Engineering Mathematics - I
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Lecture - 46
Eigenvalues & Eigenvectors

So, welcome back this is lecture number 46 and today we will continue with Eigenvalues and Eigenvectors another very important topic in linear algebra.

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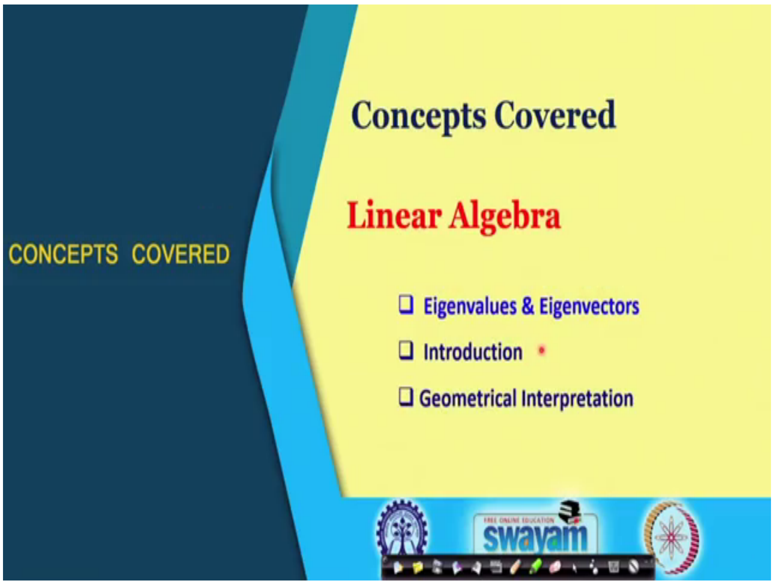

So, we will go through the introduction of these eigenvalues and eigenvectors and also their geometrical interpretation and, then some simple examples to evaluate eigenvalues and eigenvectors.

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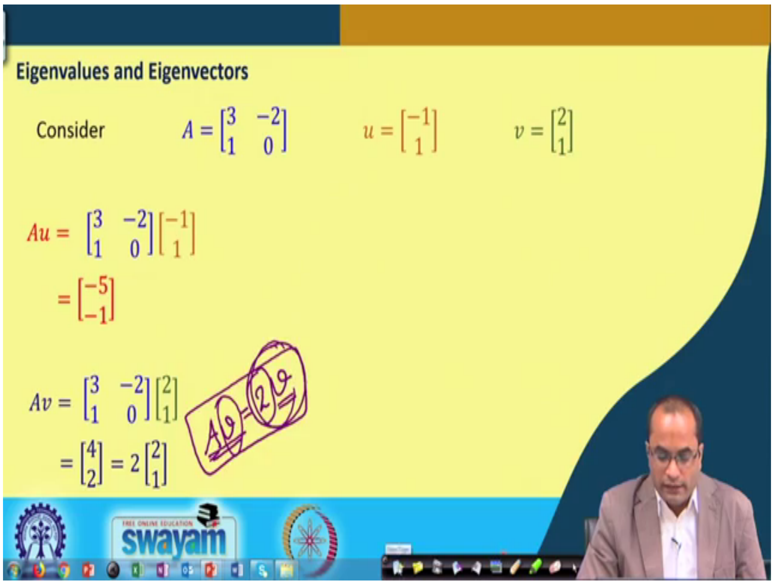

So, here what are the eigenvalues and eigenvectors of a matrix; here let us consider this simple matrix here A a 2 by 2 matrix given by this 3 minus 2 1 0. And, then we consider these two vectors one is this u which is minus 1 0 and another one is v which is 2 1. And, with this if we compute this product here A and u so, here we have this A and then we have this u. So, this product will be A minus 3 and then minus 2 that will give minus 5 and then here minus 1. So, the second component will be minus 1. So, we have this result minus 5 minus 1, but if we do this product with this vector v then what will happen?

So, A times v now so, A and this is v here 2 1 then what we will get so, this is 6 and minus 2 4 and then here we will get 2. So now, this result of this product is 4 a 2; what is interesting here now this 4 2 is nothing, but the 2 times of the vector here 2 1. So, what we observed now that this A times v in this case is this 2 times v. So, this product here is still giving us v, but it is just the 2 times here one number has come in front of this v. So, this length of this vector which was v here after multiplication this has increased or this has got double now. So, that is exactly the point which will take us to this introduction to eigenvalues and eigenvectors.

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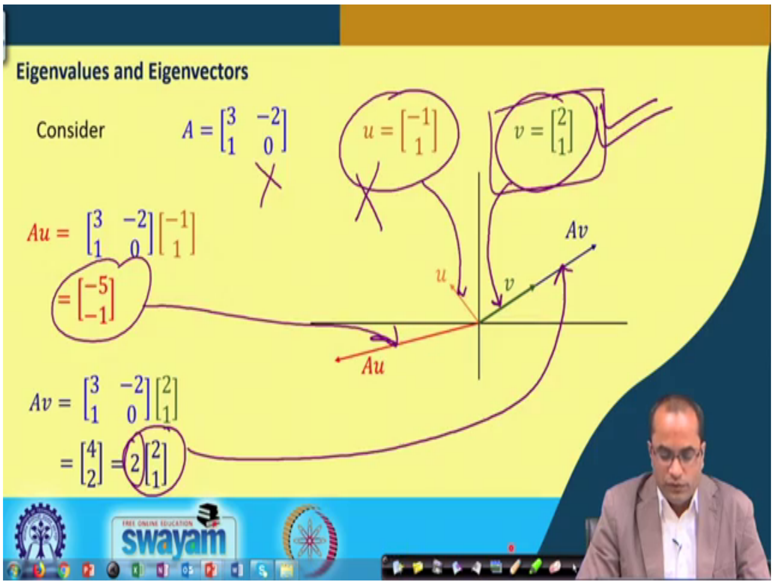

So, if we look at this geometrically what is happening. So, here we have this x axis and y axis. So, this vector v which is given here as is exactly this one and then the Av after this product it is just the 2 times. So, that is the vector this width length double of this length of this v. So, we have this vector Av; what is interesting that this Av and v they have the same direction and their magnitudes are different. So, here now it is just the double of this earlier vector v after the multiplication, but in this case of this u vector this was the vector u and this A times u has become this one.

