Engineering Mathematics - I
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Lecture - 45
Linear Transformations (Contd.)

So, welcome back. And this is lecture number 45 and we will be talking about or continue our discussion on Linear Transformations.

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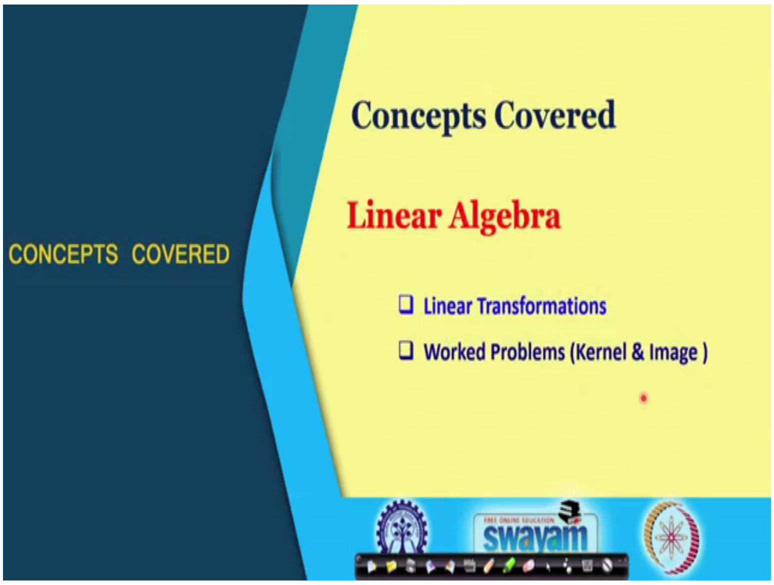

And today, we will see some worked problems where we will find out the Kernel and the image and their dimensions.

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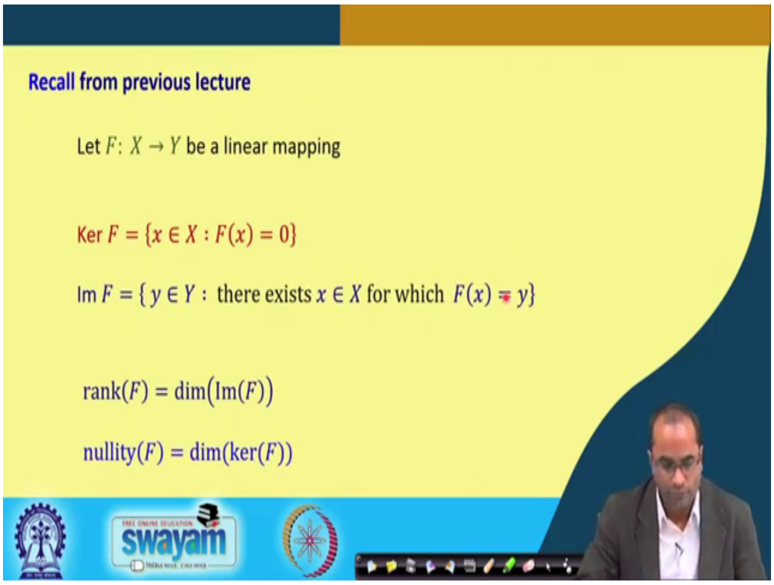

So, here just to recall from the previous lecture what we have seen there that F is a linear map than this Kernel F is defined as the vectors from X whose map is 0 in this vector space Y and the image was all the elements of y there exists Ax here in the vector space x for which exactly this y is mapped. So, F x is equal to x.

So, these were the two definitions for the Kernel of F and also for the image of F. And we have also seen that this rank of F is nothing but the dimension of the image of F and the nullity we define as the dimension of the Kernel of F.

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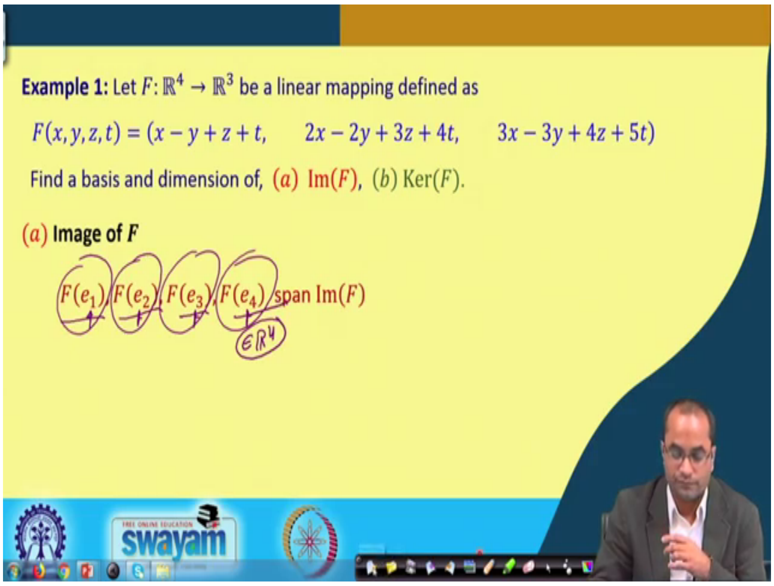

