity that is the number of linearly independent eigenvectors corresponding to a given eigenvalue here the lambda is equal to A. So, the geometric multiplicity of this lambda is equal to 8 is 1. So, here that is the number of linearly independent eigen vectors.

(Refer Slide Time: 10:48)



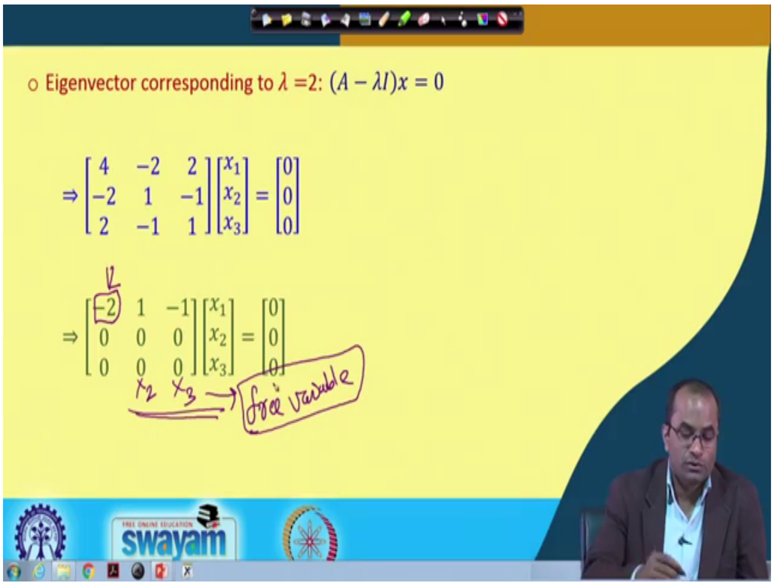
When we come to this eigenvalues lambda is an; eigenvalue lambda is equal to 2 and remember it was repeated 2 times meaning the algebraic multiplicity of this lambda is equal to 2 was 2. So, in this case we have the possibility that the corresponding eigenvectors the corresponding linearly independent eigenvectors there may be 2, I mean at most 2, but there may be 1 as well, we do not know now in advance we have to compute them.

Because, looking at this eigenvalue we cannot just tell how many eigenvectors will be linearly independent corresponding to a given eigenvalue, but what we can tell now because the multiplicity of this 2 was 2 or the algebraic multiplicity was 2 and we know that the geometric multiplicity will be less than or equal to 2. So, we know now that the number of linearly independent eigenvectors could be 1 or it could be 2 also now in this case.

So, like let us compute this. So, this a minus lambda I when we subtract from this diagonal entries this number lambda. So, we get this equation the system of linear equation and then by reducing to this echelon form. So, indeed these two rows are the same. So, we can set one of them equal to 0 immediately and out of this first row again because it is half of this is again when we add to this row number 2. So, this will become 0 and similarly row number 3, if we subtract half of the row number 1 this will also become 0. So, this is here the operation we have taken that the R 2 is nothing but R 2 plus half of R 1 and here for R 3 we have taken now the R 3 and minus the half of R 1.

So, with this two operations; two elementary operations we have written got this row reduced echelon form of this system of linear equation. And then and now we can identify that how many linearly independent eigenvectors we are going to have in this particular case.

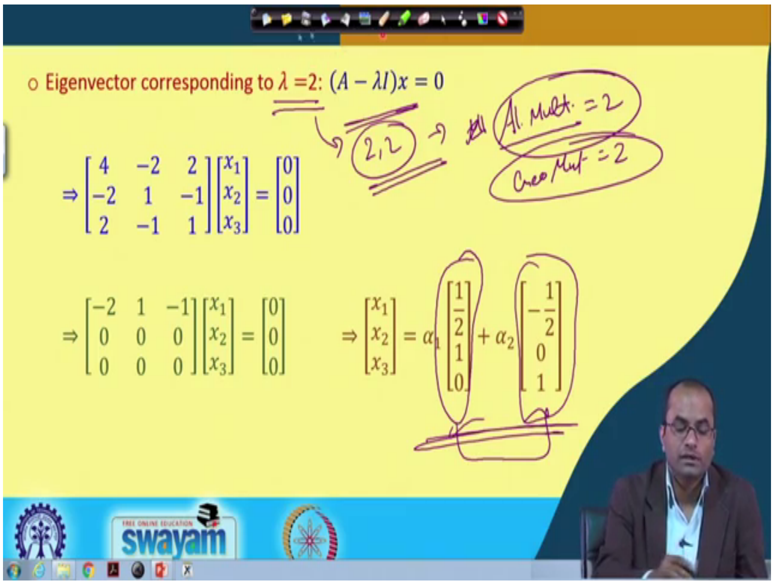
(Refer Slide Time: 13:10)



So, here this is the pivot element which is minus 2 in this case and the column number two does not have a pivot here also we do not have pivot. So, there is only one pivot that is in the column number 1.