So, we do not have such relation that after multiplication the vector direction remains the same, we have a completely different vector now Au. So, our interest is not exactly this u with this A, our interest is for such vectors whose multiplication with that matrix does not change its direction its a parallel to the original vector v. And, that is what we are looking for and these are called actually the eigenvectors and this number which has come here that will be called as eigenvalues. So, that is the topic of today's lecture and we will go little more into the detail now about these eigenvalues and eigenvectors.

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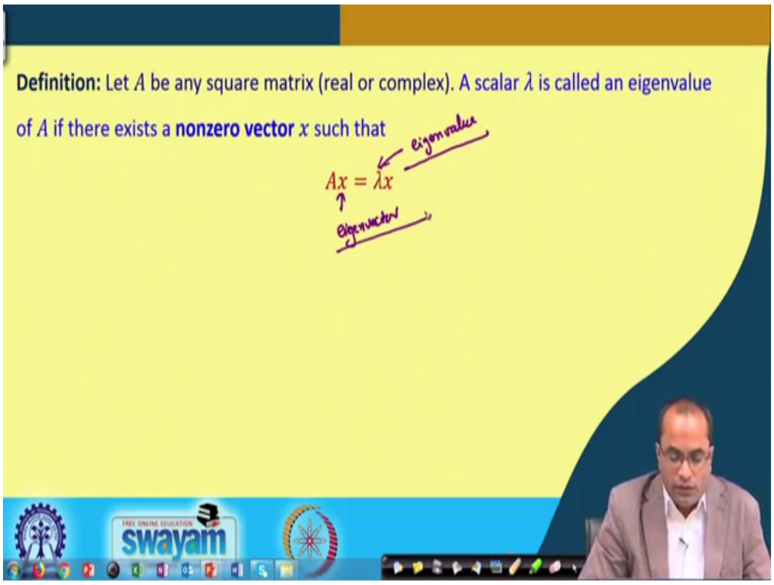

So, the definition the mathematical formal definition here: let A be any square matrix, the entries can be real or the entries of the matrix A can be complex. A scalar lambda is called eigenvalue of A if there exists nonzero vector yeah, that is also important here this nonzero; vector if there exists a nonzero vector such that such that this Ax is equal to lambda x. So, exactly it is a parallel to what we have just seen in previous examples. So, if we have such a vector x whose multiplication with this given matrix a Ax is nothing, but the lambda time x and this lambda is some scalar some real number or a complex number.

So, here this Ax is equal to lambda x that is a very important equation which we will be talking about today. So, that is the definition now of this eigenvector this x is called the eigenvector and this lambda is called the eigenvalue. So, this is the eigenvector and this has to be always nonzero otherwise, the 0 will be satisfied always. So, we are looking for the nonzero vector here which is called the eigenvector and this is called the eigenvalue ok.

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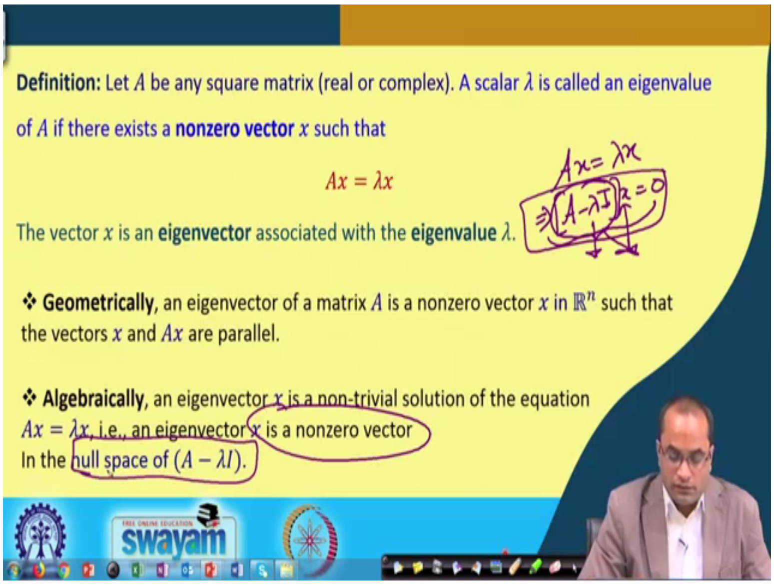

So, now the vector x is the eigenvector associated with this eigenvalue lambda and the two important points now, the geometrically which we have already seen with the help of earlier example that an eigenvector of a matrix A is a nonzero vector, x in this R n such that the vectors here x and this Ax are parallel. This is what we have seen in previous example that this vector v and the vector Av they were just the parallel, the length was different. So, here also the geometrical meaning of this eigenvalues eigenvector in general is that of this vector this x and this Ax both are parallel and their magnitude will change and therefore, we have this eigenvalue lambda.