So, with these definitions we start now here with the example, where we have taken this F linear map from R 4 to R 3 is a linear mapping which is defined by this relation F maps this element which is from R 4. So, x, y, z and t and it maps to x minus y plus z plus t the second component here 2 x minus 2 y plus 3 z plus 4 t and then this is the third component 3 x minus 3 y plus 4 z plus 5 t. So, this is the element from R 3 and this is the element from R 4 and that is what we said here this F maps an element from R 4 to an element in R 3.

And what we want to now find? We want to find basis a basis and the dimension of first the image of F and then second for the Kernel of F. So, we need to find what is the image and what is the Kernel of F. So, here first let us talk about the image of F. What we know from the previous lecture that once we have the elements in vector space x which span this vector space x. So, for instance here we have this R 4. So, we know that these a standard basis because they are simple to work with, so let us take whether standard basis here that this e 1, e 2, e 3, e 4 these are the standard basis from R 4 and we know that the theorem that result from the previous lecture that these e s span this x R 4 than this F e 1, F e 2, F e 3, F e 4 these are the vectors in R 3 and they will span actually the image of this mapping F.

So, what we know now that this F e 1, F e 2, F e 3, F e 4 they will span the image F. So, that is the result we have now that these are the vectors which span the image F. And now let us compute what are these F e 1, F e 2, F e 3, and F e 4.

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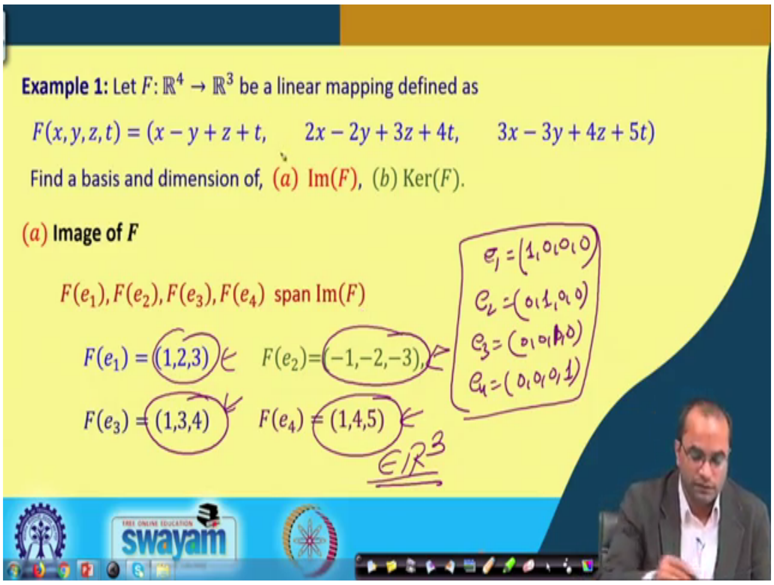

So, here F e 1, so e 1 just remember that e 1 is nothing but in this case these are the standard vectors from R 4. So, these are that is the 1 0 0 0, from R 4 the e 2 will be 0 1 and 0 0, e 3 will be 0 0 0, so here 1 now and 0, and e 4 will be all these first 3 components 0; this is the forth one. So, we have these standard basis from R 4 and now F if we apply on this e 1 here. So, what will happen? So, 1 this all 0s, so we will get one from here and then this is any way 0s, the one from here will get two all these 0 and 3. So, this e 1 will be mapped to this 1 2 3 elements. Similarly, the F of e 3, e 2 if we take e 2 then the second will survive here this minus 1, minus 2 and minus 3 that is the map of this F of this e 2.

Similarly, F of e 3 and F of e 4. So, we have these vectors here 1, 2 3, minus 1 minus 2 and minus 3, 1 3 4, and 1 4 5 from this R 3 and we know now with this with the theorem we have studied already that these vectors will span image F and what we want to find out we want to find out what is the image F, what is what are the basis of image F and what are the what is the dimension of image F. So, we already know that these 4 vectors will span image R 3, but they are they are not the basis because this is this is the spanning set, and spanning set may have some linearly dependent vectors. So, what we have to now extract out of these vectors what we have to collect only the linearly independent vectors. So, we have to see now how many linearly independent vectors we have in this spanning set. And the number of those linearly independent vectors will be the dimension of the image F and those linearly independent vectors among all these 4 will form basis of the image F.