So, here x 2 and the x 3; x 2 and x 3 will be will be free variables so, there will be free variables now free variables. So, we can assign any value to them; that means, we are going to have now two linearly independent eigenvectors because a number of free variables decide exactly how many linearly independent eigenvectors we will get.

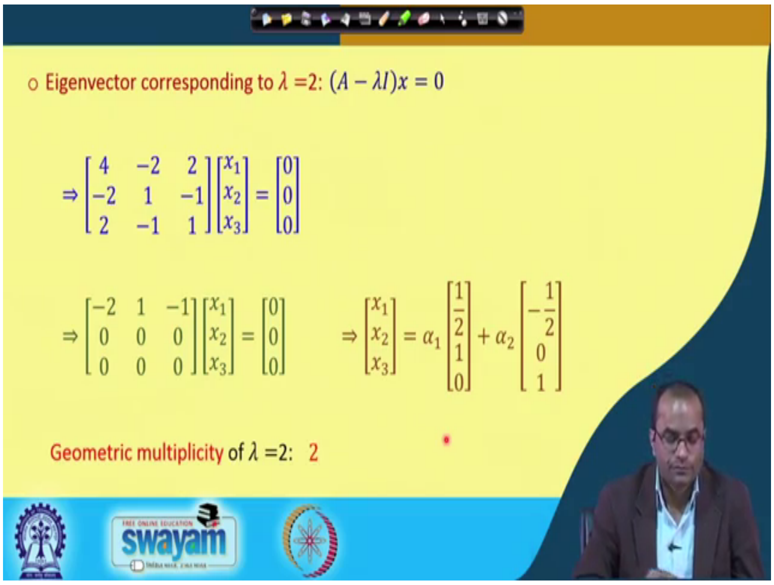
(Refer Slide Time: 13:55)



So, in this case we will get two linearly independent eigenvectors, and that is what we write. So, here x 2 is taking alpha 1, x 3 is taken as alpha 2 and then we have computed this alpha 1 from; this x 1 from this equation number 1 when we write in the vector form as this alpha 1 we have this half and 1 0, alpha 2 a minus half 0 1. So, we got this two linearly independent eigenvector. So, this one and this one, one can check where they are linearly independent.

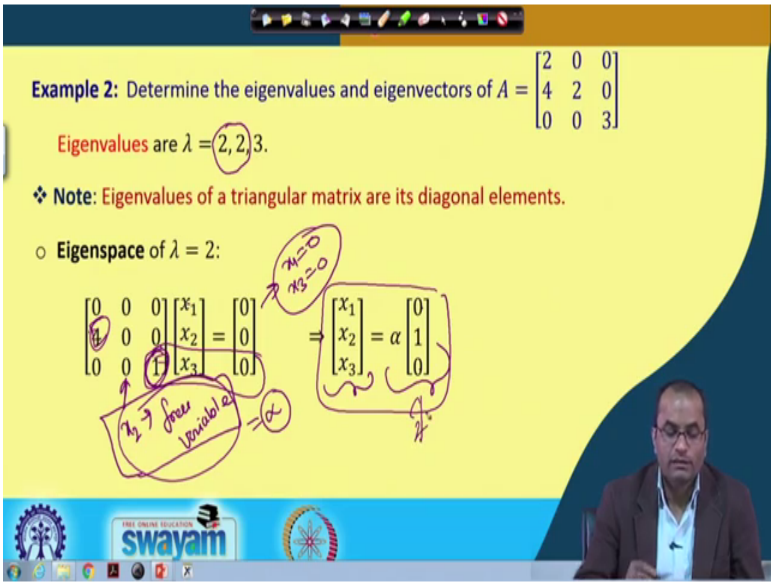
So, corresponding to those lambda is equal to 2 because it was repeated this 2 times the algebraic multiplicity was 2, this algebraic multiplicity of this was 2 and we also got now the geometric multiplicity. So, the geometric multiplicity is also 2 in this case. So, we have the algebraic multiplicity 2 and as well as the geometric multiplicity 2; geometric cannot be more than the algebraic one again, but in this case we got at least the equality.

(Refer Slide Time: 15:01)



So, the geometric multiplicity of this lambda is equal to 2 is 2, because we have two linearly independent eigenvectors corresponding to this lambda is equal to 2 ok.

(Refer Slide Time: 15:13)



So this example 2, where we determine the eigenvalues and eigenvectors of this A the matrix is given here 2 4 0 0 2 0 0 0 3 and in this case one can compute easily the eigenvalues will be the diagonal entries because it is a lower triangular matrix and for the triangular matrices, we have all the eigenvalues sitting on the diagonals here.

