Engineering Mathematics - I
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Lecture - 47
Eigenvalues & Eigenvectors (Contd.)

Welcome back and this is a lecture number 47, we will continue our discussion on Eigenvalues and Eigenvectors.

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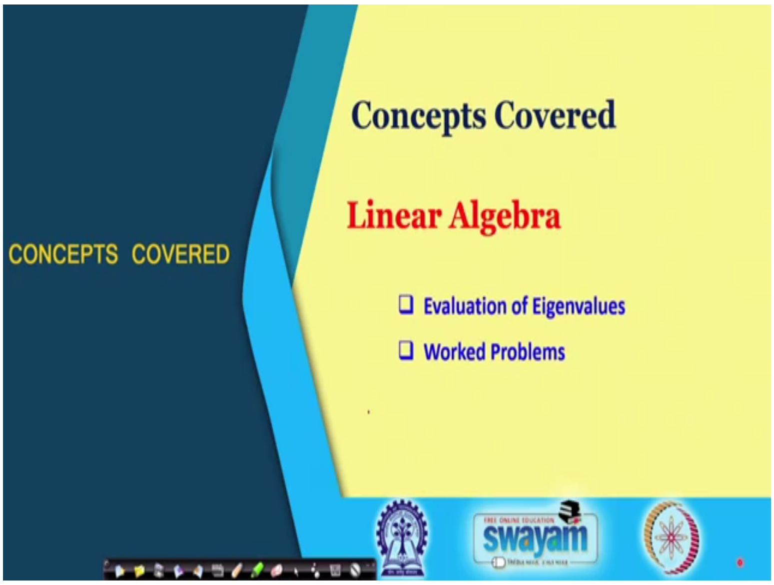

In particular today we will focus more on evaluation of the eigenvalues and some good worked out problems will be discussed.

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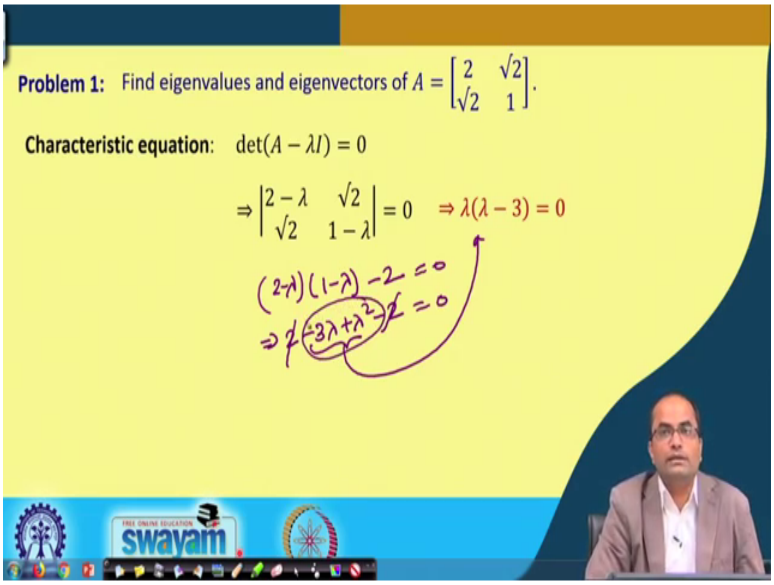

So, let us start with this problem here find eigenvalues and eigenvectors of this matrix, again a very simple matrix we have taken 2 square root minus 2 and then square root 2 and again here square root 2 and this one there. So, the characteristic equation we have to write down as a first step to find the eigenvalues; that means, the determinant of this A minus lambda I is equal to 0 that is the characteristic equation we have.

So, here the lambda will be just subtracted from the diagonal entries otherwise this is just the matrix a. So, 2 minus lambda and the square root lambda again here the same element is square root 2 and then 1 will be 1 minus lambda is equal to 0 and now we can simplify this easily. So, we can multiply 2 minus lambda 2 this 1 minus lambda and then this minus this 2 so that can be simplified to give that minus 2.

So, this is the 2 minus lambda and then we have 1 minus lambda and minus 2. So, what do we get here? The 2 and then this will be minus 2 and here minus 2, so 3 minus 3 lambda and plus this lambda square and then we have minus lambda minus 2, so this 2 gets cancelled and we have exactly this characteristic equation which is lambda times this lambda minus 3. So, here we have this characteristic equation which tells that there are 2 eigenvalues lambda is equal to 0 and lambda is equal to 3.

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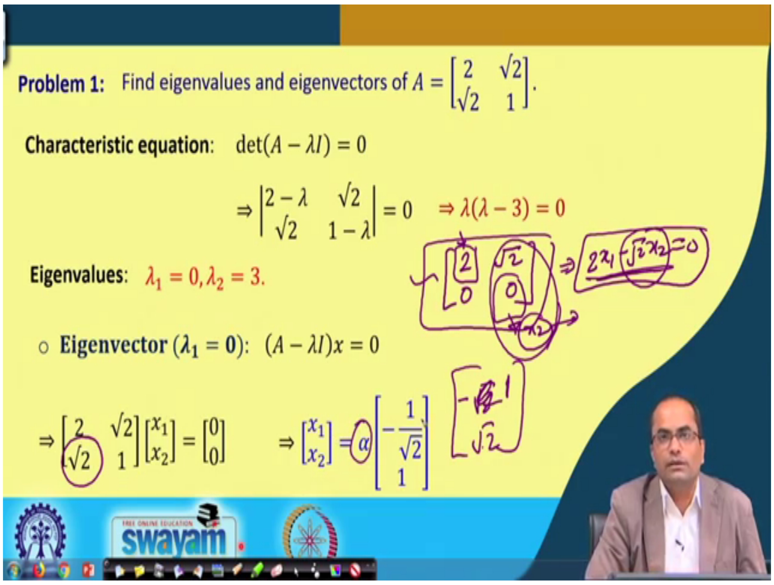

So, the eigenvalues here are lambda is equal to 0 and lambda is equal to 3. So, now, we will compute the eigenvector corresponding to lambda 1, the first eigenvalue that is 0 here. So, we have to solve the system of linear equation A minus lambda x is equal to 0. So, the lambda is 0, so this is simply this a so we are solving basically this A x is equal to 0 or we are trying to get the null space of this A x is equal to 0 and the generator of the null space or the basis of the any basis of the null space will be the eigenvector.