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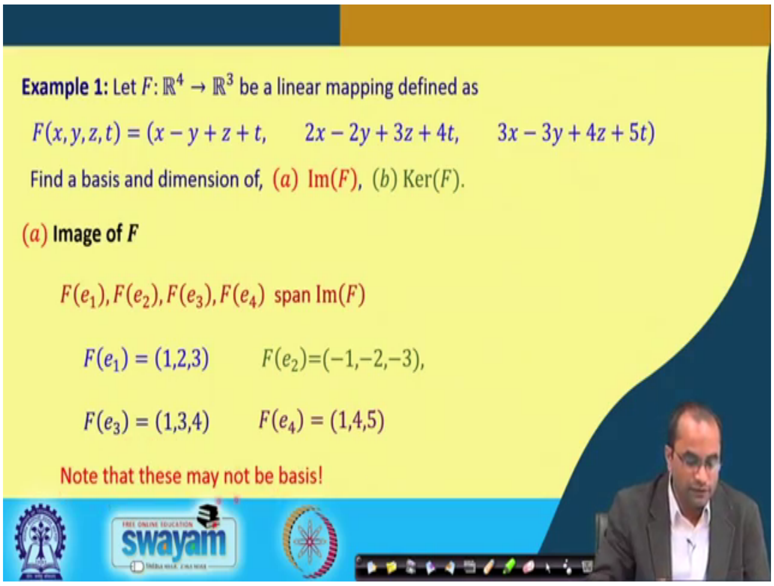

So, here now let us do this. So, these may not be the basis now we have to just find out that how many of these vectors are linearly independent.

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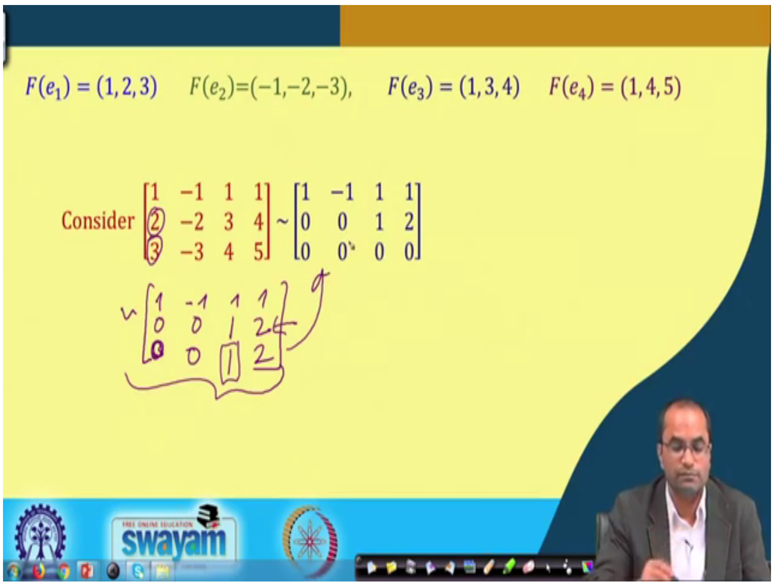

So, to do so, so we have this 1 2 3 these 4 vectors we will the idea is to have we can put them in the column for instance. 1 2 3 the second vector here minus 1 minus 2 3 and 1 3 4 and then 1 4 5. Having this vector here whose columns are the given vectors we can now reduce it to this row reduced echelon form and from that row reduced echelon form we can find out that which vectors are linearly independent and which vectors are linearly dependent and we have seen one such example before in previous lectures.

So, now to find out the linearly independent vectors out of this what we can do we can place them in the in this matrix here as the columns, and then we can reduce it to the row reduced echelon form which should not be difficult here in this case because as a first step we will set these two elements to 0. So, the first will first row will be as it is, the second one here will become 0, the two times of that this is also 0, two times this will give 1, here two times will get 2, here now the 3 times of the row 1 we are subtracting here so this will be 0, this will be again 0 and the 3 times this will be 1 and 3 times this will be 2. So, this is the one step of this row reduction process and now we will said this 0 with the help of this row 2 and we will get this matrix which is given already there.

