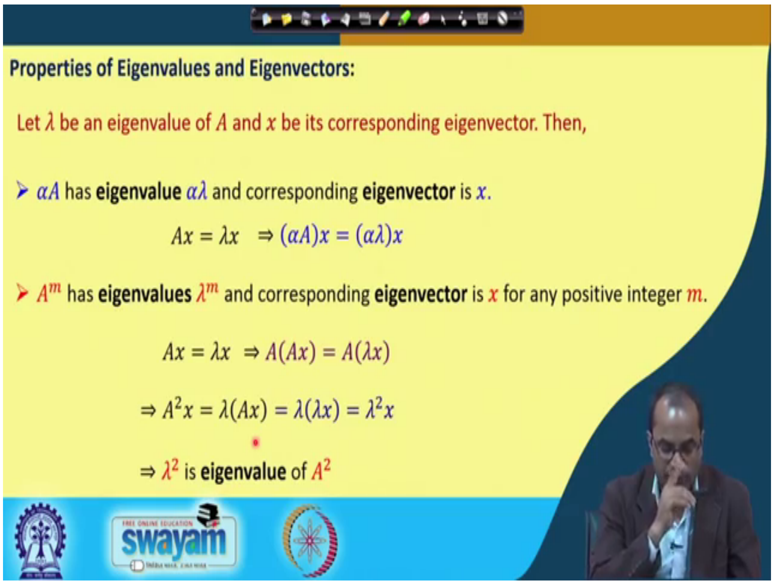
Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 48
Eigenvalues & Eigenvectors (Contd.)

So, welcome back and this is lecture number 48 and today we will talk about the properties of Eigenvalues and Eigenvectors.

(Refer Slide Time: 00:26)



So, here the first property so if we have a lambda as an eigenvalue of A and then x be its corresponding eigenvector; then, this alpha A has eigenvalue lambda A and the corresponding eigenvector is x. So, what is this property that if lambda is the eigenvalue of A and x is its corresponding eigenvector in that case the alpha times a alpha is some constant some scalar quantity from the set of real numbers.

For instance so here the alpha A will have the eigenvalue alpha lambda. So, this alpha will be multiplied simply to lambda and the corresponding eigenvector will remain the same that is x this property we can easily verify. So, here Ax is equal to lambda x that is the relation we have for eigenvalues and eigenvectors.

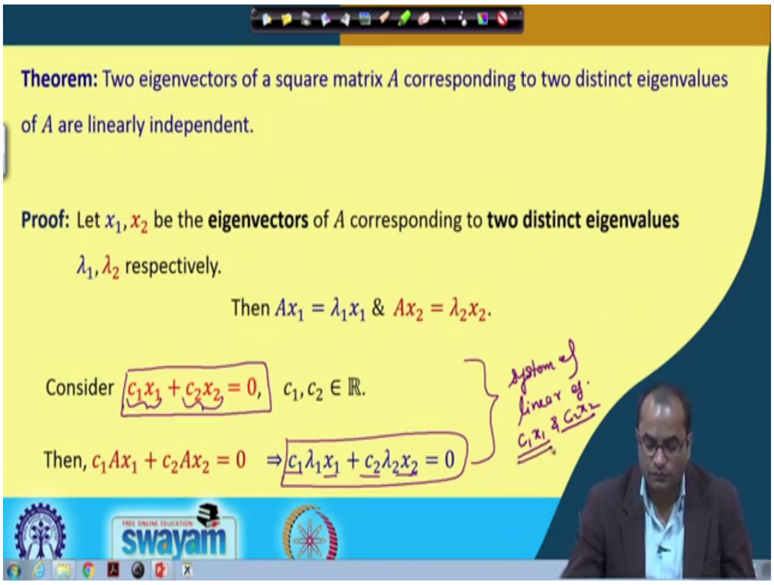
So, if you multiply by alpha here both the sides so alpha A and this will be alpha and then this alpha a we are treating as matrix. So, alpha will be multiplied to each of the entries of A and here alpha times lambda. So, with this relation what do we see that this alpha A has the eigenvalue lambda alpha lambda and x as the eigenvector from this relation the matrix times x should be some scalar into x.

So, that is from here we conclude that this alpha lambda is the eigenvalue of this matrix alpha A with the eigenvector same as before that is x. So, another property of the eigenvalues eigenvectors we have if A power m, so we are multiplying this m times so A power m has eigenvalues lambda power m and the corresponding eigenvector is x again for any positive integer m. So, if we have for instance A square so its eigenvalue will be just the lambda square with the same eigenvector as the A has.