So, here getting to this A here we have to square root 2 and square root 2 and 1 here and then the vector x 1 x 2 is equal to 0. So, we need to solve this equation, so how to get this one? Let us just take a while here, so this 2 and square root 2 that is the first row which we can keep as it is. The second one is square root 2 and 1, so we want to make this 0, so we can divide the first row for instance by a square root 2 and this will be 0 then and here 1 1, so that would be also 0.

So, this is the rho reduce a echelon form of the given matrix. So, we have basically this first equation which says that 2 x 1 minus square root 2 x 2 is equal to 0 and we have one free variable which we can take whichever we want, so let us take this x 2 is a free variable because as per our notations here this is the first the p both here. So, this x 1 is not free and then we can take as x 2 free, but any case in any case we can take any one of them and the other one will depend on the given chosen number. So, here the x 2 we can choose and then we can get the x 1.

So, out of these many possibilities we can just pick one vector that we have done here for instance. The x 2 was taken as the free variable alpha and then x 1 is coming 1 over square root 2 with the minus sign or any multiple of this we can for instance x square or 2. So, we can also take the vector here the square root 2 with minus sign and the square root. So, here when we multiply 2 square root 2 to both, so this will become 1.

So, this vector also we can take minus 1 and square root 2 or we can take 1 and minus square root 2. So, we can multiply by any number here that will be the characteristic a vector here or the eigenvector of the given equation. So, here the one characteristic vector or the eigenvector we got corresponding to lambda 1 is equal to 0.

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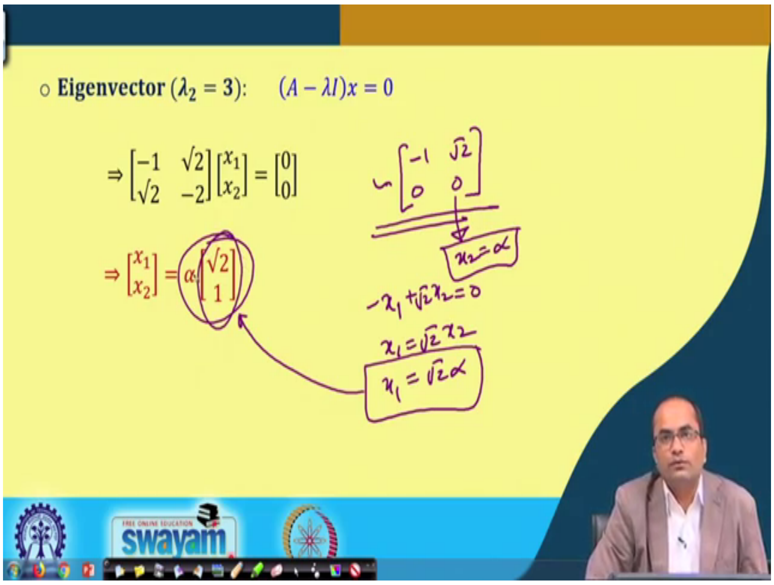

Similarly, we can also look for the eigenvector corresponding to lambda 2 is equal to 3. So, again we have to solve the system of equation a minus lambda I x is equal to 0 this time and then, so again this lambda will be subtracted from this matrix A.

So, that will be the matrix here for the coefficient matrix for the system of equation and again we have to get the solution of this one which we can we can again choose one of because the second row can be made to 0 when we multiply the first equation by square root 2 and then subtract from 2, so the second row will be 0.

So, the corresponding to this the row reduced form will be minus 1 and square root 2 0 0 and from here now we can choose this x 2 variable s as alpha and then here the x 1 from this first equation minus x 1 plus square root 2 x 2 is 0; that means, the x 1 is square root 2 x 2, x 2 is alpha. So, we have taken here x 1 as square root 2 alpha which we have written in this form the solution x 1 is square root 2 alpha and x 2 is alpha. So, this is another vector here any multiple of this is square root 2 1 we can take as an eigenvector.

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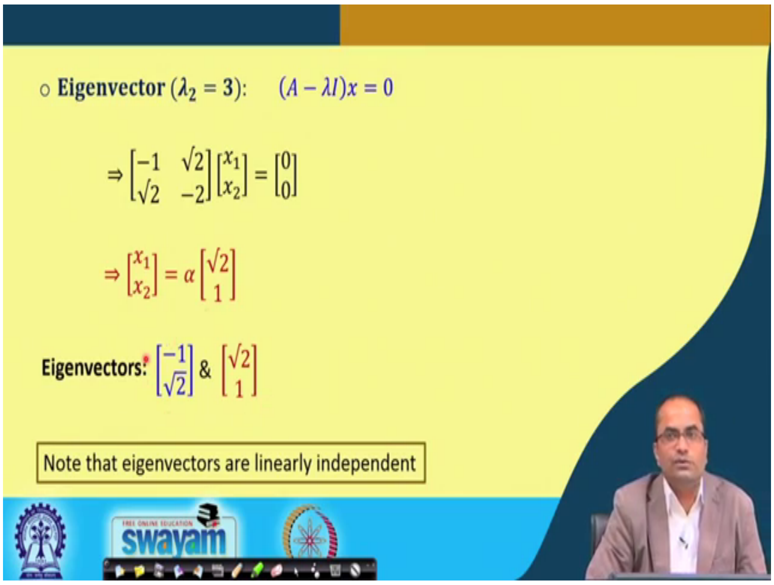

So, in this case the eigenvectors corresponding to the 0 was this minus 1 and square root 2 and here we have this is square root 2 and 1. Again we can note that these eigen vectors are linearly independent and as I said in the previous lecture that we will also proof formally this result that corresponding to distinct eigenvalues we have the eigenvectors, the set is linearly dependent the set of eigenvectors is linearly independent or the eigenvectors corresponding to distinct eigenvalues are always linearly independent. So we can check here also that these 2 vectors are linearly independent.