So, we have this 2 2 3 there these are the eigenvalues and the eigenvalues of a triangular matrix are always it is a diagonal element; so, we have these eigenvalues 2 2 3. The eigenspace now we will compute for lambda is equal to 2. Again I mean here, the if we want to know the algebraic multiplicity so it is 2 4 2 and the algebraic multiplicity of 3 is 1. So, here the eigenspace you want to compute now to get the geometric multiplicity.

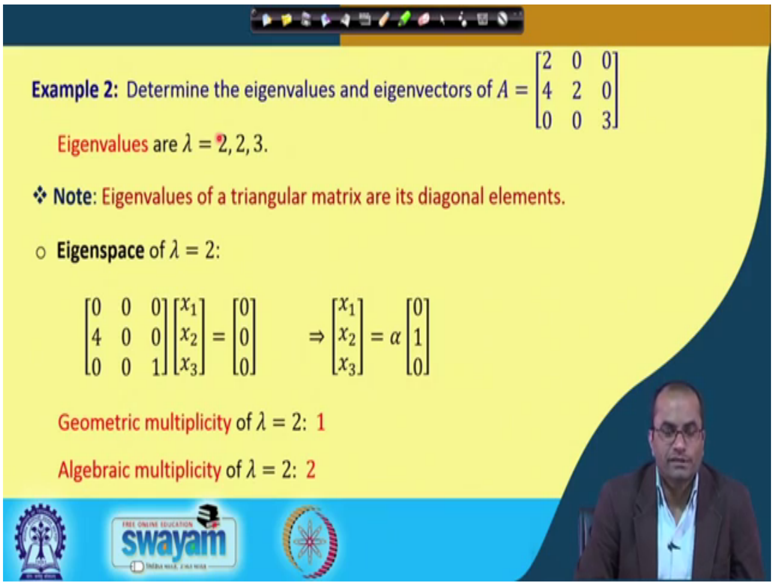
So, here the eigenspace; so, A minus lambda I so here 2 will be subtracted from the diagonal entry. So, we will get 0 there, 0 there, 1 there. So, this is the now the matrix A minus lambda I and x is equal to 0. So, what do we see here, we can actually just take this we can interchange the row and then we have this echelon form; row reduced echelon form the 0 we can bring to the bottom if we like.

So, we can easily convert to this echelon form here and then we will see there will be 2; there will be 2 pivot elements here. So, this will be the pivot element and this will be also the pivot element when we convert into this echelon form. And this middle one so, here the first column will have a pivot and the third column has a pivot and this x 2 is going to be the free variable. So, this x 2 is going to be the free variable; that means, only one free variable and we will get only one linearly independent eigenvector.

And surprisingly here that, when we compute this x 3 is equal to 0 that is straight away from this equation and from this equation, we will get this x 1 is equal to0. So, out of this we are getting x 1 is equal to 0 also x 3 is equal to 0 and this x 2 will be the free variable which we can take as alpha, and then this x 1 x 2 x 3we can write down as alpha times 0 1 0. So, here in this case what we observe though the algebraic multiplicity of this lambda is equal to 2 was 2, but now we got the geometric multiplicity as 1.

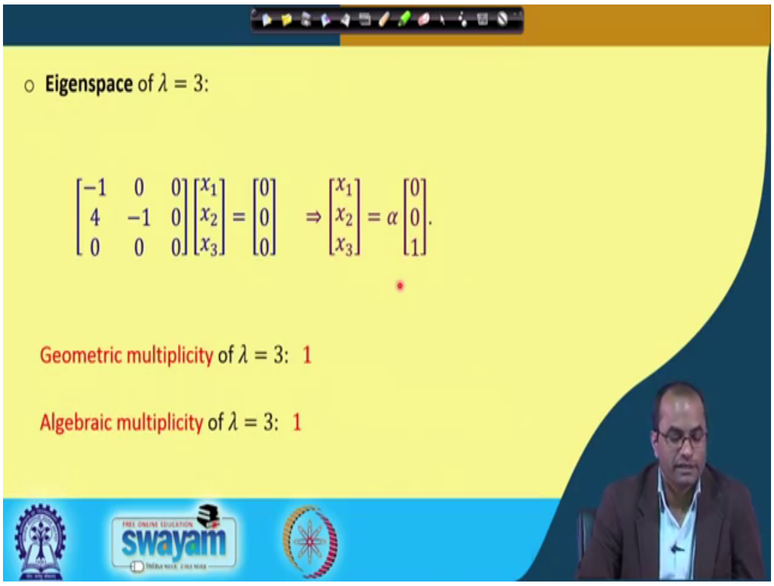
So, not surprising as I said before that for given eigenvalues we cannot predict in advance at how many eigenvalue vectors will be linearly independent so, we have to compute them. What we know from that result that algebraic multiplicity is less than or equal to the, or the geometric multiplicity is less than equal to the algebraic multiplicity that the algebraic multiplicity of this 2 was 2. So, we know that there will be at most 2 linearly independent eigenvectors. There cannot be three linearly eigen linearly independent eigenvectors for example, in this case, but we do not know whether there will be 2 or there will be 1. So, what we have observed? In the previous example though the algebraic multiplicity was 2 and the geometric multiplicity was also 2. In this case we have the algebraic multiplicity 2, but geometric multiplicity is just 1 in this case.