So, this also we can easily see because we have this Ax is equal to lambda x and now we can multiply for example, just to see this result for A square. So, we multiply here A both the sides and the right hand side here this lambda it is a constant here we can bring to the outside. And then we have this Ax there and the Ax we can replace again by lambda x by this relation Ax is equal to lambda x. So, if you replace your Ax by lambda x then what will happen we have this lambda into lambda x that is lambda square x.

So, what relation we have now here A square x that was the multiplication of this A with A. So, A square x is equal to lambda square x which tells that this square the matrix a square has the eigenvalue here lambda square and lambda was the eigenvalue of A. So, with this relation we can easily get the eigenvalues of the A square or A cube or A power any integer m. Because it will just the power will go to the eigenvalue and the remaining the eigenvector will be also the same which was the eigenvector of this lambda for matrix A.

(Refer Slide Time: 04:17)



Now the next property so here we have seen this lambda square is the eigenvalue of A. Now, the two eigenvectors of a square matrix A corresponding to two distinct eigenvalues of A are linearly independent. So, what we will prove here that two eigenvectors corresponding to two distinct eigenvalues are always linearly independent. This observation we already have seen for numerical examples, but we can prove here for more general matrix for any matrix we can prove this result theoretically.

So, what do we consider now so let us say x 1 and x 2 be two eigenvectors of A corresponding to two distinct eigenvalues lambda 1 and lambda 2. So, this lambda 1 and lambda 2 these are two distincts eigenvalues we have assumed and their corresponding eigenvectors are denoted by the x 1 is corresponding to lambda 1 and x 2 is the corresponding to this eigenvalue lambda 2.

So, with then meaning is that we have this relation they satisfy this relation that Ax 1 is equal to lambda 1 times x 1 and Ax 2 is equal to lambda 2 x 2 because they are the pair of this eigenvalues eigenvectors. And now what actually we want to show that this x 1 and x 2 are linearly independent.

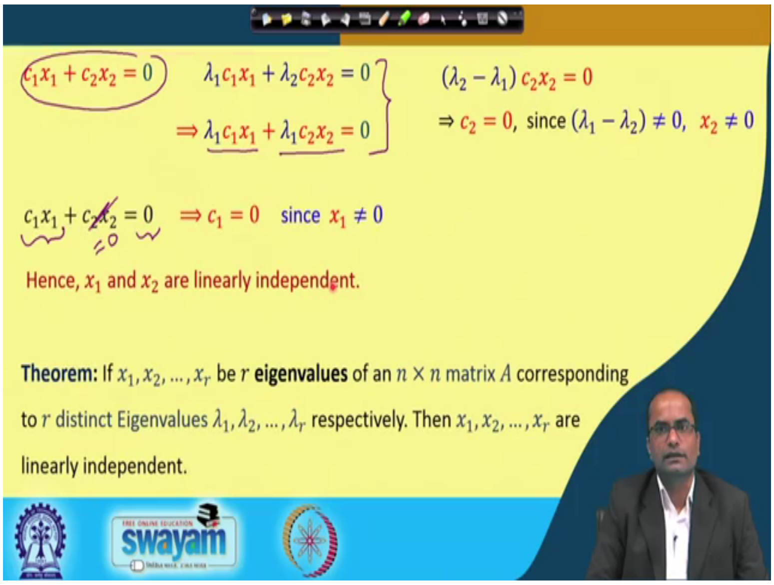
So, for that you will consider this linear combination c 1, x 1 plus c 2 x 2 is equal to 0 and. So, this is the 0 vector here the right hand side and then we will show that this will imply that this is true this linear combination is 0 this is true only when x c 1 is 0 and c 2 is 0, that shows that this x 1 and x 2 are linearly independent.

So, to do so we will consider here the c 1 and this x 1. So, we have multiplied basically by the matrix this A by the given matrix A. So, we have c 1, Ax 1 because c 1 was a constant, so we have taken out here. So, c 1 times Ax 1 and then c 2 times Ax 2 that is so c 1 times Ax 1 and the c 2 times Ax 2 this is the relation we got from this equation by just multiplying this to A.

Well, so the next having so here now we have Ax 1 and we have Ax 2 right there which we can replace by lambda 1 x 1 and the lambda 2 x 2. So, we have basically these two equations now, one is the c 1 x 1 plus c 2 x 2 is equal to 0 and the another equation we have here c 1 lambda 1 x 1 plus c 2 lambda 2 x 2. Indeed this unknown we can consider as c 1, x 1 and the another one c 2 x 2.