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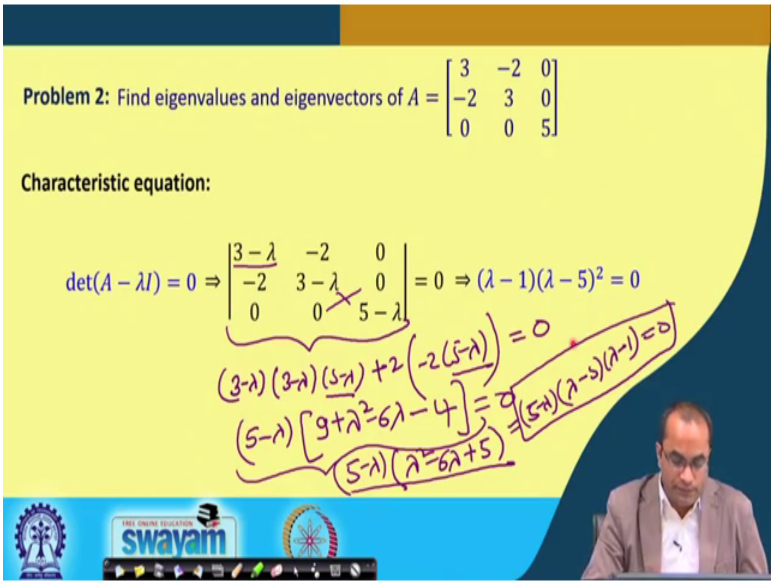

Another problem this is 3 by 3 matrix, so 3 minus 2 0 and so on. So, here again we want to find the eigenvalues and eigenvectors of this given matrix which is 3 by 3 now. So, we have to again follow these steps that first we have to write down the characteristic equation from there we can get the eigenvalues. So, the characteristic equation will be determinant of this a minus lambda I, so we have to subtract this lambda from all the diagonal entries which is 3 3 and 5 there.

So, that is the characteristic equation of this matrix and then we have to evaluate this determinant which is not so difficult. So, we have this 3 minus lambda here and that will be multiplied now with this 3 minus lambda into this 5 minus lambda and minus 0, so this is done and then we have this minus minus plus 2 here. So, this will be then minus 2 times this 5 minus lambda and then the rest will be 0.

So, here and then again, so this is the value of this determinant which should be 0, then this 5 minus lambda we can take as common from both the terms. So, here we will get this 9 and then we have also minus 3 lambda minus 3 lambda, so minus 6 lambda plus lambda square, so minus 6 lambda plus lambda square. So, there is 3 minus lambda whole square, so and then we have here minus this 4 term 2 2 the 4 here. So, this is the characteristic equation here with again we can factorize the second term, so your 5 minus lambda and this is lambda square minus 6 lambda and this is 5.

So, we got here this 5 minus lambda this factor and then this is lambda minus 5 and lambda minus 1, so is equal to 0. So, that is the characteristic equation from there we can get the roots of the equation, so we have this 5 minus lambda or lambda minus 5 this whole square and thus we have this lambda minus 1 the another factor. So, the characteristic equations suggest now that we have basically 2 distinct eigenvalues the one is lambda is equal to 1 the other 1 is 5 which is repeated two times.

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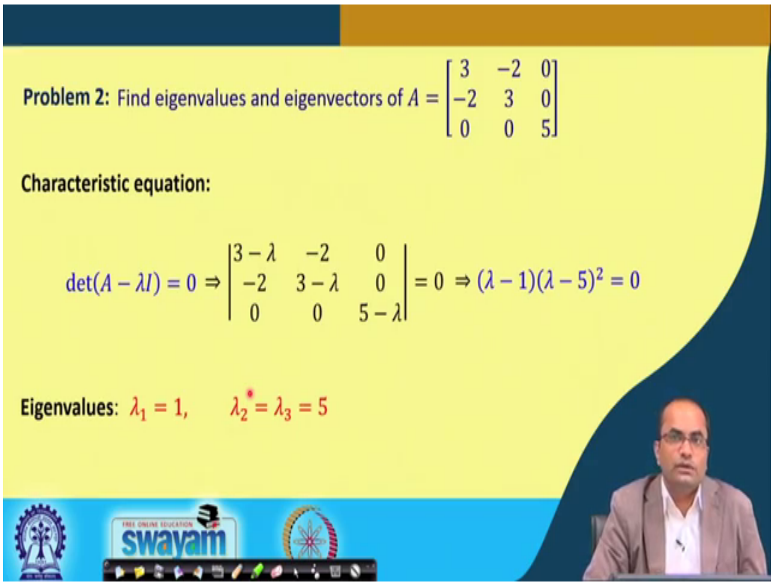

So, eigenvalues are now 1 and 5, these are the 2 eigenvalues and we need to compute the eigenvectors corresponding to each.

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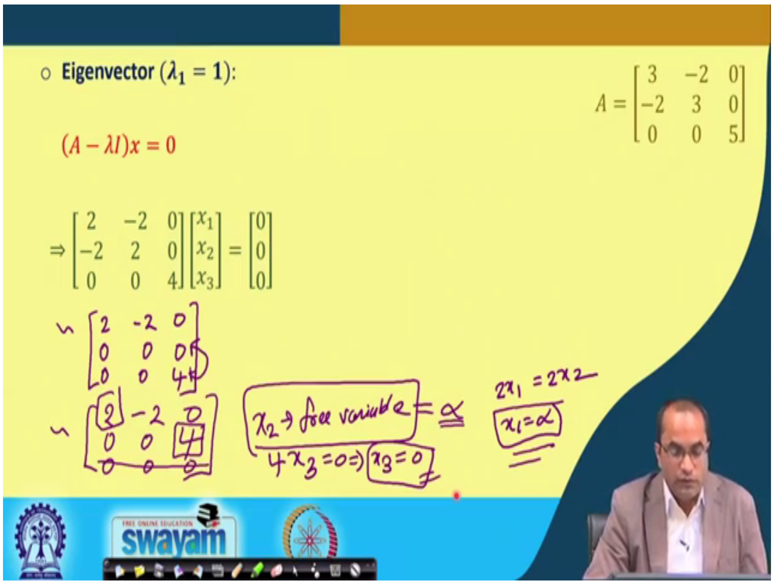

So, when we take this lambda 1 is equal to 1 what do we get this system of equation a minus lambda I x is equal to 0 and our a was this here 3 minus 2 0 and minus 2 3 0 0 0 5. So, here a minus this lambda I and lambda is 1 now, so 1 will be subtracted from the diagonal and that will be our matrix here with this 2 minus 2 0 and this is again 2 here the diagonal entries is reduced by 1. So, we have this matrix which we want to now solve the system here and we need to get this reduced to echelon form, so here 2 minus 2 and 0 here we want to make it 0, so we can just add the rule number 1.

So, this is 0 and this will become 0 and this is 0, so here 0 0 4, this is the one step of the row reduced form and now we can interchange these 2 rows. So, we will get 2 minus 2 0 0 0 and this 4 and the last row will be the 0 row. So, we have this, so we have the pivot here in the first column, we have pivot in the third column the second, so corresponding to x 2 here we can choose as a free variable and then x 1 and x 3 will be computed based on this free variable.