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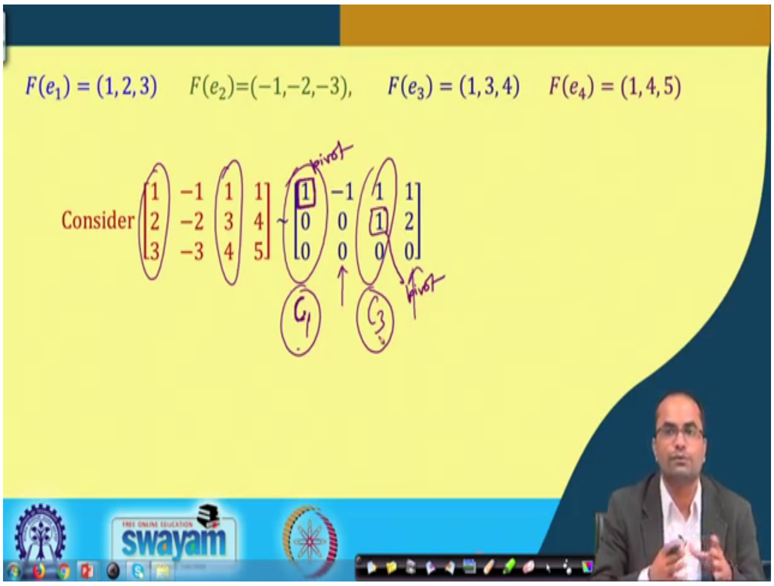

So, what we will have that this is the row reduced echelon form and in this case now, we will find out these pivots here. So, we have this one as a pivot and this is also pivot here, only there are two pivots. So, this column number 1 and the column number 3 they have the pivots. So, a straight forward we can pick the column number 1 here and the column number 3 and they will be linearly independent. These column here does not have a pivot this does not have a pivot and one can show that those two columns depend on this column number 1 and column number 3. So, this was a very systematic approach to found out the linearly independent vectors here

So, now, with this we have we can take this column number 1 which has the pivot here, so the corresponding this column number one and we can take this column number 3 and they will form the basis.

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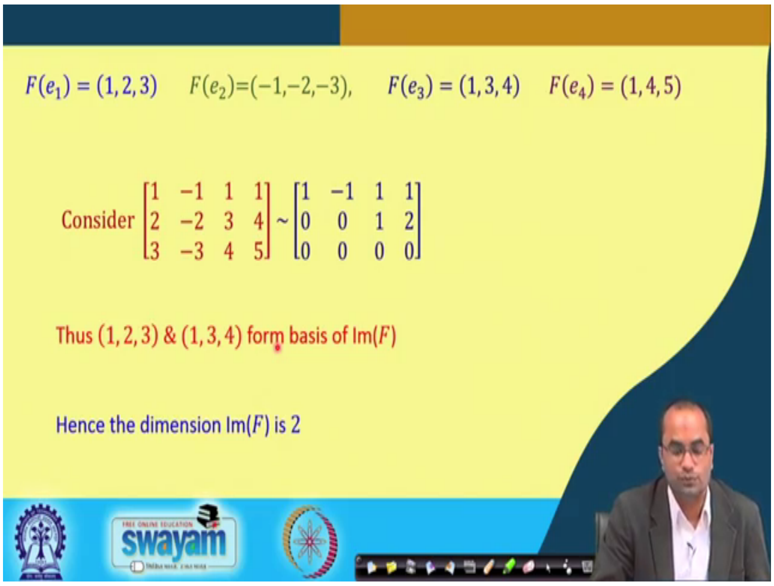

So, the dimension of this image is going to be true and the basis these two vectors. So, 1 2 3 and 1 3 4 these will form the basis of the image F and the dimension of the image F is 2.

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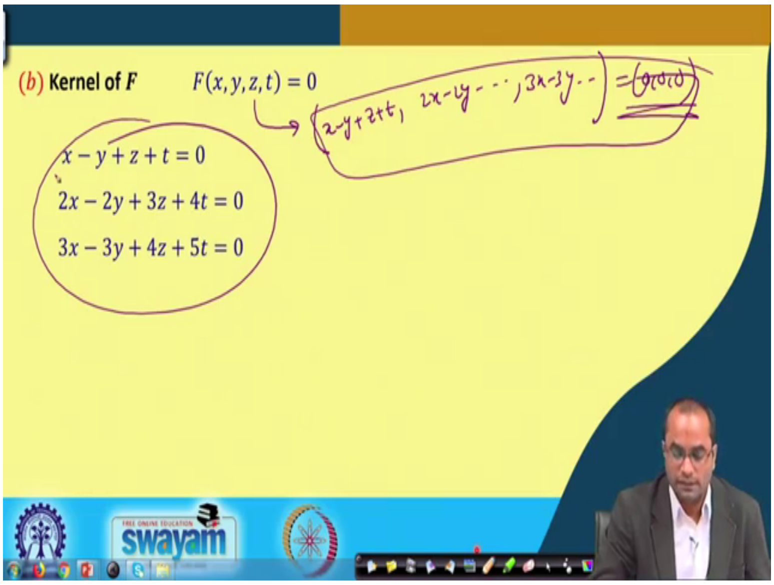

Now, getting to the Kernel of F; so, what was the Kernel of F? Now, we are looking for the elements in x whose map is 0 in y, so that means, we are looking for these elements x, y, z t I mean these vectors or such vectors whose image is 0. And now to look into this now what is F x, y, z t from the definition we have that x minus y plus z plus t must be 0 and this was the second component of this and this was the third component of this vector in R 3 which is the map of this F there and that should be equal to 0, so the 0 0 0.