So, here also we have c 1 x 1 and here also we have c 2 x 2. So, these two forms a system of a linear equation; system of this linear equation and with unknown here the unknowns are the c 1 x 1 and c 2 x 2. So, these are the two linear equations or they form the system of linear equation with unknowns so c 1 x 1 and c 2 x 2.

(Refer Slide Time: 07:54)



With this so we have two equations now this c 1 x 1 plus c 2 x 2 is equal to 0 and with this lambda 1 c 1 x 1 plus lambda 2 c 2 x 2 is equal to 0. So, we want to solve now for this c 1 x 1 and c 2 x 2. So, what we do we multiply this equation number 1 here. So, we multiply this equation by lambda 1. So, if you multiply here then we have lambda 1 c 1 x

1 and lambda 1 c 2 x 2 and now with these two equations we can subtract this equation number 2 from this equation 1.

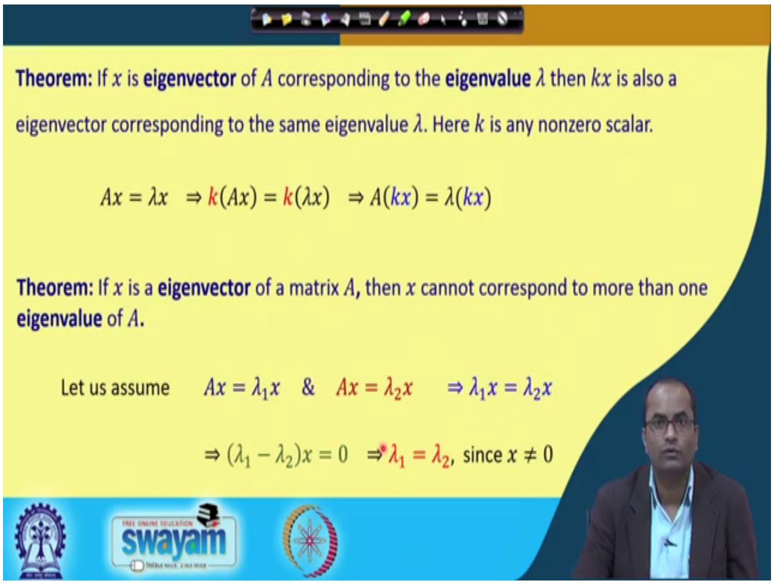
And then what we will get this lambda 2 minus this lambda 1 because this term will cancel out. So, we will get lambda 2 minus lambda 1 with this c 2 x 2 is equal to 0. And now what we see that this equation will imply simply that c 2 equal to is equal to 0 because this lambda 2 minus lambda 1 cannot be 0.

Because we have two distinct eigenvalues and this x 2 is the eigenvector which again is a non-zero vector. So, this equation implies that c 2 must be 0 as this lambda 1 minus lambda 2, or lambda 2 minus lambda 1 is not 0 and x 2 is also not 0. And then from this equation number 1 again here c 1 x 1 plus c 2 x 2 if we substitute this c 2 equal to 0.

So, this term will be will be 0 and then we have this relation that c 1 x 1 is equal to 0 and again with the same argument because this x 1 cannot be 0. So, here again this implies that c one is equal to 0, since this x 1 is not equal to 0. So, with these we have our now the c 1 0 and c 2 0 and that was the aim to show that in this linear combination c 1 x 1 plus c 2 x 2 equal to 0 is possible when this c 1 is 0 and c 2 is 0, meaning that these eigenvectors are linearly independent. So, the eigenvectors here x 1 and x 2 both are linearly independent.

So, this was the case when we have considered two distinct eigenvalues, but we can also generalize this case for more eigenvalues for instance here, we have eigenvalues x 1, x 2, x 3, x r are corresponding to our distinct eigenvalues here. So, these are the eigenvectors corresponding to these eigenvalues lambda 1 lambda, 2 lambda respectively and in that case also we can use the similar trick to prove that these eigenvectors are linearly independent. So, we have this very nice result that corresponding to distinct eigenvalue the eigenvectors are linearly independent.

(Refer Slide Time: 10:53)



So, another result we have they said if x is an eigenvector of A corresponding to the eigenvalue lambda, then this kx is also the eigenvector corresponding to the same eigenvalue lambda. So, this we have also seen before that for the given eigenvector you can multiply by any constant and that will also remain the eigenvector and this is what we will see here, and more formally theoretically that this is true for any matrix.