(Refer Slide Time: 18:57)



So, here the geometric multiplicity is 1, because we have 1 linearly independent eigenvector and the algebraic multiplicity of 2 is 2 because this 2 was repeated 2 times.

(Refer Slide Time: 19:11)

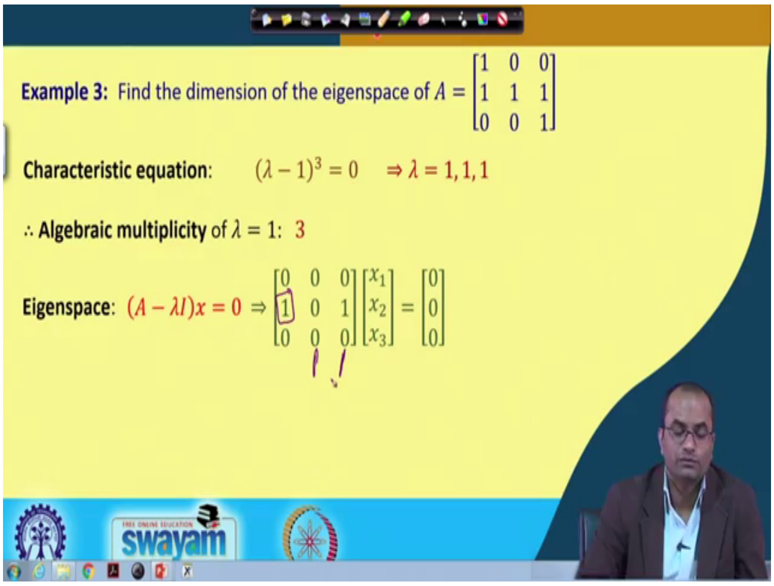


Coming to the eigenspace of this lambda is equal to 3. So, we know already that there would be only one. So, in this case we know that there will be only one free variable definitely because the algebraic multiplicity is 1. So, we cannot have more than 1 linearly independent eigenvector.

So, here if we compute this a minus lambda I x is equal to 0. So, we have this and then when we solve this system. So, we will observe that there are 2 pivots in this case, when we just we can just make this to 0 and then this will become also a pivot because this will not be 0 in that case. So, you will have 2 pivot elements and this x 3 will be the free variable in this case.

So, therefore, this alpha is corresponding to x 3 and this x 2 will be 0 and x 1 will be also 0 from this structure of the matrix. So, we will get the solution alpha x 0 1 1 and as expected or there is only one linearly independent eigenvector corresponding to this lambda is equal to 3. So, the geometric multiplicity of this lambda is equal to 3 is 1 and the algebraic multiplicity of this lambda is equal to 3 was also 1 in this case.

(Refer Slide Time: 20:35)

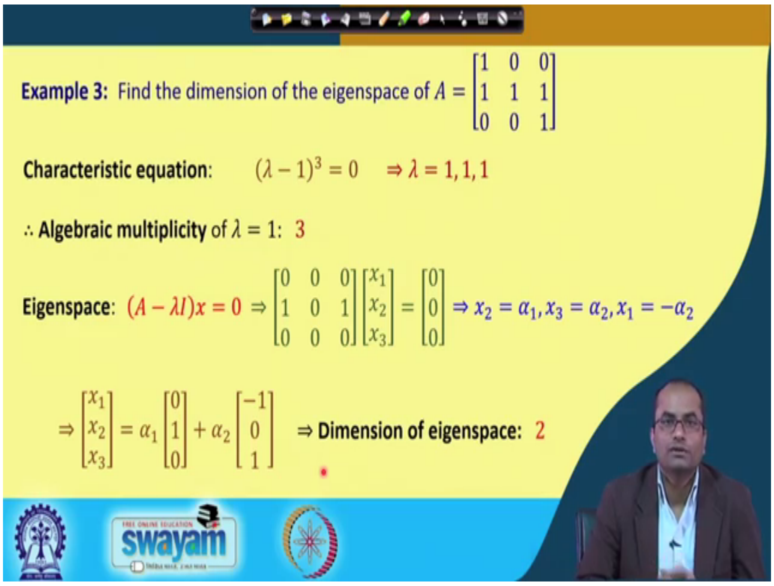


Another example we will find the dimension of this eigenspace of this lambda is equal to this very special matrix here 1 0 0 1 1 1 and 0 0 1. So, in this case again we need to write the characteristic equation. So; that means, a minus lambda I is equal to 0; so, 1 minus lambda here, 1 minus lambda and 1 minus lambda and that determinant.