So, here the second equation; however, suggests that the x 3 is 0 because this third second equation is now 4 times the x 3 is 0. So, which tells us that x 3 is anyway 0 always and then x 2 is a free variable. So, we can take any alpha and then from this equation number 1 which says the 2 x 1 is equal to 2 x 2, so this x 2 is free variables. So, here x 1 is also alpha because x 1 and x 2 both are equal. So, we have here x 1 alpha x 2 alpha and x 3 is 0 in all the cases, so what do we get?

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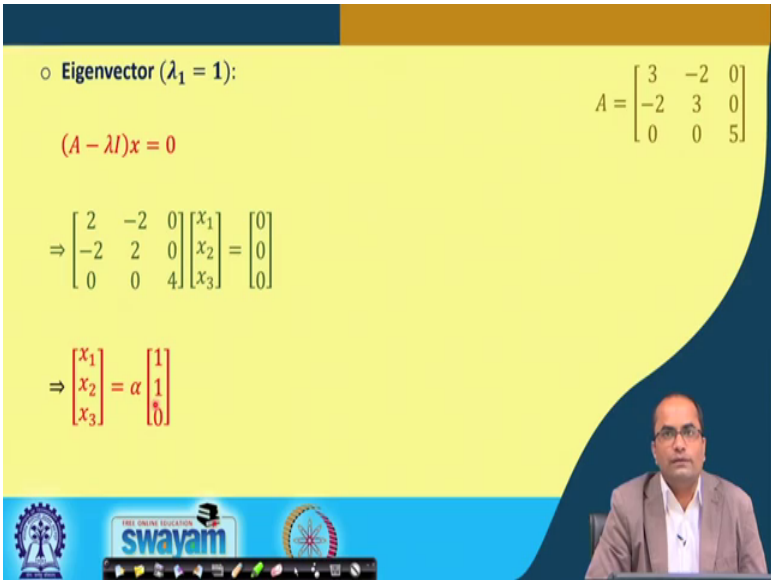

We get this as the solution here of this equation that x 1 x 2 x 3 is the alpha times the 1 1 0 because x 1 was alpha x 2 was also alpha and the third component x 3 is coming to be 0. So, in this case the solution of this system is alpha times 1 1 0 and that is exactly the generator of the null space of this matrix.

So, we can pick any vector, any nonzero vector from this null space. So, we can take 4 instances 1 1 0 or any multiple of it that will be the eigenvector corresponding to this eigenvalue lambda 1 is equal to 1.

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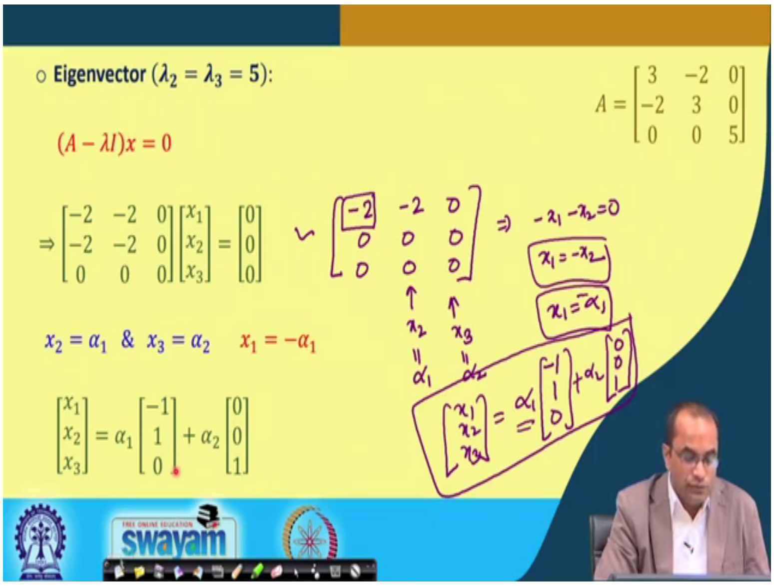

Moving to the other one, so we have this lambda 2 is equal to lambda 3 the 2 eigenvalues are equal there which is 5 and then if we set up the system of equations for this given A which will be in this case because the 5 will be subtracted from the diagonal entry, so we will get minus 2 here minus 2 and the last entry the diagonal entry will be 0. So, we have this system and this is interesting now to see that we can reduce to this row reduce echelon form very easily. So, when we subtract row number 1 from row number 2, so we will get 0 0 and 0 here again this is 0.

So, this is the row reduced echelon form corresponding to this matrix which suggests now we have here this pivot element and that is the only pivot element, so we can choose x 2 and we can choose x 3 whatever we like. So, here let us take this alpha 1 and here if we take this l alpha 2, then what will happen from this equation 1 we have the minus this x 1 and minus this x 2 is equal to 0 or minus 2 x 1 minus 2 x 2 is equal to 0, meaning this x 1 is minus x 2 that is coming from this equation number so this x 1 is x 2 means this minus alpha 1.

So, we have now the vector here the x 1 x 2 and x 3 with alpha 1 and alpha 2. So, alpha 1 with x 1 we have minus 1 and x 2 also we have a component here this is 0 with alpha 2, again we have in x 1 component there is no alpha 2 in x 2 or so there is no alpha 2 and here we have this 1. So, this is the solution of this system of equation x 1 x 2 x 3 is alpha 1 times minus 1 1 0 and alpha 2 times this 0 0 1. Having this, so, what we have exactly this is written here, so x 2 we have chosen this free variable x 3 also free variable and x 1 is coming as this minus alpha 1 which is we have evaluated and this is the solution now which we have already written.

So, here we have these 2 generators minus 1 1 0 and 4 alpha to the 0 0 1. So, what do you observe in this case? That we are getting this is either 1 is satisfying the given differential equation the given system of the question, so here minus 1 1 0 satisfies the given equation, also 0 0 1 is satisfying the given equation. And these two are also linearly independent which we can see from the structure itself, any linear combination like alpha 1 plus alpha 2 is equal to if we set to 0 the alpha 1 alpha 2 has to be 0.

So, that says easy check for the linear independency in for this case. So, what we are getting corresponding to this lambda is equal to 5 which was the repeated eigenvalue and we are also getting here the 2 linearly independent vectors which satisfy this A minus lambda I x is equal to 0. So, basically corresponding to lambda is equal to 5 we are getting 2 eigenvectors or rather to say 2 linearly independent eigenvectors, we are getting in this case.