So, here Ax is equal to lambda x that is a relation, so it tells us that lambda is one of the eigenvalue and the corresponding eigenvector is x and then this k time, so we multiplied the equation by k here both the sides and then we have k times this Ax is equal to k times this lambda x and then we can combine this like A into this kx and is equal to lambda times this kx there.

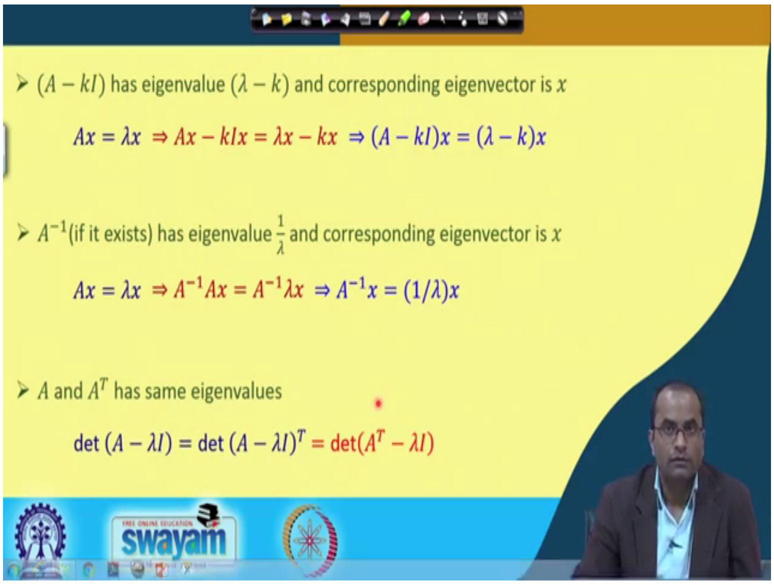
So, we have this relation that A some vector here is equal to lambda the same vector kx which tells us that this kx is the eigenvector again, if the x was the eigenvector the k times x is also the eigenvector for any this k and nonzero scalar. Here if x is the eigenvector of the matrix A, then x cannot correspond to more than one eigenvalue of A. So, another important result that this eigenvector is like a unique.

So, if you have an eigenvector corresponding to let us say the lambda, then this x cannot correspond to any other eigenvalue, so it is a unique in that sense. So, here if we assume that for a given matrix here we have Ax is equal to lambda 1 x, this is our assumption and we also assume that this Ax is equal to lambda 2 x, meaning we have assume that this x the eigenvector x corresponds to two eigenvalues; that means, the lambda 1 and lambda 2.

So, these two eigenvalues correspond to the same in vector x this is our assumption and we will see now that this is not possible. So, having this relation we have actually the lambda 1 x is equal to lambda 2 x because they have the same value here of vector Ax; Ax, so they both are same. So, lambda 1 x is equal to lambda 2 x which tells here the lambda 1 minus lambda 2 times x is equal to 0 and this x is a where eigenvector, so it cannot be 0.

So, naturally we should have here that lambda 1 minus lambda 2 is equal to 0. So, lambda 1 minus lambda 2 is equal to 0 meaning say lambda 1 is equal to lambda 2 since this x is a eigenvector and so our assumption here is this somehow says now if we have taken that there were two eigenvalues. So, these two eigenvalues have to be the same eigenvalues you cannot have 2 distinct eigenvalues which can correspond to the same eigenvector.

(Refer Slide Time: 14:07)



So, here A minus KI has eigenvalue lambda minus k and the corresponding eigenvector is x, so another result which says that this A minus KI so if we subtract this k from the diagonal entries here. So, the eigenvalues will be lambda minus k of this new matrix and the corresponding eigenvector will remain as x. So, to see this again we start with this standard result on the eigenvalues eigenvector; that means, Ax is equal to lambda x.

Having this Ax is equal to lambda x we cannot subtract this kx from both the sides, so here from lambda x we have subtracted this kx and here also the same thing the k x, so this is nothing, but the kx because this ix is just simply x. So, here also we have kx, here also we have kx both the sides we have subtracted this kx here for x we have written ix because we have also the matrix together, so it will be easy now to combine.

So, having this now we can take this x common from this left hand side, so A minus this KI into x and is equal to here also this lambda minus k into x. So, with this relation tells that if this A minus KI has the eigenvalue lambda minus k, so this A minus KI vector your matrix sorry is has the eigenvalue A lambda minus k with the same eigenvector x as before.