Algebraically, an eigenvector x is a non-trivial solution because we are looking for the eigenvector x which is nonzero. So, meaning this non-trivial solution because x is equal to 0 will always satisfy this equation. So, we are not interested in the 0 solution, our interested in nonzero solution; meaning the non-trivial solution of this equation Ax is equal to lambda x. Or, this eigenvector x here is a nonzero vector in the null space of this A minus lambda I, because this equation which we have a is equal to l Ax is equal to lambda x. So, this Ax is equal to lambda x what we can do we can bring this lambda x term to the left hand side. So, we have A minus this lambda with the identity matrix because, we need to subtract from this A. Then we have to also introduce this identity matrix and then x is equal to 0.

So, basically what we are looking for, we are looking for the non-trivial solution. This x of this equation A minus lambda I or rather the system of linear equations A minus lambda x is equal to 0 and this is exactly the definition of the null space of this A minus lambda I. So, here the matrix is A minus lambda I and if we look for the null space of this A minus lambda I.

So, this x vector which we are looking for is in the null space of this matrix A minus lambda I. And, since it is a nonzero I mean the null space also has a now has a 0 vector, but we are looking for the nonzero vector in the null space of this. So, x is a nonzero vector in the null space of this A minus lambda I. Now, the natural question is how to compute this eigenvector and how to compute the eigenvalues associated with this with the matrix of order n.

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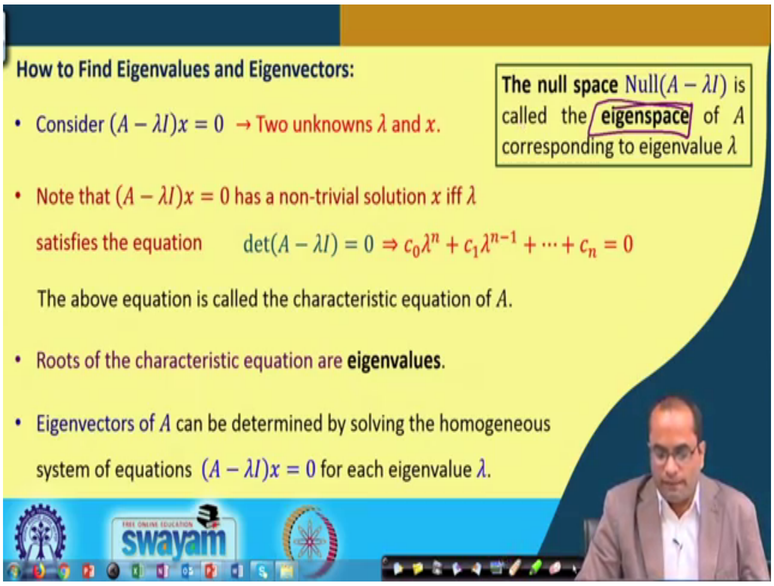

So, how to find the eigenvalues and eigenvectors we will discuss now; so, consider this equation A minus lambda I x is equal to 0, that is Ax is equal to this lambda x equation written in this form. And, there are two unknowns here, the unknowns are the lambda we need to compute lambda and also we need to compute x. There is a one equation here A minus lambda I x is equal to 0 or it is a system of linear equation and we have these two unknowns the lambda and x. So, how to compute these two unknowns so, that those unknown satisfy this equation A minus lambda I x is equal to 0? The lambda is a scalar and this x is a vector whose components will be exactly equal to this n, if the matrix is n cross n matrix. So, what is other information we have that we are looking for this non-trivial solution x.

So, this equation A minus lambda I x has a non-trivial solution which we have already studied in previous lecture. If and only if this satisfy the equation, which equation that the determinant of this A minus lambda I; if this determinant is 0 then we will have a non-trivial solution. If the determinant is not equal to 0 then there will be a unique solution and that will be the trivial solution meaning this x will be 0.

But, we are looking for a non-trivial solution of this equation A minus lambda I x is equal to 0 and in that case we have this condition that this determinant of this A minus lambda I matrix this must be 0. So, we got another condition which is leading us to at least get now something out of this condition determinant of A minus lambda I is equal to 0. So, when we expand this determinant because, this lambda is the unknown now A is a given matrix and I is the identity matrix.

So, here this lambda is unknown. So, when we expand this determinant, determinant is nothing, but this polynomial equation there c 0 lambda power n plus c 1 lambda power n minus 1 and so on c n, these are the coefficients and they will be naturally the given when this A is given. So, we have this polynomial equation and then if we solve this equation we will get at most these n roots of this equation; that means, the n values of the lambdas. They may be distinct and they may not be distinct, they may be real, they may not be real so whatever. So, the solution of this equation which is called the characteristic equation. So, this equation is called characteristic equation of the matrix A and after solving this equation we can get the possible lambdas.

Once we have the lambda here then we have this equation A minus lambda I x is equal to 0 and for each lambda we can find the solution of this A minus lambda I x is equal to 0. And, note that our lambda which we will get with this condition that the determinant is 0. So, naturally we will get non-trivial solution of this A minus lambda I x is equal to 0 because, the trivial solution will come when this determinant is not equal to 0. But, we have we will get our lambda such that this determinant A minus lambda I will become 0. And therefore, automatically for these lambdas we will get the non-trivial solution of these A minus lambda I x is equal to 0.

So, here the roots of this characteristic equation are exactly called the eigenvalues because this lambda is the eigenvalue. So, once we solve this characteristic equation we will get all the eigenvalues and the eigenvectors of A can be determined by this solving the homogenous system of this linear equation; that means, this A minus this lambda I x is equal to 0. So, we need to solve again the system of linear equations. So, from the beginning of this lectures in linear algebra I am emphasizing again and again on this system of linear equations because at each and every step finally, we are solving the system of linear equations. So, it is very important now and here for each value of this lambda we need to solve the system of equations.