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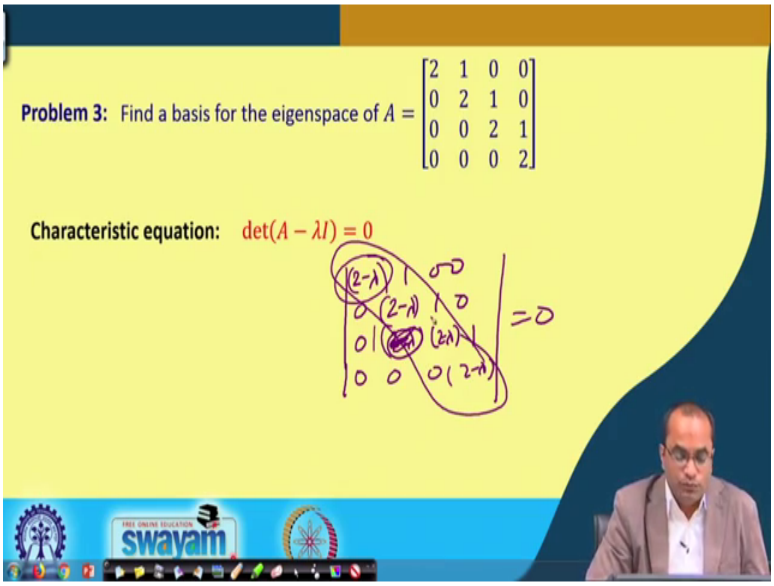

Now, if we look into another problem here the eigenspace of this matrix that is 2 1 and 0 0, then we have this is actually 4 by 4, so little bigger matrix, but the calculations are, so in this case if we set up the characteristic equation again A minus lambda I x is equal to 0. So, the determinant of this will give the characteristic equation and what is the determinant here?

So, 2 minus lambda 2 minus lambda and so the determinant is this 2 minus lambda 1 0 0 and here also 0 this 2 minus lambda 1 and 0 and to minus lambda and then we have no the second component is 0, then third one is 2 minus lambda and this 1 and 0 0 0 at this 2 minus lambda. What is interesting now that if we compute this determinant, so this and then the rest determinant again we will take, so this is 0 here. So, again this, so this is coming that 2 minus lambda power cube nothing else.

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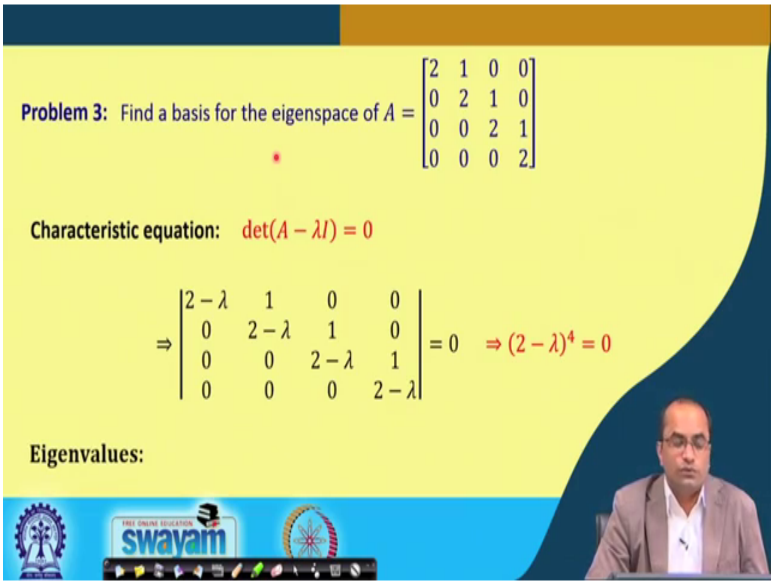

So, in this case this determinant is nothing, but these 2 minus yeah the determinant is here when we get we will solve for this determinant we are getting this 2 minus lambda cube; so, 2 minus lambda sorry for because there are 4 entries here, so we will get just the product of the diagonal entries for such matrices.

So, we get this 2 minus lambda power 4 is equal to 0, so this is another special case where all the eigenvalues, all these 4 eigenvalues are same and that is the 2 or the value is 2. So, if we compute now because we want to compute the eigenspace means the eigenvectors set of all eigenvectors and when we include 0 or so that is called the eigenspace corresponding to the given eigenvalue.

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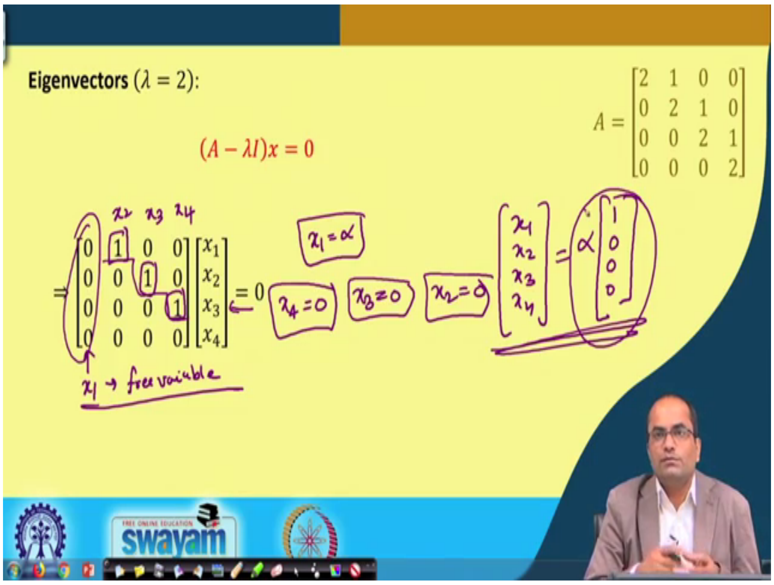

So, here the eigenvalue is 2 which is repeated 4 times, so to get the eigenvector. So, we have to set up this a minus lambda x is equal to 0 this system of equation corresponding to this lambda is equal to 2. So, what do we get now corresponding to this one because this 2 will be subtracted from the diagonal entries and our matrix there was having this 2 in the diagonal entries. So, this diagonal entries will become 0 now. So, we got this as our system of equation and here we do not have to do anything to get the reduced form because we can clearly see this structure here this is already the reduced form the reduced echelon form.