So, this is another result that if we have this new matrix which is just the A minus KI, then we know about the eigenvalues from the eigenvalues of A. And this A inverse again important here that if it exists of course, then only we are talking about this result. So, if A inverse exists for a matrix, then this A inverse will have eigenvalue here or eigenvalues one over lambda. So, if we have lambda 1, lambda 2 eigenvalues for instance of A, then A inverse will have 1 over lambda 1, 1 over lambda 2 as an eigenvalue.

So, here the A inverse will have eigenvalue 1 over lambda and the corresponding eigenvector will be x, so eigenvector will not change only the eigenvalue will change for this inverse matrix. To see this result we have this Ax is equal to lambda x and if you multiply by A inverse both the sides, so we have A inverse into Ax the right hand side also we have A inverse into this lambda x. So, we have multiplied both the sides by this A inverse and then what we have this A inverse x is equal to, so A inverse x from this side what we have here? A inverse x.

So, here we have A inverse x and this lambda it is a constant term we can take to the left hand side, where this A inverse A n is just the identity matrix and identity matrix with this x will give us x here and this lambda goes to this left side, so that we will get 1 over lambda. And this A inverse x remains here, so what relation we have now, that A inverse x is equal to 1 over lambda times x. That means, this 1 over lambda here, 1 over lambda is the eigenvalue of this A inverse matrix, so A inverse x is equal to 1 over lambda times x.

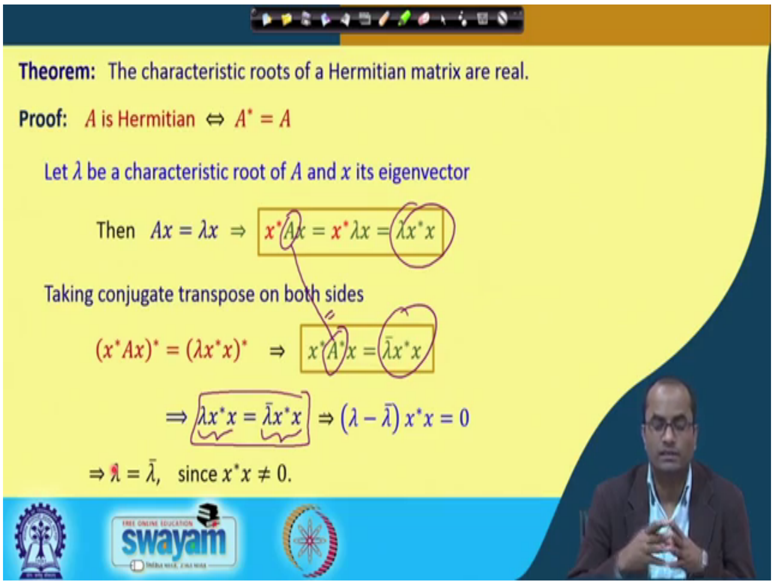
So, A and A transpose have the same eigenvalues, A and A transpose have same eigenvalues and which we can again easily see because the determinant of a minus lambda I that is the characteristic equation which actually gives the eigenvalues. So, here this characteristic polynomial which is A minus lambda I we know the property of the determinant, that the determinant of this matrix A minus lambda I will be the same as the determinant of a minus lambda I transpose.

So, the transpose does not change the determinant of a matrix, so that property we have used here that the determinant of A minus lambda I is equal to determinant of this a transpose of that matrix A minus lambda I. Now the property of the transpose says here that A minus lambda A transpose will be A transpose minus lambda and I transpose which is again I.

So, here this is equal to the determinant of A transpose minus lambda I and that source itself there the determinant of this. So, this characteristic polynomial here A minus lambda I same as the characteristic polynomial of A transpose minus lambda I. And this relation says that we have the same characteristic equation for A and A transpose; that means, they will lead to the same eigenvalues.

So, here the result is that A and A transpose have the same will have the same eigenvalue. So, A and A transpose have same eigenvalue, so that is another important result which easily we can find out with the help of this determinant property.

(Refer Slide Time: 20:02)



Next theorem, so here the characteristic roots I mean the eigenvalues so sometimes we also call the characteristic roots. So, the eigenvalues of Hermitian matrix are real. So, what is the Hermitian matrix? So, we know that A is Hermitian when A star is equal to A meaning the conjugate transpose, A star means here that we are taking the transport and also we are taking the complex conjugate of the matrix A.

So, this complex conjugate of this transpose is equal to A, then we call that A is Hermitian matrix. So, what is this result that for Hermitian matrices the roots are real because what we have also seen that though the matrix having all real entries, but we can get the characteristic roots as complex number we have seen in previous lectures one of the examples where we had a very simple 2 by 2 matrix with real entries. And its characteristic polynomial or I mean the characteristic roots the eigenvalues were non real so the complex.