So, if again here the F of this was nothing but these vectors F, this was the first one and this 2 x minus 2 y and so on and then 3 x minus 3 y and so on and this we want to be equal to 0 0. So, from here we get actually these 3 equations, these 3 equations.

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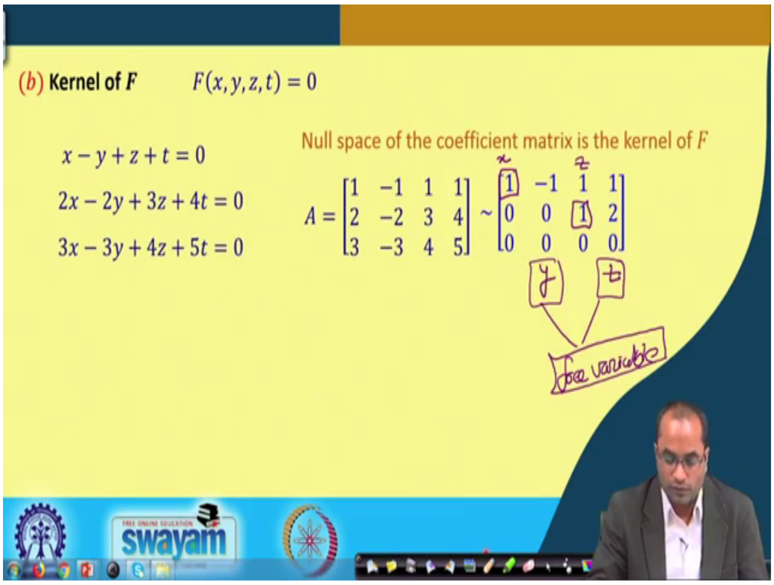

And we want to now solve these equations to get all possible values of these x, y, z and t and this is nothing but the null space of the coefficient matrix of the coefficient matrix of this of this system of equations and that is nothing but the Kernel of F. So, the null space here of this coefficient matrix will be the Kernel of F, because what is the Kernel of F all these vectors here from R 4 whose image is 0 they are not 3. So, what we need to solve? We need to solve this system of linear equations and we will find out how many solutions are there or how many linearly independent solutions are there or in other words we call the generators of the solution of this system of homogeneous equations and from there we will find out what is the what is the dimension of the Kernel, what is the what are the basis of the Kernel.

So, we need to solve finally, this system of equations. So, let us put in this matrix form. So, the matrix is 1 minus 1 and 1 1 the coefficient matrix 2 minus 2 3 4 and this 3 minus 3 4 and 5. And again we need to just convert into the reduced echelon form which we have done just before and now we can easily find it out because this is the pivot here and this is the pivot. So, corresponding to this x this is corresponding to y this is corresponding to z and this is corresponding to y and this is corresponding to t. So, here these are the columns where we do not have pivots. So, these are the free variables which we call, so there are two free variables and that will define exactly the nullity or the dimension of the null space that will be 2 here. So, the dimension of this Kernel is clear that is going to be 2 and we have to find the basis for that we have to write down solution now.

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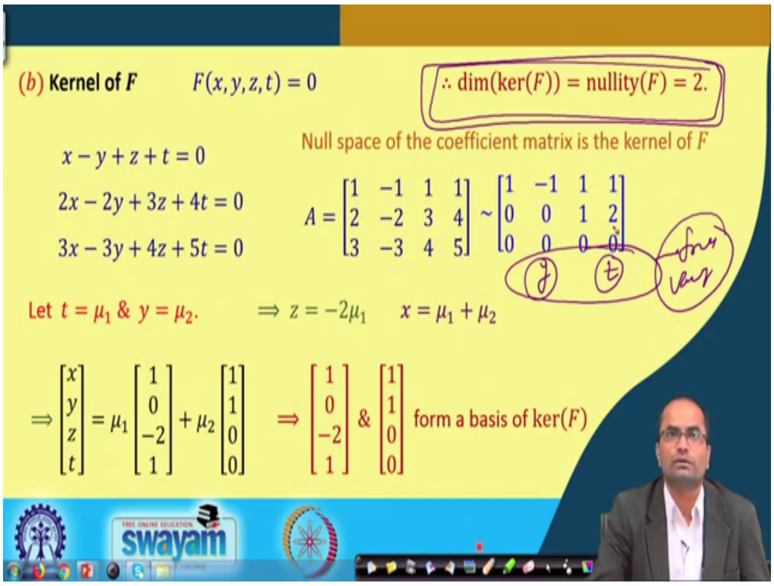