So, what we will observe in this case that the characteristic equation is lambda minus 1 power 3 is equal to 0. So, we have these 3 roots; so, 1 1 1. That means, this algebraic multiplicity of this lambda is equal to 1 is 3 now. So, we have an example where all these we have the same eigenvalues, but repeated three times. When we compute the eigenspace here meaning we have to compute the eigenvector so with this equation a minus lambda ix is equal to 0. So, what will happen in this case? That when we take this minus lambda I so, minus 1 from the diagonal entries.

So, this diagonals will be also 0 and we have this very a simple example and in this case, there will be only 1 pivot. So, this first column will have pivot and the second and the third one will be the free variables. So, this is going to be the pivot element and then nothing else. So, we have the free variable we have the free variable. So, there are two free variables, meaning 2 linearly independent eigenvectors.

(Refer Slide Time: 22:12)

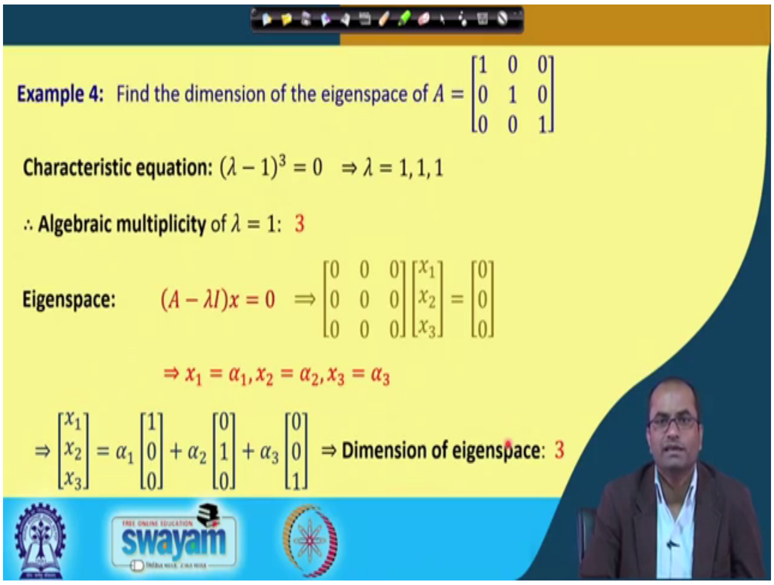


So, here for x 2 be sine alpha 1, x 3 we sine alpha 2 and from this equation which says x 1 plus x 3 is equal to 0; means, x 1 is equal to minus x 3. So, we get here x 1 is equal to minus alpha 2. So, when writing in this vector form we have x 1 x 2 x 3 as alpha 1. So, this component here is 0; so 0 and then the second place will be 1 and then 0 for alpha 2 at 1 place you have minus 1 and then 0 and x 3 is alpha 2 here so 1.

So, we have this x 1 x 2 x 3 as alpha 1 times is 0 1 0, alpha 2 times minus 1 0 and 1. So, there are 2 linearly independent eigenvectors corresponding to this eigenvalue 3; so, eigenvalue 1 here, which was repeated 3 times. So, the algebraic multiplicity of this lambda is equal to 1 was 3 and the geometric multiplicity of this lambda is equal to 1 is 1 oh sorry 2.

So, they are 2 linearly independent vector. So, the dimension of this eigen space is 2 or the geometric multiplicity of this lambda is equal to 1 is 2, in this case. So, again though it was repeated 3 times, but we got only this dimension as 2 not 3 and not 1, but the possible values here could be 3, it could be 2 as this is the case here, but it can be 1 as well.

(Refer Slide Time: 23:49)



In this example again we will take this identity matrix. So, very simple to evaluate so we have, we want to find the dimension again of the eigenspace of this A is equal to this identity matrix. And if we write down its characteristic equation, we will get this lambda minus 1 power 3 is equal to 0; so, again we have this 1 1 1 which the algebraic multiplicity of this eigenvalue is just 3 now.

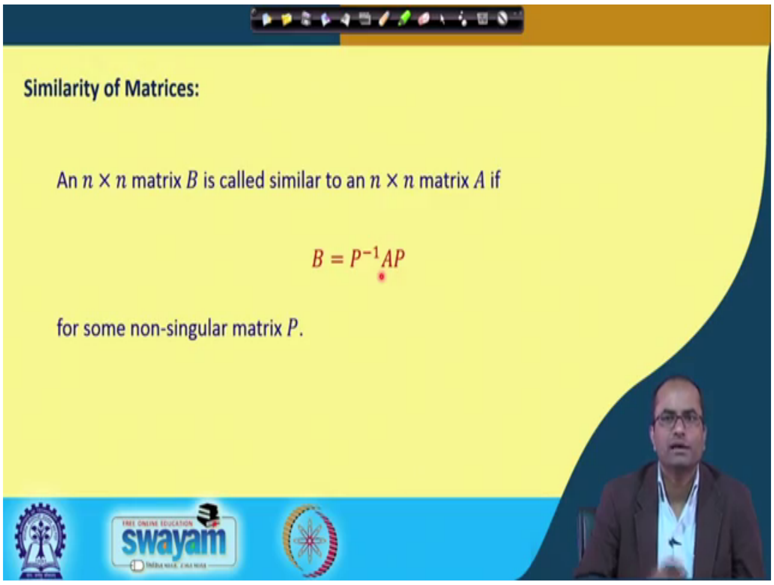
So, corresponding to this 1 so; algebraic multiplicity is 3 and if we compute the geometric multiplicity now that is interesting. So, the eigenspace will be computed by this A minus lambda I, x is equal to 0 and therefore, when we subtract from the diagonal entries this eigenvalue 1. So, what we will get this 0 matrix here and x 1 x 2 x 3is equal to again the 0 matrix.