So, this is the pivot element and this is another pivot element this is also a pivot element. So, how many pivot elements are there this x 2 is the dependent variable, x 3 will be the dependent variable, x 4 will be the dependent variable and here we have this x 1 which we call the free variable. So, our free variable is this x 1 which is corresponding to this column number 1 because we do not have pivot in this column. So, we can choose this x 1 as a free variable, so x 1 is not taken as alpha and now we can get the other variables.

So, from this equation number third, from the third equation what we are getting, so we are getting this x 4 is equal to, so no dependency on alpha indeed. From the second equation we are getting this x 3 is equal to 0 and from the first equation we are getting x 2 is equal to 0. So, that is the solution here x 1 we can choose any vector alpha as alpha we have taken here, and now if we write down the solution x 1 x 2 x 3 and x 4.

So, in terms of the alpha only the first component has alpha all others are 0, so this is the solution of this system of equation; that means, here we have only one free variable at though the 2 was repeated several times the 2 was repeated 4 times in this particular example, but what we are getting here we are getting only 1 linearly independent eigenvector. Whereas, in the previous example the eigenvalue was repeated two times and we were also getting two linearly independent solution of this equation, but now though the 2 was repeated 4 times, but we are getting only 1 eigenvector corresponding to this 4 times repeated eigenvalue 2.

So, what we want to discuss now with this example that that anything is possible just based on this eigenvalue whether it's repeated several times we cannot claim anything about I mean directly looking at the eigenvalue, we cannot claim that how many linearly independent eigenvectors we will get as this is the case here the eigenvalue was repeated 4 times, but we are getting only 1 linearly independent eigenvector and that is this 1 0 0 in this case.

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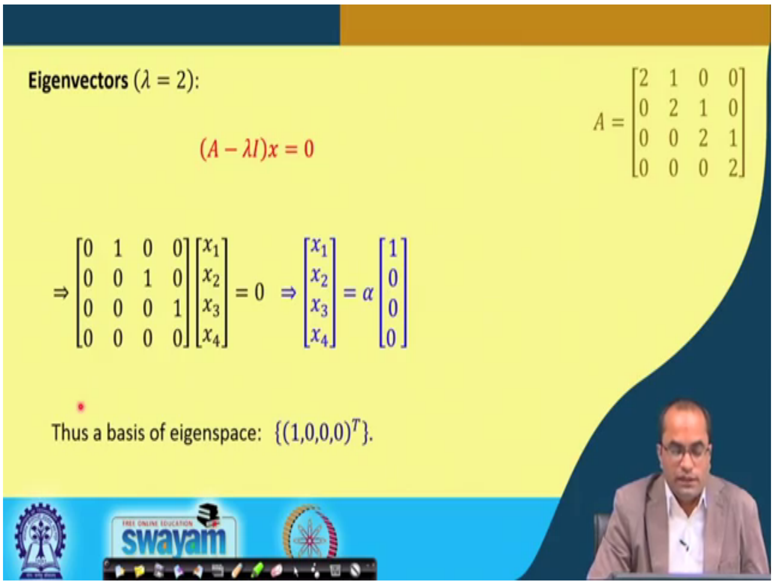

So, and in some cases if it is repeated for example, 3 times you may get 3 linearly independent eigenvectors also, so the other way around is also possible ok. So, thus a basis of this eigen space is this 1 the three 0's here the transpose is used here, so that is the basis for the eigenspace or an any vector from this eigenspace other than 0 is the eigenvector right. So, here this is also the basis for the null space of this matrix here 1 0 0 0 and this the transpose to show that this is a column vector ok.

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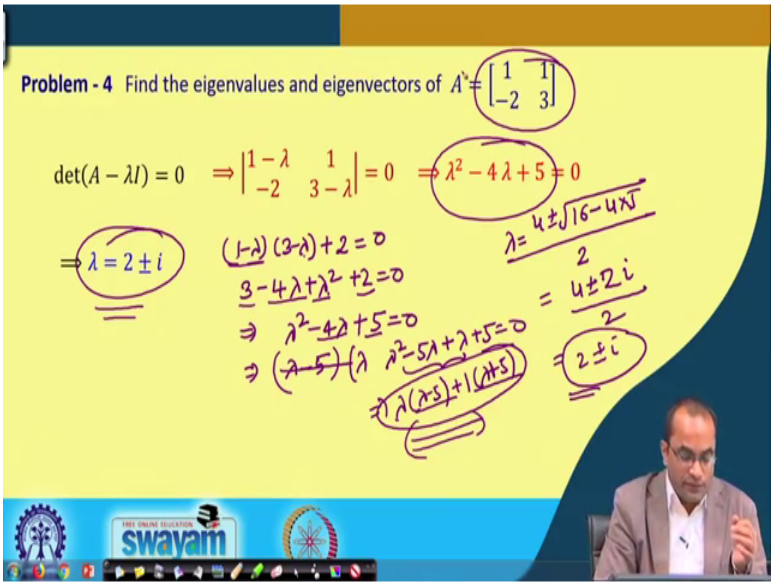

So, the another example where we will look for the eigenvalues and the eigenvectors of this matrix A which is given by this 1 1 and minus 2 and 3. So, here if we compute this determinant of a minus lambda I is equal to 0, so what will happen? So, we have 1 minus lambda and then 3 minus lambda minus 2 and 1 remain, intact. So here if we compute this, so we have 1 minus lambda and multiplied by this 3 minus lambda and minus minus this becomes 2 is equal to 0. So, let us do the calculation so 3 and then we have 4 lambdas also we have minus 4 lambda and then this lambda square plus this 2 is equal to 0.

So, we are getting this equation lambda square minus this 4 lambda and plus this 3 plus 2 this 5 is equal to 0. So, we are getting this lambda minus 5 and lambda plus so the plus. So, once more here, so we have the 3 and then we have a minus 3 lambda also minus lambda. So, will be minus 4 lambda here the minus minus, so lambda square and then plus 2.

So, we have this lambda square term as it is and minus 4 lambda and plus 5 is equal to 0. So, we get this roots here for this equation, so we have lambda square then minus 5 times lambda and plus this lambda plus 5 is equal to 0, so this minus 4 lambda we have written as minus 5 lambda plus lambda. So, here this lambda and this lambda minus 5 and plus this 1 and lambda minus 5 lambda plus 5. Now this is not working, so let us just check here that the equation is lambda square minus this 4 lambda n plus 5 exactly and the roots here because it does not have the real roots.