So, writing the solution out of this. So, first we will assume these free variables. So, that was y and this t these are the free variables. So, we will take some values for this free variables and then, so let us take for this t is equal to mu 1 and for y we take mu 2 some number arbitrary number because we can choose these y and t whatever we like these are the free variables. So, we have chosen this mu 1 and mu 2 and then we can compute the others. So, for example, the z form this equation z plus 2 t is equal to 0. So, z will be minus 2 t of minus 2 times mu 1, so that will be the z. And from the equation number 1, we can get this x that will be coming out as mu 1 plus mu 2.

So, we have this t, z and x all and also y here with mu 2. So, we can write down in a matrix in this vector form x, y, z, t will be this mu 1 the constant here and this we can collect now for x, x is one the coefficient here for y it is 0, for z it is minus 2 and from t it is 1. So, similarly for with mu 2 also we can also collect the coefficients here. So, in x we have mu 2, so with 1 and then here y also we have this mu 2 which is 1 and this z also has this mu 1 which is minus 2, no no it does not have mu 2. So, here 0 and also this t does not have mu 2. So, with this coefficient is 0.

So, we have written this x, y, z the solution in terms of this mu 1 and mu 2 we can play now mu 1 mu 2 can give any values to mu 1 and mu 2 and we can keep on generating the solution here of this system of equation and the solution that is nothing but the Kernel of F. So, here these are the two generators which generates the solution or the they span the solution, also these two vectors are linearly independent. So, we have this linearly independent vector which can span the whole Kernel of F. So, naturally these two vectors will form the basis and the dimension of this basis here will be 2. So, these two vectors form the basis of the of the Kernel F and the dimension or the nullity which we call is already there the dimension of the Kernel or the nullity of this F which was clear also from the number of these free variables which are 2 here. So, we can just get the dimension or the nullity and also the basis simply.

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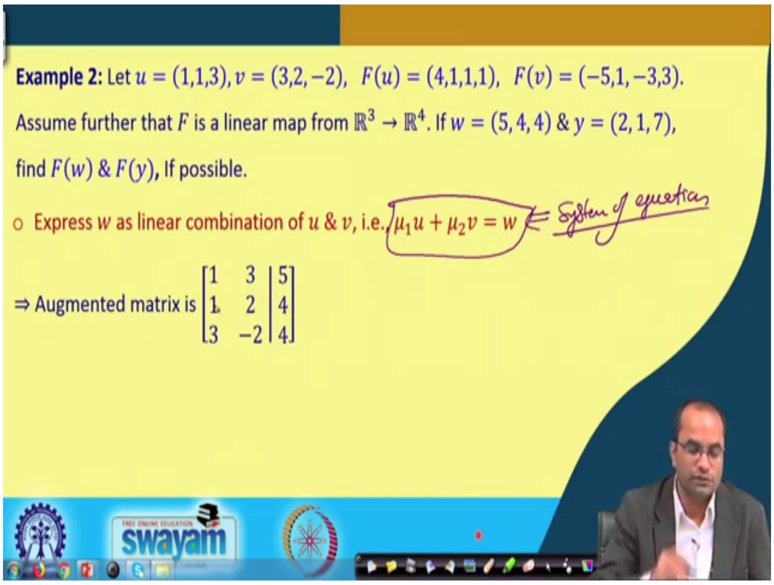

Now, the another example where we have u v and it is also given that the F u some map on u is 4 1 1 1 and also F 1 v it is a minus 5 1 1, minus 5 1 and minus 3 3. And we assume that this F is a linear map from this R 3 to R 4 because it maps this element R 3, this u and v are from R 3 and the output is in R 4. So, this maps from R 3 to R 4. And what is the question now that if w is given as 5 4 4 and y is given as 2 1 7, then can we find this F w and F y, yeah. We know only that F u is given and F v is given, but we want to find now what will be the F w the w vector is given as 5 4 4 and this y is given as 2 1 7 and we want to find this F w and F y and F is a linear map. So, what do we know about the linear map this additive property; that F on u plus v is equal to F u plus F v.

So, what we can think here? That if this w we can write down in terms of this u and v if. So, if we can write down this 5 4 4 in terms of the vectors u and v then we can find out what is the F of w. If we can if we can write this w as a linear combination of this u and v then we can also find out the F and similarly with y also if we can write down this y as a linear combination of u and v, then we can apply this transformation F on y and we can write down in terms of u and v and that will be the image of this y.