So, what do we see now in this case? That there is no there is no pivot here; there is no pivot here and all the variables x 1 x 2 x 3 they are the free variables. So, we can choose, we can assign any value to x 1 x 2 x 3 they are free here and that is a very special case which we have just seen now, that we got the 0 matrix here as A minus lambda I and then we have the possibility of choosing this x 1 x 2 x 3 freely. So, whatever we like and we have taken alpha 1 here alpha 2 and there alpha 3 because all three are free variables.

And then this x 1 x 2 x 3 we can write down in terms of these alpha 1, alpha 2, alpha 3 as this combination alpha 1 this 1 0 0, alpha 2 0 1 0 and alpha 3 will be 0 0 1. So, we have three linearly independent eigenvector in this case corresponding to this lambda is equal to 1. So, the algebraic multiplicity of lambda 1 was 3 and also the geometric multiplicity which is the dimension of the eigenspace that is also 3 in this case. So, we have seen in this example that, if it is repeated 3 times it is also possible that we can get the full dimension, here the dimension of the eigenspace that is 3.

As many times as the lambda was repeated, but what that result says that it cannot be more than 3. And naturally, that is the case here because the dimension that is the full dimension because the elements belongs to this R 3 and we cannot have the dimension more than 3 in that sense also we can conclude here.

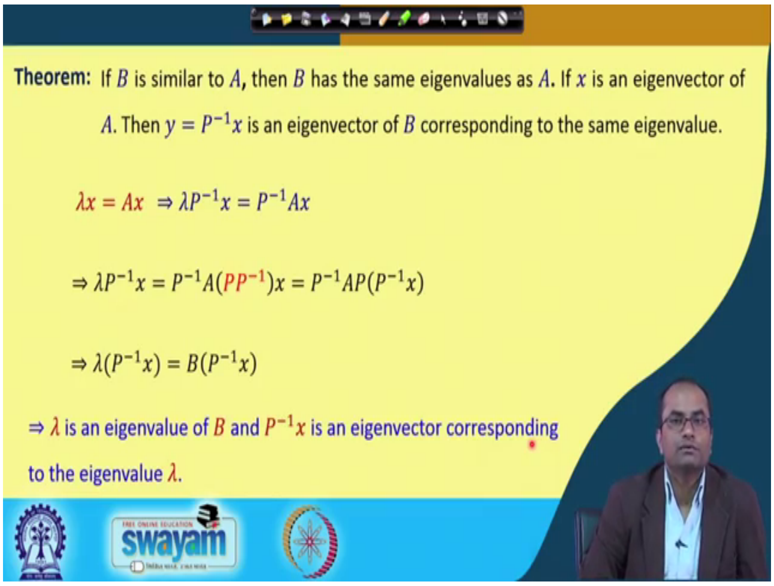
(Refer Slide Time: 26:38)



There is a concept here the similarity of matrices which will introduce here and we will continue for the discussion in the next lecture. So, an n cross n matrix B is called similar to an n cross n matrix A, if we have this B is equal to P inverse AP. If we have this relation between the between the matrix A and B, then we call this P is similar to the matrix A or A is similar to the matrix B. And what to we have to we this P what is the P before some non singular matrix P, if there exists a matrix here this P inverse I mean, this non singular matrix P therefore, that P inverse make sense.

So, if we have this relation between the 2 matrices here B and A that P inverse AP gives the B the other matrix, then we call that these two are similar. Why do we use the similar words some of the properties we will check today itself, that they share away many common properties this B and A in terms of the eigenvalues, eigenvectors and there are other considerations as well which we will continue in the next lecture.