So, that is the point here we are not getting any factorization in this case we are not getting this factorization here lambda plus 5 is coming and then we cannot take common. So, the factorization was not possible because this equation does not have a real root. So, when we solve this equation for this lambda it is a quadratic equation. So, we can easily do that. So, minus b; that means, 4 b square minus 4 ac and then we have here 2 a, so divided by 2.

So, this is 4 plus minus and then here also 4, so this will becoming 2 i and then we have a 2 there. So, this is a 2 plus minus I is the eigenvalue here, so there are 2 eigenvalues what is interesting in this example that we are getting 2 complex values; 2 complex value for the eigenvalues though the matrix here was real matrix. So, we had the entries 1 1 minus 2 and 3, but we are getting here 2 complex values for the eigenvalues in this case.

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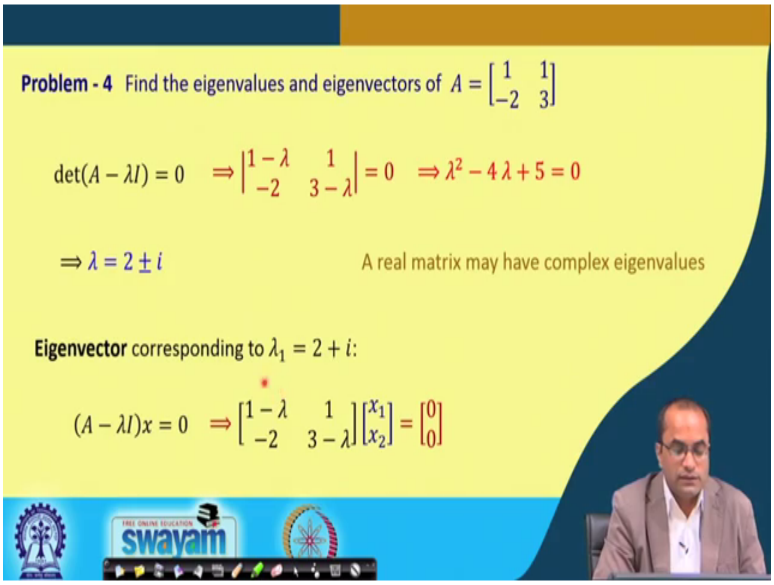

So, that again shows that the matrix entries may be real we can have a real matrix, but still we may get the complex roots of this characteristic equation meaning the eigenvalues may be complex. So, a real matrix may have complex eigenvalues that is the remark and then eigenvectors corresponding to now this 2 plus i the first eigenvalue. So, we have 1 eigen value 2 plus i another one is 2 minus i and we will see later on that these eigenvalues will always appear in conjugate form as the root always appears there and not only that we will have something more interesting for the eigenvectors corresponding to these conjugate eigenvalues.

So, if we take this 2 plus i and again the same system of equation this a minus lambda I x is equal to 0 we need to solve. So, that will be this 1 minus lambda, 3 minus lambda here and x 1 x 2 is equal to 0. So, this equation and now for lambda 1 this 2 plus i we need to substitute for this lambda.

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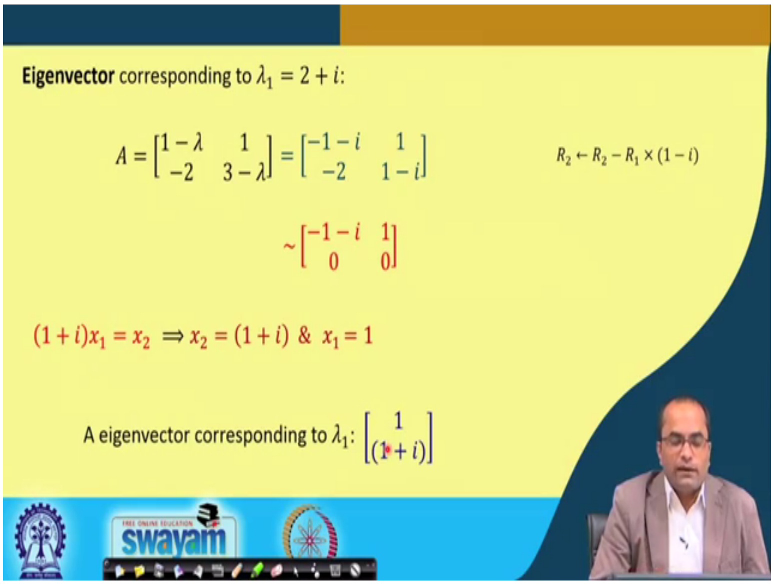

And what we will get, so corresponding to this we have now this A now the coefficient matrix of this eigenvector equation. And now we will make this transformation that this R 2 will be now R 2 minus R 1 multiplied by this 1 minus i. So, if you multiply this equation by 1 minus i and subtract from this equation number 2, what we will get? We will get the reduced form of the echelon form and this will be precisely 0 here.

So, when we multiply by this 1 minus i this will be minus 2 and this will become 1 minus i. So, when we subtract this is exactly what happening here the 0 and 0 we are getting the 0 row, so this is already the reduced row reduced echelon form and then what we we have here? We have this first equation which is 1 plus i times this x 1 and is equal to x 2 this is the first equation and we can take any arbitrary value for x 2 and we can get the corresponding x 1 or we can get any arbitrary value for x 1, we can get the x 2 because there is only one equation.

So, one variable we have to choose arbitrarily. So, by doing, so if we take this x 2 1 plus I then we are getting x 1 is equal to 1. So, that is the one possible eigenvector we are getting here. So, the eigenvector corresponding to this lambda 1 is nothing, but 1 and 1 plus i that is the eigenvector corresponding to this eigenvalue 2 plus i.