So, if we have 3 distinct lambdas for example so, for each lambda. So, for 3 times we have to solve the system of linear equations and they will be different equations because we have the different lambda. So, this matrix is going to be different and we will get different eigenvectors. So, we will be talking more on this now and this null space of this A minus lambda I. So, the null space means these x I and which includes the 0 also because the null space will also include the 0. So, the null space here called the eigenspace of A corresponding to eigenvalue lambda.

So, what is in the null space? In the null space carries all the eigenvectors plus the 0 vector because the 0 is not the eigenvector. So, here in the null space which is also called the eigenspace, it contains all eigenvectors including 0 vector because 0 is naturally the solution of this A minus lambda I x is equal to 0 or 0 will be there in the null space. But, 0 is not the eigenvector because the eigenvector we define as the nonzero, nonzero x the non-trivial solution of this equation. So, another terminology here which we may use later that is eigenspace; so, eigenspace is nothing, but the set of all these eigenvectors including the 0 vector ok.

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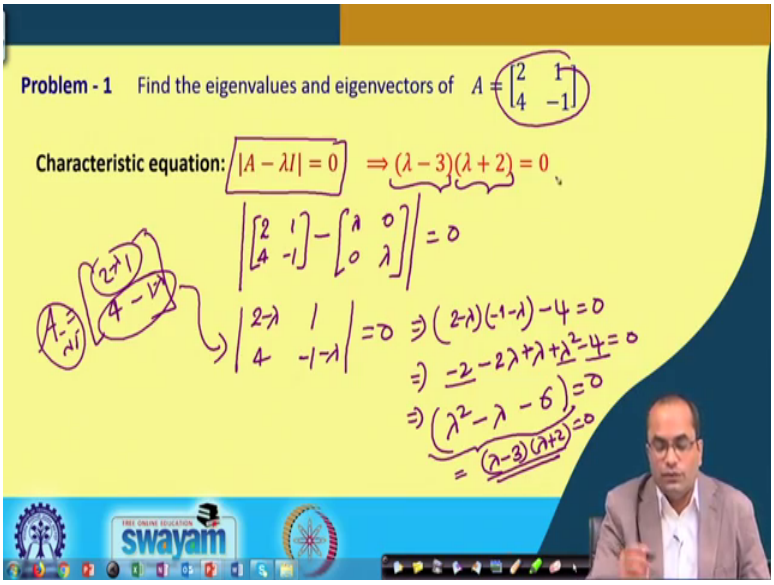

So, let us go through the problem here. The problem number 1 it is a find the eigenvalues and the eigenvectors of this matrix A is equal to 2 1 4 n minus 2; it is a very simple example we start with the evaluation of these eigenvalues and eigenvectors. So, here first we have to write down the characteristic equation and we need to solve the characteristic equation always to find the eigenvalues. And, then for each characteristic value or each eigenvalue we have to solve that system of equation that now we will get in that way the eigenvectors. So, here the characteristic equation is the determinant of this A minus lambda I.

So, A minus lambda I; that means so, the eigenvalues are nothing, but the this solution of this characteristic equation. So, we have this A minus lambda I. So, A here 1 2 1 and this 4 and minus 1 and then we have this lambda I that will be the determinant at the end. So, lambda I means the lambda 0 0 and the lambda that is the product of lambda and this identity matrix and this we want to solve know the determinant here. So, what is the matrix? The matrix here is 2 minus lambda and then we have here 1 and here we have 4 and then minus 1 and the minus lambda, that is the determinant here which we can directly always we can read write down for this given matrix A.

So, the A was given this 2 1 and 4 minus 1. So, how to write the characteristic equation? Just the determinant of this matrix subtracting lambda from the diagonal; so, 1 minus lambda I so, lambda from the diagonal here how we will get this 2 minus lambda and minus 1 minus lambda is equal to 0 and now the determinant value here this product. So, which will be 2 minus lambda and multiplied by this minus 1 minus lambda and this is minus 4 here is equal to 0. So, we can get this product that will give minus 2 and then here minus 2 lambda plus lambda and plus lambda square and then we have minus 4 is equal to 0.

So, we get this minus 6 here they send this then we have this lambda square and then we have minus lambda. So, this is the characteristic equation which we can factorize here easily and that is my lambda minus 3 and this lambda plus 2 is equal to 0. So, that is the characteristic equation from there we can get the roots of the equation. So, that is what given here the lambda minus 1 and lambda plus 2 is equal to 0. So, that is our characteristic equation here.

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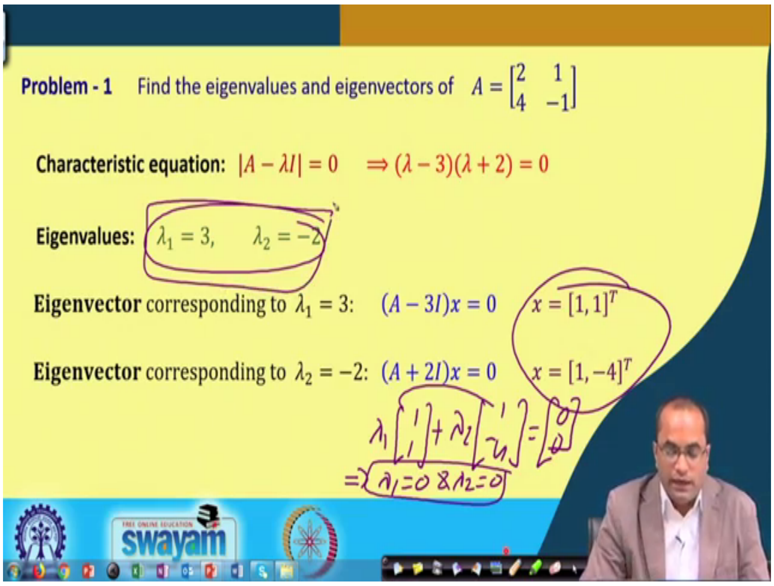

And its root that will be the eigenvalue; so, eigen values are the 1 eigenvalue here which we are calling lambda 1 that is 3 because, this is the solution of this equation. And, the another eigenvalue will be lambda 2 which is minus 2 and the eigenvector now corresponding to each here. So, first let us take this lambda 1 is equal to 3. So, while taking this we have to now form the system of equation A minus lambda I and times this x here so, that will be the system of equations. So, we have to subtract this 3 from the diagonal entries and that will be our matrix here.