(Refer Slide Time: 27:46)



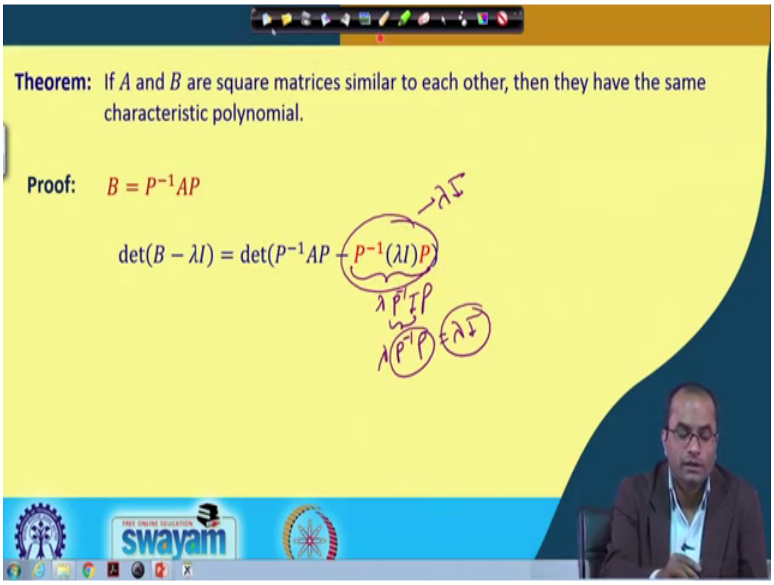
So, today we will see that if B is similar to A, then the B has the same eigenvalues as A and if x is an eigenvector of A. Then this y is equal to P inverse x is the eigenvector of B corresponding to the same eigenvalue. So, meaning if we know the eigenvalues and eigenvector of one, we can get the eigenvalues, eigenvectors of the other. In fact, they same they have the same eigenvalues and the eigenvector also will be just the P inverse x where P we have introduced already in the similar t definition. So, what we take that let us say this lambda is the eigenvalue of this matrix A and the similar to A, we have the B matrix. So, first of a relation we have for this A that x is the eigenvector and lambda is the eigenvalue.

So, we have this relation Ax is equal to this lambda x. And what we do now? We multiply by this P inverse here. So, the right hand side we have P inverse, P is that matrix which we are talking about the similarity there. So, we have P inverse e Ax and here also P inverse. So, the lambda is constant so, we have P inverse x there. And then what we do, we have here the lambda P inverse x the same, the P inverse A again we have introduced this identity matrix.

So, here we have introduced identity matrix which we have written as P and P inverse x and then what we do, this we combine here P inverse AP; P inverse AP and then we have P inverse x. So, what do we see now, this P inverse AP as per the definition of the similarity that A similar to the or B is similar to A; that means, this B we can write as P inverse AP. So, this we have this lambda P inverse x is equal to B times, this is B P inverse x. So, what we observed now from this relation, that this is the eigenvector P inverse x and this lambda is the eigenvalue of this B.

So, if this B is similar to A the B will have the same eigenvalue as A because this lambda was the eigenvalue of A and the eigenvector will be this P inverse x. So, we can get the eigenvector and eigenvalue of the similar matrices if we know for one. So, lambda is an eigenvalue of B and P inverse is the eigenvector corresponding to the eigenvalue lambda.

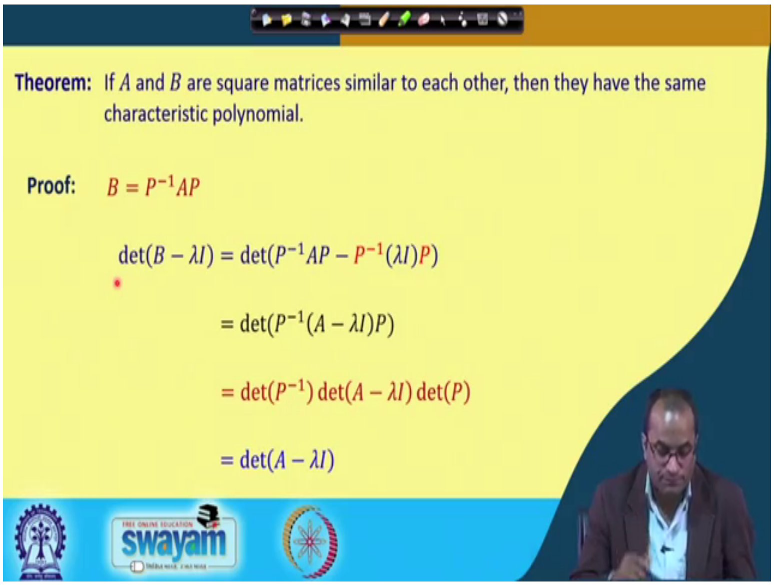
(Refer Slide Time: 30:29)



Another result which actually we have seen already in this first result A B are the square similar matrices, then they have the same characteristic polynomial. So, eventually we have seen already that they have the same eigenvalue. So, if we have the same eigenvalues meaning they have the same characteristic polynomial, but this is just another way of looking at it.