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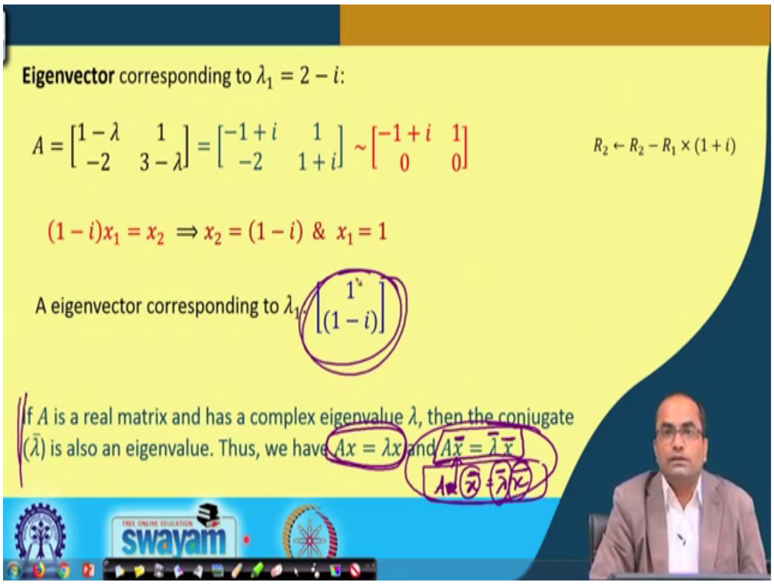

The same calculation we can repeat for the other root which was 2 minus i and again what we will get here this is our matrix now for this system of equation and by using this transformation here that are the row transformation that R 2, we will now get R 2 minus when we multiply 2 R 1 by 1 plus i and again the same situation here, so we can get when we multiply 2 1 plus i, in this case we will be getting again this minus 2 there and here 1 plus i 1 when we subtract, the second row will become 0.

So, we have this row reduced form by this transformation and again we have this equation 1 minus i x 1 is equal to x 2 from where we can just set this x 2 is equal to 1 minus i and we can get this x 1 as 1. So, a eigenvector corresponding to this lambda 1 is now this 1 and the 1 minus i, so we have 1 and 1 minus i as the eigenvector. So, if A is a real matrix this is the result not related to this example the general result what we have now because what we observed in this example, the earlier eigenvalue was 2 plus i and the corresponding eigenvector was 1 plus i and now we have this 2 minus i and the eigenvector here what we get 1 minus i.

So, the eigenvector or one of the eigenvectors here we are getting exactly the conjugate of what we have got for the eigenvector corresponding to its conjugate there. So, that is a nice result here in general we have this then A is a real matrix and has complex eigenvalues. Then the conjugate ; then the conjugate this lambda bar, so lambda is an eigenvalue, then the conjugate will be also the eigenvalue and, but that is natural which is coming from this characteristic equation. So, the conjugate will be always there as the root.

So, here and what is interesting? The interesting is that this we have from this Ax is equal to lambda x if we take the conjugate both the side what we will get because a is a real matrix, so A will remain as it is and this will be the conjugate here for x and the right hand side the conjugate of lambda and the conjugate of x. So, again we have this equation, the eigenvalues eigenvector equation is satisfied for this x x bar, the conjugate of x and the right hand side is just the lambda bar that is the eigenvalue now and x bar.

So, what this tells now that this x bar here the conjugate of x will be the eigenvector corresponding to this conjugate of lambda bar. So, indeed we do not have to compute this since we know this result from this calculation that once we have eigenvector corresponding to one complex value of this lambda and its conjugate will be will be the eigenvector of the other conjugate eigenvalue. So, here that is the interesting result we have about the conjugate roots or about the conjugate eigenvalues complex conjugate here.

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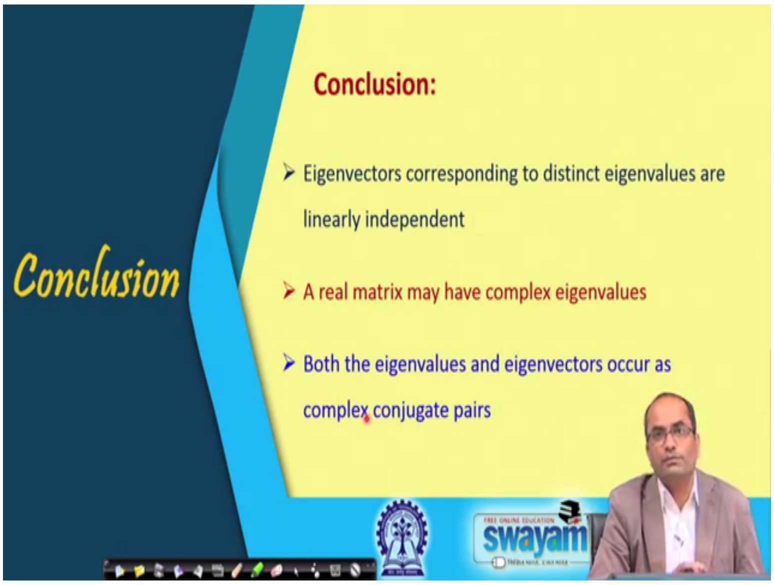

So, in this case the eigenvectors corresponding to distinct eigenvalues are linearly independent this is what we have observed at least for the calculations we had in numerical calculations and also what we have observed here that a real matrix may have a complex eigenvalues. So, that is interesting here and both the eigenvalues and eigenvectors are correct complex conjugate pair.

So, the once we have the complex value as the eigenvalue its conjugate will be there and not only that the eigenvectors will be also the conjugate pairs. So, that is interesting result we have seen and that is true in general not for the example we have just shown.

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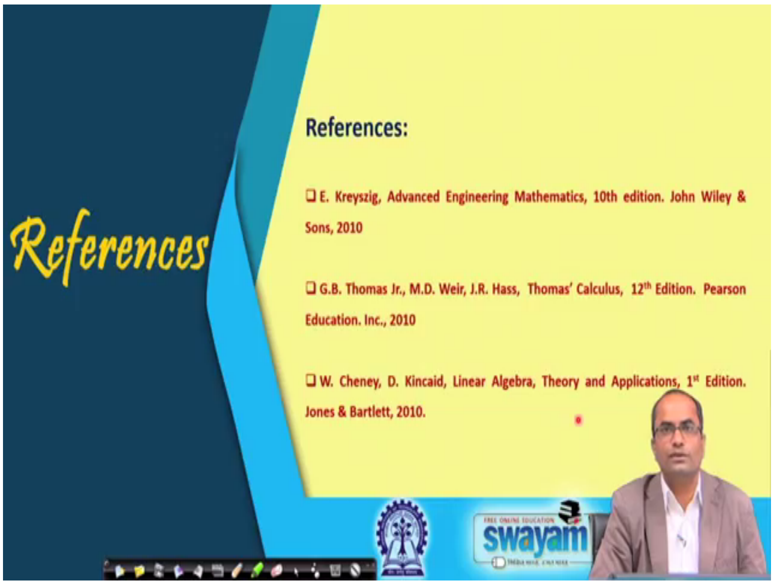

So, these are the references used. Thank you for your attention.