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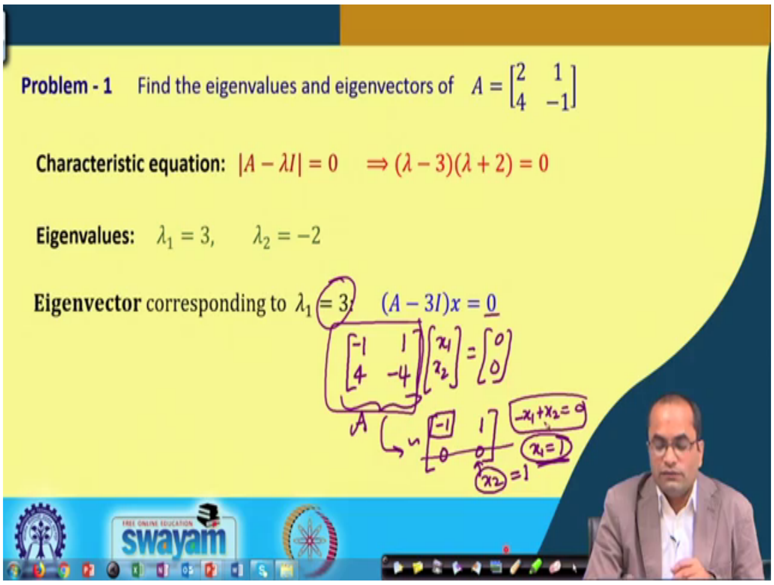

Meaning so, if we do so, so here A minus 3 I. So, A was 2 here so, if we subtract 3 from the diagonal entries we will get minus 1, here 1 and here 4 and this is minus 1 and then minus 1 so, minus 3. So, this will be minus 4 and then we have these x 1 x 2 4 x, it has 2 component and the right hand side is the 0 vector this one. So, we have this system of equation which we need to solve to get the eigenvector corresponding to this lambda 1 is equal to 3 and this is simple because, our this matrix here which I am again calling this A. So this is the matrix which we can easily get to the reduced echelon form.

So, row reduced echelon form for this matrix will be minus 1 and this 1 and here we can multiply by 4 and add it to this equation so that will give 0 0 and that is the row reduced echelon form of this matrix from where we can easily identify the solution. So, here this is our pivot element the first one minus 1 and here we have the 0 rows. So, naturally we will get a non-trivial solution because, the lambdas were obtained with that condition that we will a non-trivial solution; meaning always you will get for solving such system some free variables.

So, here for instance this x 2 we can call as free variable. So, this x 2 is a free variable we can choose a value whatever value we like. So, if we choose x 2 1 then this equation will give us; so, minus x 1 plus x 2 is equal to 0 that is the equation here and if we choose x 2 is equal to 1 so, x 1 will be also 1. So, one solution of this and there are infinitely many possibilities of the solutions. So, here the one possibility of the solution is that x 1 is 1 and x 2 is also 1. So, that is a solution what we get out of this, that is a one solution any multiple of this so, we can have 2 2 or any number we can multiply to this one that will be the solution of this equation meaning eigenvector.

So, eigenvector is never unique here corresponding to this lambda 1 is equal to 3 we got this vector 1 1 a 1 vector that is also need to be mentioned because, the 2 2 will be also the solution and that will be also the eigenvector. So, any multiple of this will be the eigenvector because that is a solution of this and this is the kind of generator of the solution or the basis of the null space of 4 of this A minus 3 I matrix. So, that is the basis here for this null space of this A minus 3 I and that is the vector here x which we call the eigenvector. So, same similar steps we will have to repeat now for lambda 2 is equal to minus 2 and maybe I can skip that. So, here we have this a plus 2 I now.

So, this 2 will be added now to the diagonal entries. So, we will have A plus 2 I will be this 4 1 and 4 minus 1 as the matrix and from there we can get as again 1 vector here 1 minus 4, but any multiple of this will be also the eigenvector. So, by solving this equation A plus 2 I is equal x is equal to 0 we got the another vector here which is 1 and minus 4. What is also interesting and we will note it down later on that there were two different eigenvalues here 3 and minus 2 and, their eigenvalues 1 1 or any multiple of this 1 and 1 4. What we can also check and we can indeed easily see here in case of these two vectors that these two are linearly independent vectors. We cannot get like lambda 1 times this 1 1 and then lambda 2 times this 1 minus 4.

If you want to set this linear combination to 0 the only solution will be that lambda 1 is 0 and lambda 2 is 0, there is no other possibility here. So, these vectors 1 1 and 1 minus 4 they are linearly independent vectors. And, later on in the theoretical result also we will see that whenever we have two different or n different eigen values then corresponding the eigenvectors will be linearly independent, that we can theoretically prove. We will prove actually in future lectures. So, here these are the 2 eigenvectors which we have easily evaluated corresponding to each of the eigenvalues.