So, we take this B is P inverse AP this relation and then if we get this determinant B minus lambda, if that is the characteristic polynomial B minus lambda I the characteristic polynomial of this B here is equal to the determinant this B, we will replace by this P inverse AP minus the same thing this P inverse P that is the identity matrix we have introduced here. I mean, you can see easily that, this is nothing but the lambda I because lambda we can take common, then we have P inverse I P and then P inverse I is equal to nothing but the P inverse P lambda times and this is I. So, lambda times I so, we have again here this is nothing but the lambda times I only, but we have rewritten in this form that P inverse lambda I and P.

(Refer Slide Time: 31:46)

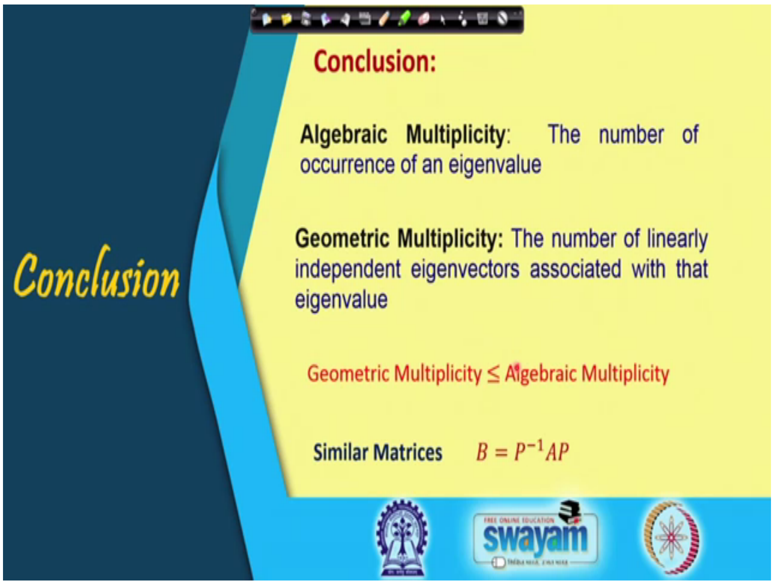


So, here the determinant we have P inverse let us take common from both and then we have A here and minus this lambda I and P from this right hand side, we can take as common. So now, this A minus lambda I and then, this we can use the property of this determinant here the product of these three matrices; that means, the determinant of P inverse determinant of this middle one A minus lambda I and the determinant of P.

So, here the determinant of P inverse and determinant of P will cancel out each other, we will get just one here and what we will get that is the property of this determinant P and P inverse they are just the reciprocal and here we have determinant of A minus lambda I.

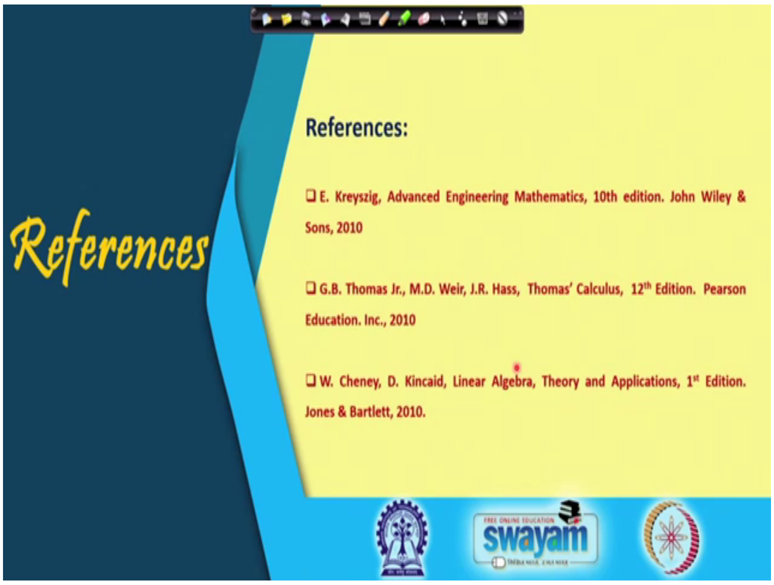
So, what we have seen that the determinant of this B minus lambda I is equal to determinant of this A minus lambda I so, they have the same characteristic polynomial. In other words, we can say again that this A and B will have the same eigenvalues and we have seen again in the previous slide here the relation for the eigenvectors as well ok.

(Refer Slide Time: 32:56)



Coming to the conclusion so, in this lecture we have talked about the algebraic multiplicity and that was nothing but the number of occurrence of an eigenvalue. And we have also seen the geometric multiplicity that was the number of linearly independent eigenvectors associated with that eigenvalue. And always this is the case that geometric multiplicity is less than equal to the algebraic multiplicity and we have also talked about the similar matrices; that means, B and A are called the similar to each other they are the similar matrices, if we have this relation that B is equal to P inverse AP for some invertible matrix P.

(Refer Slide Time: 33:40)



So, these are the references used to prepare these lectures and thank you for your attention.