So, here we have at least the results for the Hermitian matrix that all the characteristic roots are real in this case. So, if lambda be a characteristic root of A and lambda the corresponding eigenvector and then we will show that this lambda has to be real, how? So, we have this Ax is equal to lambda x that is the property of the relation of the eigenvalues eigenvector again.

We multiply here by this x star term, so what is x star again the transpose of x and its complex conjugate. So, we have multiplied by this vector both the sides and then this lambda is a constant term, so we can always take into the front here, so this is x star x. So, we have x star Ax is equal to lambda x star x that is a one relation and now we take the conjugate transpose both the sides of this equation here x star Ax is equal to lambda x star x.

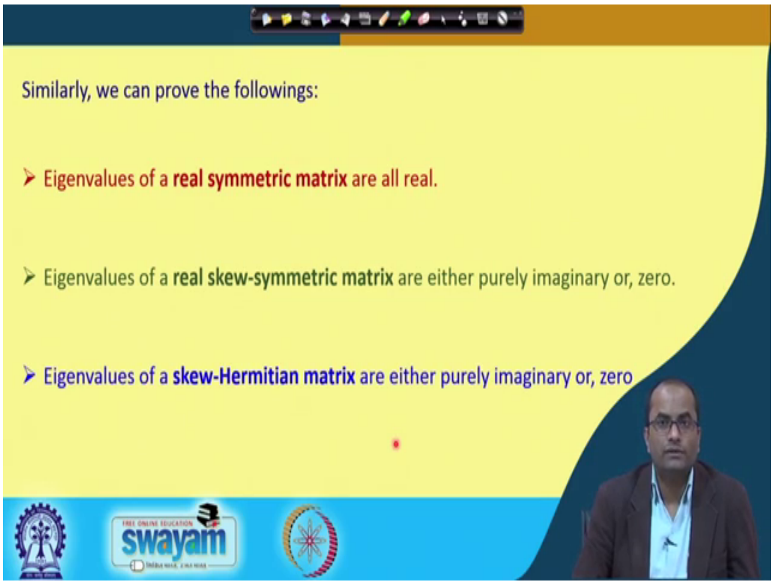
So, what do we get here? X star Ax complex conjugate and here also we take this conjugate transpose again this x star and then we have the properties here that this will be x star and A star and again x star there. So, I mean the x star or star that will be x and here also we will have the same scenario the x star and then the x star star will become x and this lambda will have its conjugate there lambda bar.

So, we have this relation and we have also this relation x star Ax is equal to lambda x star x, we have this relation x star Ax is equal to lambda bar x star x. Just by taking the complex conjugate from this equation we got this equation and now we have these two equations here whose left hand side will be the same because A star is A, so here A here also this A star is A.

So, with these two equations what we can conclude that this right hand side should be equal to 0; that means, lambda x star x is equal to lambda bar x star x, the reason is that this A star and this A are the same here, they are the same. So, naturally the right hand side will be also the same here and we have lambda x star x is equal to lambda bar x star x and this one now what it tells? That this lambda, so we can bring to the left hand side. So, lambda minus this lambda bar x star x is equal to 0 and x is the eigenvector, so that cannot be 0 x star x cannot be 0.

So, here the lambda must be equal to lambda bar because this quantity cannot be 0, so this has to be 0. So, here what we have seen that the lambda is equal to lambda bar and that is what we want to see here that the lambdas are real. So, if lambda is the eigenvalue of Hermitian matrix, then the lambda is equal to lambda bar meaning it is a real number, it cannot be a complex number ok.

(Refer Slide Time: 24:39)