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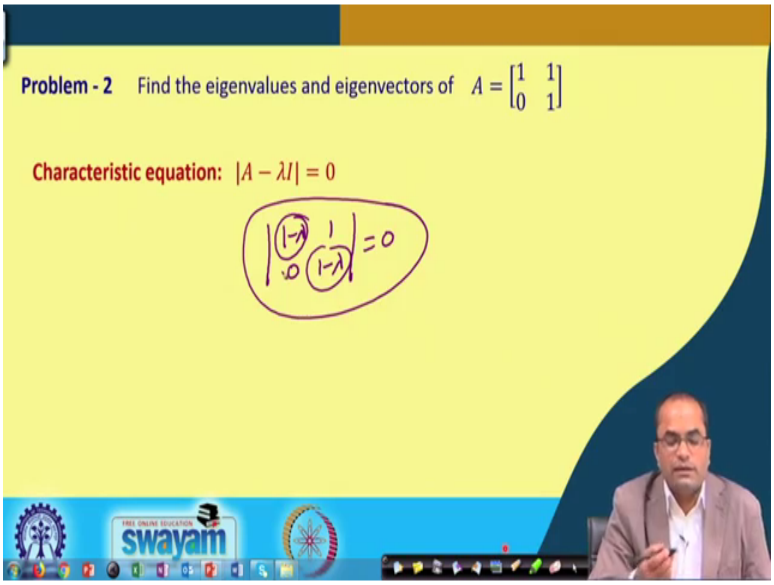

So, another example here find the eigenvalues and these eigenvectors of A which is given here again a very simple matrix we have taken just for demonstration 1 1 and 0 minus 1. So, if we write down the characteristic equation. So, that will be A minus this lambda I is equal to 0 so; that means, we will subtract here from the diagonal entries lambda. So, that will be a 1 minus lambda 1 0 and this 1 minus lambda is equal to 0.

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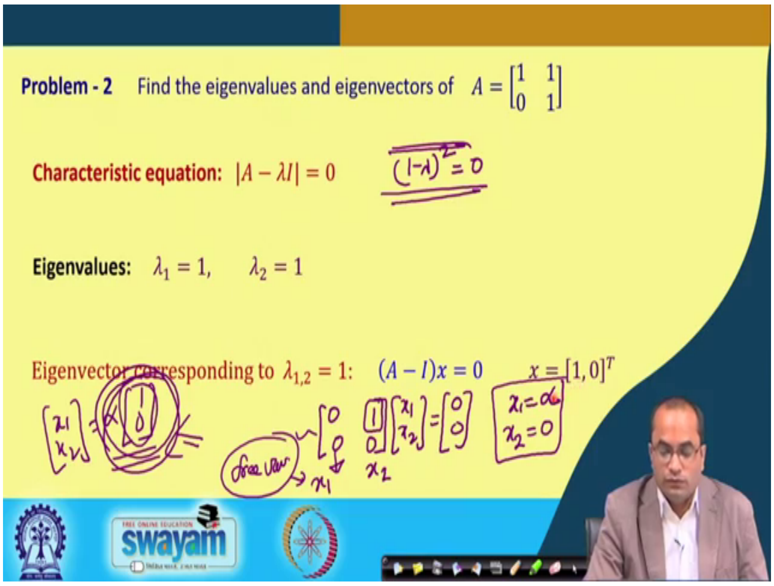

So, which the product here 1 minus lambda square is the characteristic equation and their roots will be then lambda 1. So, here the characteristic equation is coming 1 minus lambda whole square is equal to 0. So, there are two roots of the characteristic equation and these two roots are the lambda 1 is equal to 1 and lambda 1 2 also 1. So, it is a repeated root the case of the repeated root we have only this one eigenvalue which is repeated 2 times and now corresponding to this eigenvalue we need to compute the eigenvector. So, the eigenvector corresponding to corresponding to these values here 1 1 we can again form the system of linear equation that is A minus this 1 lambda times 1.

So, lambda is 1 here so, A minus 1 x is equal to 0. So, this system of equation we have to solve now and what is the system now A minus 1. So, here 1 will be subtracted from the diagonal entries and what we will get; so, let me just first see here. So, our matrix will be now the 0 and 1 and 0 and then here also this 0 that is our system of equation x 1 x 2 and is equal to this 0 0 the right hand side vector. So, this is the row reduced echelon form already and we have here this pivot element; that means, this x 2 is not a free variable, but here this x 1 because the first column does not have a pivot element.

So, this is what we call the free variable and this free variable we can assign any value we like. So, this x 1 we are assigning some alpha for instance and this x 2 equation that is already given from the first equation that x 2 is 0. So, we do not have even dependency on this alpha, the x 2 is always 0 whatever alpha we take that is the freedom we have now. So, our solution of this system of equation is x 1 and x 2 this alpha and this 1 0. So, any alpha we can we can choose of course, here and that is the solution. So, this 1 0 is the is the generator here for the solution and now that is what we have this 1 0 is one of the eigenvectors corresponding to 1 and any multiple of this will be also the eigenvector.

So, here we have seen there were two different eigenvalues, but we get only one eigenvector. So, that is also possible which can be seen here with the help of this simple example; more on this we will be talking about later that what are the possibilities corresponding to 1 this eigenvalues how many eigenvectors are possible and so on.