So, let us try now how we can write we can express this w as a linear combination of u and v; that means, the w can we write as mu 1 u and mu 2 v for some scalars mu 1 and mu 2. So, to do so, we have to now again form this augmented matrix to solve this system of equations here for mu 1 and mu 2. So, here u was 1 1 3, so that will be placed here in the column when we write down these system of equations. So, the first equation we will get this one times mu 1 2 3 times mu 2 is equal to this the first component of w that is 5 there, the second equation will be mu 1 2, mu 2 is equal to 4 the third equation will be 3 mu 1 plus minus 2 mu 2 is equal to 4. So, out of those that is system of equations here from this linear combinations. So, from those, these system of equations we can write down this augmented matrix and this augmented matrix is given here.

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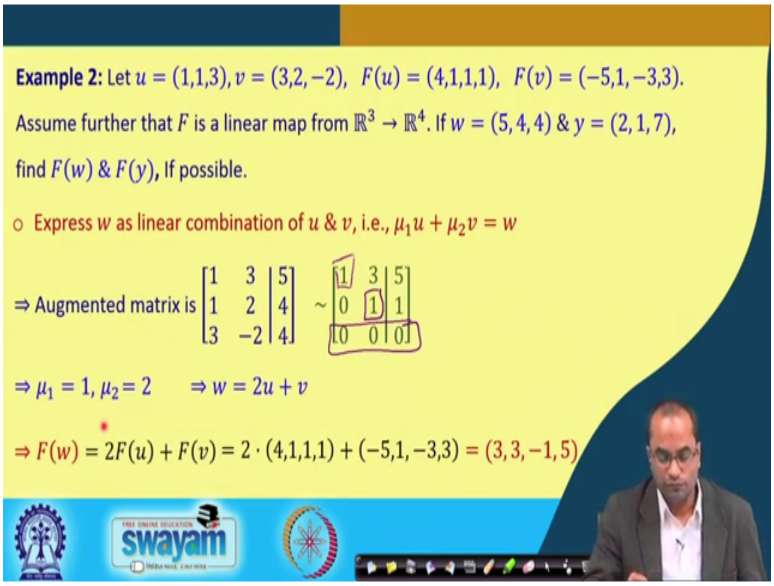

So, now out of this augmented matrix, we have to get this row reduced forms. So, this 1 3 5 and this is simple here. So, we can subtract it and then further. So, we will get this reduced form, and from this reduced form we will now observe that this is the pivot here this is also the pivot.

So, the first column has a pivot, second column has a pivot and everything is constant here the full row is 0. So, we can find out the mu 1 and mu 2. So, here the mu 1 is coming to be 1 and mu 2 is coming to be this two out of these two equations now; we have solved this we have solved that this w we can represent in terms of u and v as u times 2 v. So, the w is 2 u plus v and once we have this relation we can apply now the linear transformation. So, F w will be two times the F u plus F v and then the two times the F u v know and this F v we know and then we add it to get this here 3 3 and minus 3 5. So, and the key point here was that if we can represent this w as a linear combination and then we apply the linearity of this map here to get this value of this F w.

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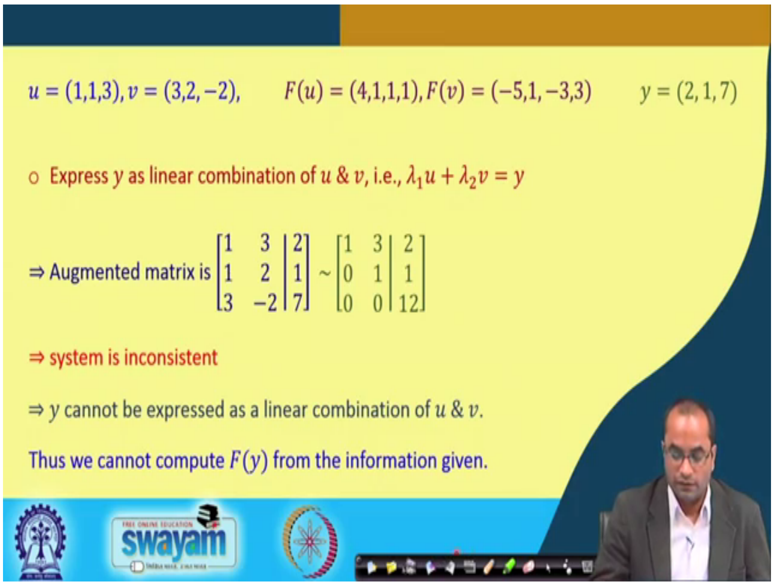

Now, if we try to do for the y again the same steps so now the y is 2 1 7 and we have this u and v. So, again we will try to express as a linear combination of u and v to this y and again we will get this augmented matrix that is 1 1 3 and then we have 3 2 minus 2, the right hand side that is y here, so that is 2 1 7. And again, we will convert into the row reduced form, so we got this row reduced form now.