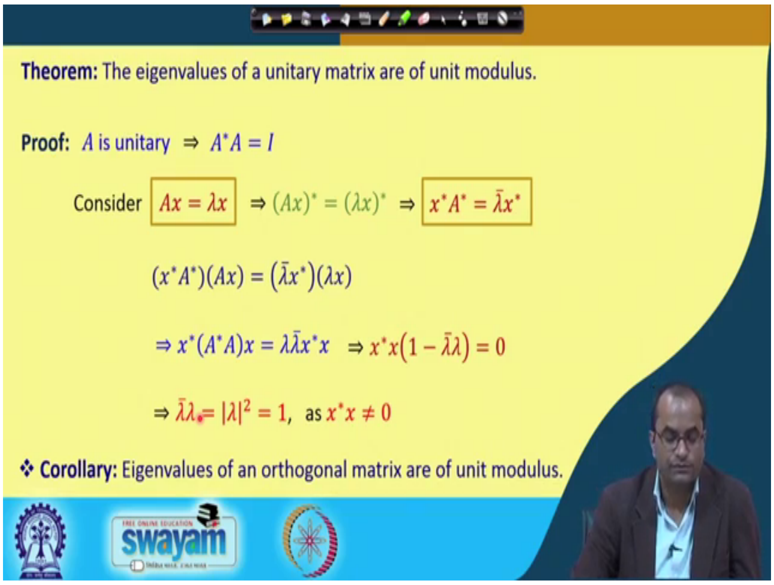


So, another similarly we can prove these following results which have the similar lines of the proof which we have just done, that the eigenvalues of this real symmetric matrix are also real that is what we can also do and the eigenvalues of real a skew symmetric matrix. So, here we have; that means, this a transfer is minus of the A. So, here this real a skew symmetric matrix are purely imaginary.

So, this is also interesting here that eigenvalues of such matrices are either purely imaginary or 0 that is the two possibilities, which again if you follow the earlier proof we can also do this one and the eigenvalues of the a skew Hermitian matrix. So, for Hermitian matrices we have seen, but now there is a skew Hermitian matrix; that means, this A star is equal to minus A.

So, for those cases the eigenvalues are purely imaginary or 0 again. So, these are the consequence of the earlier proof which we can easily see here.

(Refer Slide Time: 25:50)



Now, another important result that the eigenvalues of unitary matrix are of unit modulus, so this also we can prove in general that if you have this unitary matrix; that means, this A star A is equal to is equal to identity matrix, so such matrices are called the unitary matrix.

So, if we have unitary matrix then we will prove now the eigenvalues the modulus of the eigenvalues is 1; that means, if you consider here Ax is equal to lambda x and then taking the complex conjugate here again Ax star is equal to lambda x star what we will get so this again we will use this property.

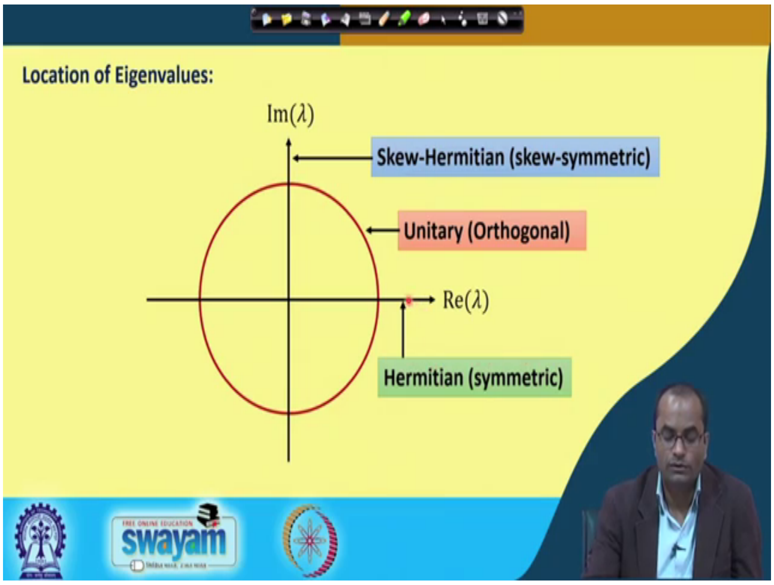
So, x star and the A star is equal to this will be lambda bar x star and from this A star Ax is equal to lambda x and from this equation x star you know A star is equal to lambda bar x star. We will now continue the product here we will take the product, so here x star A star the product with this Ax these two these two vectors.

So, x star A star multiplied by this vector Ax is equal to this lambda bar x star and multiplied by this lambda x. So, we have done just the product here of these two and then what we see here x star and this associativity, so we can use this aced a together and then x here lambda bar this lambda is a constant, so we can easily take out.

And then we have x star x there and then we can take this common because this A star A is equal to I. So, here we have the identity matrix meaning this term is nothing, but this term here is nothing, but the x star and x. So, we have x star x here also we have x star x. So, we take common this x star x and we get 1 minus this lambda bar lambda and that is equal to 0 and with this we got this result that this lambda bar lambda is equal to 1 or lambda bar lambda is nothing, but the absolute value of lambda square. So, this absolute value of lambda square is equal to 1 because this cannot be 0. So, this has to be 0 which tells us this lambda bar square is equal to 0.