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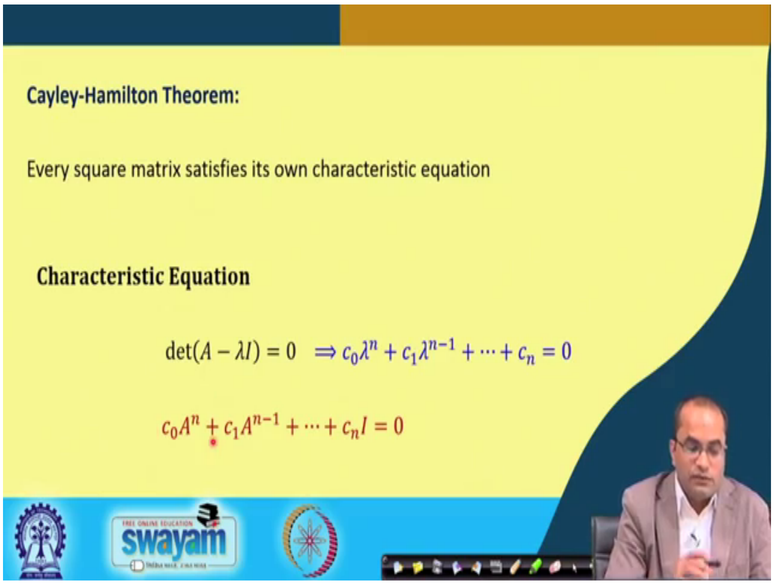

So, there is a Cayley-Hamilton theorem which says that every square matrix satisfy its own characteristic equation, that is another result which directly coming from the characteristic equation that every square matrix satisfies its own characteristic equation. So, we know what is the characteristic equation that is the determinant of this A minus lambda I; meaning this such polynomial equation is the characteristic equation. And, this result says which we will not go through the proof that this u this matrix itself will satisfy this characteristic equation; that means, if instead of the lambda if we replace this by A.

So, here A power n minus 1 and then this and so on up to c n this will be in that case c n into I, I will be the identity matrix of the same size. So, that this Cayley-Hamilton theorem says that every square matrix also satisfies its characteristic equation and that means, that A will also satisfy this equation. So, it is just the lambda is replaced by this A here and with the c n to make it consistent because, now we are working with the matrices. So, there should be a matrix here of the same order.

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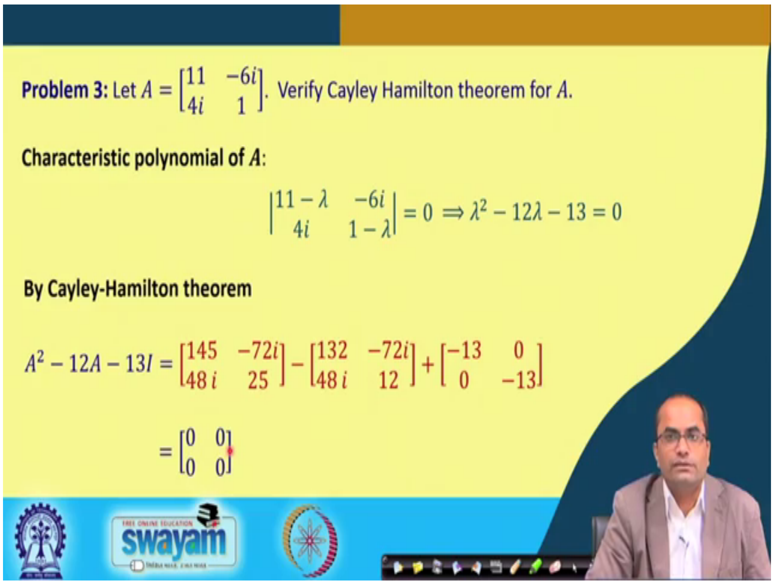

So, that is the result of the Cayley-Hamilton theorem which we can verify for instance for this very simple example which have taken 11 minus 6 i 4 i and 1 and we will verify this characteristic polynomial by this or de Cayley's-Hamilton theorem. So, the characteristic polynomial characteristic equation we have to write corresponding to this A which will be just by subtracting this lambda from the diagonal entries and now we can write down in terms of this I mean this determinant here. So, that we will get this equation lambda square and minus 12 lambda minus 13 that is just the product of 11 minus lambda 2 1 minus lambda and then the minus minus plus here 24 i square.

So, again minus 24 so, that we can simplify we will get this equation lambda square minus 12 lambda minus 13. And, now what we will see the Cayley-Hamilton theorem says that this A square minus 12 A minus 13 I this should be just 0. So, this should satisfy the characteristic equation and if we compute this A square that is coming to be here. So, the product of this A with the matrix A that will be coming this one here minus this 12 times the A. So, 12 is multiplied to each of the entry of the A and then we have minus 13 I here.

So, when we when we determine this one so, here this minus so, these two will be added and that this will be subtracted; we are getting actually 0 matrix. And, that is what the characteristic equation and this Cayley-Hamilton theorem says that every square matrix satisfy its characteristic equation. So, we have just replaced here lambda by this A and then we have seen that this right side instead of the 0 we got the 0 matrix.