So, meaning this we got that this absolute value of lambda has to be 1. So this unitary matrix the eigenvalues of the unitary matrix are of unit modulus. Same results we can also use for the orthogonal matrices because they are also having the same property a transpose A is equal to I. So, for orthogonal matrices also now we can prove thus the similar all absolutely all same steps here and we can again prove the there eigenvalues are also of unit modulus.

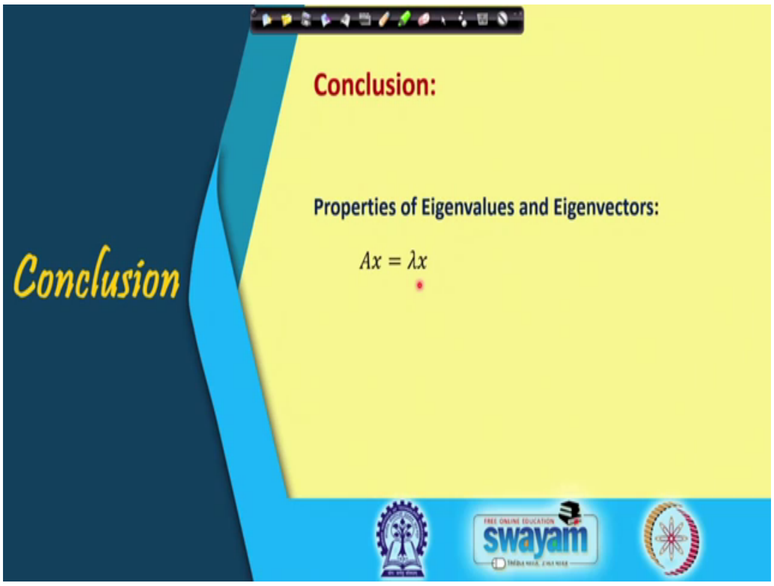
(Refer Slide Time: 29:00)



The location of the eigenvalues now what we have just seen in previous slide. So, if you have a skew Hermitian matrix their eigenvalues are imaginary, purely imaginary here the unitary matrix they lie on this modulus 1 and for the Hermitian matrix or the symmetric matrix the values are sitting on the real axis, so meaning they are the real numbers.

So, here for a skew Hermitian and exclusive metric the same thing unitary and orthogonal we have the same result, that they are of a unit modulus for Hermitian and symmetric we have also the same result for both that they are the real entries.

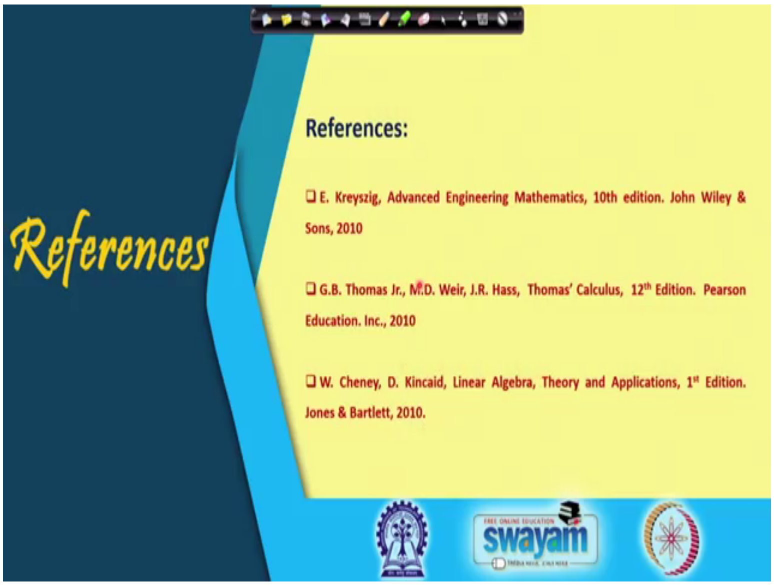
(Refer Slide Time: 29:48)



Getting to the conclusion, so we have seen several properties of this eigenvalues and eigenvectors of a matrix, we have considered different; different types of matrices where we can tell about whether the eigenvalues will be real imaginary if your imaginary you are 0.

So, here in all these properties the simple idea was to use this Ax is equal to lambda x and we played with this equation only to prove all these properties. And now they can be used now without doing all these numerical calculations we can compute directly also with the help of these properties.

(Refer Slide Time: 30:26)



So, these are the references we have used to prepare these lectures.

Thank you for your attention.