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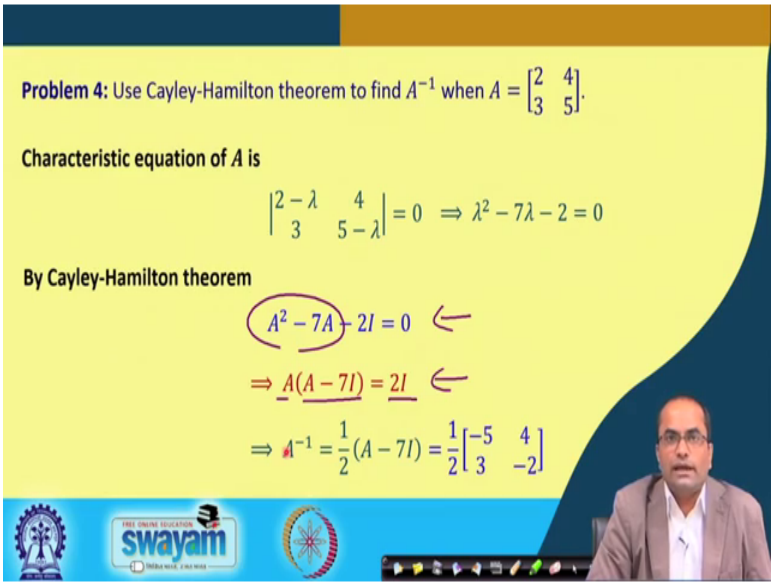

So, that is what the Cayley-Hamilton theorem is; another use of this Cayley-Hamilton theorem we can quickly look from this example we can use this Cayley-Hamilton theorem to prove this A inverse. So, again very simple example we have started with this A is equal to 2 4 and 3 5. So, if we write down its characteristic equation. So, again this lambda is subtracted from the diagonal and we can simplify this determinant. So, we will get this lambda square minus 7 lambda minus 2 is equal to 0 that will be the characteristic equation.

And, by Cayley-Hamilton theorem we know that this A square minus 7 I minus 2 I will be equal to 0 and which we can write we can take this common here in these first two terms this A. So, A into A minus the 7 the identity matrix will be introduced and is equal to this 2 I. And, now what we can do here we can multiply by A inverse again a point to be noted that we can multiply by the inverse when the A inverse exists.

So, up to now this is the Cayley-Hamilton theorem says that every square matrix satisfy its own characteristic equation, but if you are now multiplying by A inverse here; so, that is only possible when A inverse exists. So, we should not do such calculations when A inverse does not exist. So, we should verify first whether A inverse exists or not in this case naturally it exists. So, here we multiply them by A inverse and what we will get the right hand side will be A inverse and this 2 we can divide here. So, we will get 1 by 2 A minus 7 I that should be the A inverse. So, very easily we got this A inverse here. So, we need to do just a simple calculations to get this A inverse A minus the 7 times I. So, that is the value of this A inverse using the Cayley-Hamilton theorem.

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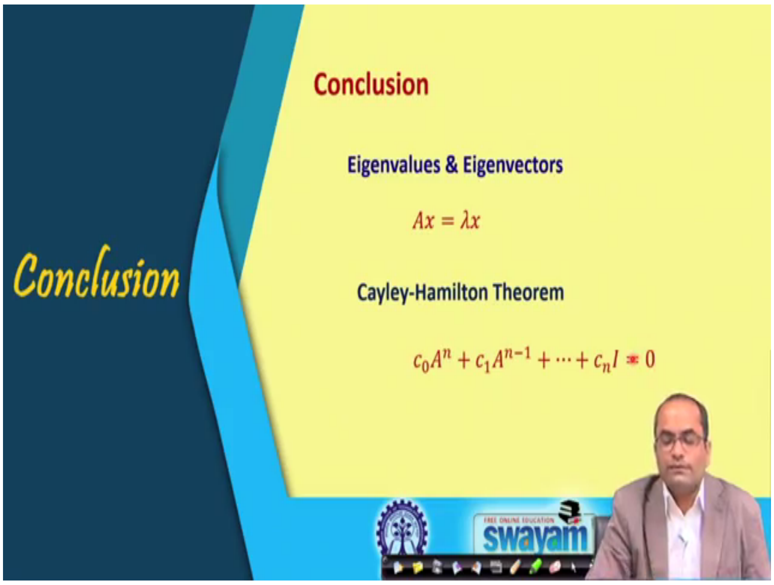

So, coming to the conclusion what we have done here, we have studied, we have introduced the eigenvalues and eigenvector and basically this equation was very important this Ax is equal to lambda x. This lambda is the eigenvalue and the corresponding eigenvector will be given by x. And, we have also studied this Cayley-Hamilton theorem which says that every square matrix satisfy its own characteristic equation. So, here the right hand side this is a 0 matrix so, the same order having all the entries 0.

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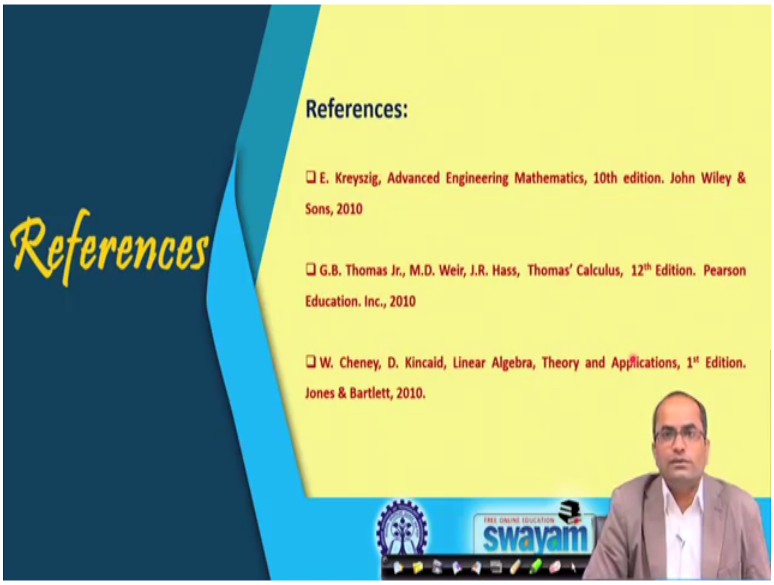

And, these are the references used for this for preparing these lectures.

Thank you very much for your attention.