Now, the problem here is the inconsistency the last column the right most column here has a pivot element here this 12 which should not happen. In this case and now because of this we have this inconsistency in the system and which leads to the nonexistence of the solution. So, here we cannot express this y as a linear combination or this u and v and therefore, the system is inconsistence, it is inconsistent and this as a reason that we cannot we cannot compute this F of F of y because the information given is not sufficient here. So, we cannot compute this F y from the given information.

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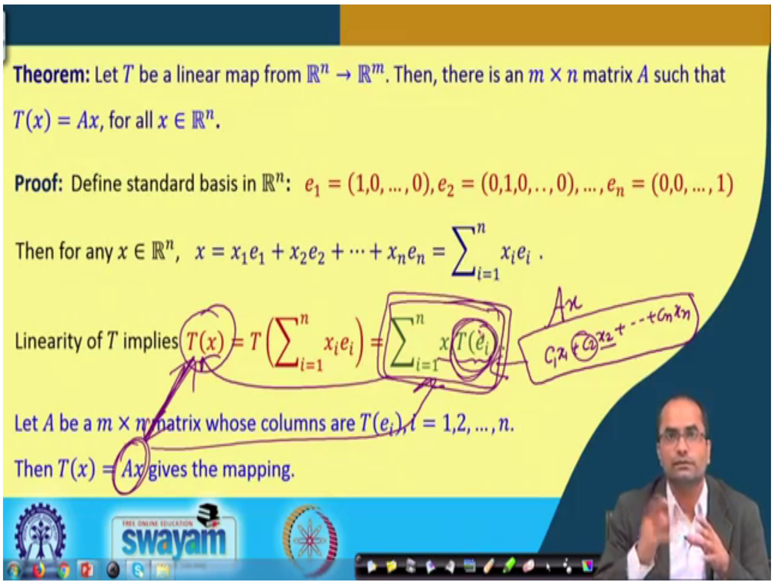

Well, so now, there is another nice result that if T is a linear map from this R n to R m T is a linear map from R n to R m then there is an always m cross n matrix a such that this T x you can represent in terms of this matrix A. So, whatever we do with the T x that is linear map we can actually work with the matrix A just multiply to this x because working with the matrices are much easier and we know many operations and many other properties of the matrices. So, in speed of working with linear map which is not given in the in the form of the matrix it is better to have a matrix form and then we can work with the matrix we can do all operations whatever we want on this map with this matrix here A.

So, that is a proof is very simple here we can define the standard basis in R n which given as this 1 0 0, and this one 0 1 and 0 0 and so on. These are the standard basis from this vector space R n. And this T is a for any x now in R n, so any vector we take from this x from R n what we can because these are the basis here these are the standard basis we can represent this x in the form of this basis; that means, this x can be represented as x 1 e 1. So, x 1, x 2, x 3, x n are the components of this x vector. So, we can represent with the help of these standard basis this x as x 1 e 1 plus x 2 e 2 and so on plus x n e n or in short we can write down in the form of summation that i goes from 1 to n and x i e i.

So, this x which is a general element in R n we have represented as a linear combination of that those standard basis and now if we apply this T, the linearity of T will implies that T x T of x will be the T of this here the summation and basically we can take we can apply on this T i's. So, we have this i 1 to and n, this T is applied on this vector i here because of the linearity this is the linearity property that T of this x 1 e 1, x 2 e 2 and so on will be x 1 into T e 1 plus x 2 into T e 2 and this x n into T e n. So, that will be the will be the relation here and how we define now the A which will be exactly corresponding to this map T or will function as this map T there. So, A now we can define whose columns are these vectors here T e 1, T e 2, T e n. So, these columns if we put into this matrix A then this Ax will be nothing but exactly this product which is the T x.

So, if you remember from this definition of this matrix product which we have seen several times. So, that is nothing but this is column 1 or will be multiplied to this x 1, the column 2 will be multiplied to x t2wo and so on. The column n will be multiplied to x n that is another way of looking at the matrix vector product and here exactly we have used that idea that this x 1 multiplied by this vector T e 1. So, T e 1 if we place as a first column in our matrix and similarly this x 2 and then this T e 2 placed in the matrices second column. So, this product here Ax will be exactly this one which is nothing but the T x. So, the T x is exactly the Ax.

So, by this simple idea what we have seen that any linear map which maps the elements of R n to R m we can have matrix here corresponding to this T e map we can have a matrix A of order again this m cross n and whose columns we can easily compute with the help of this T e i's and this will give exactly the map which is given as this from R n to R m.

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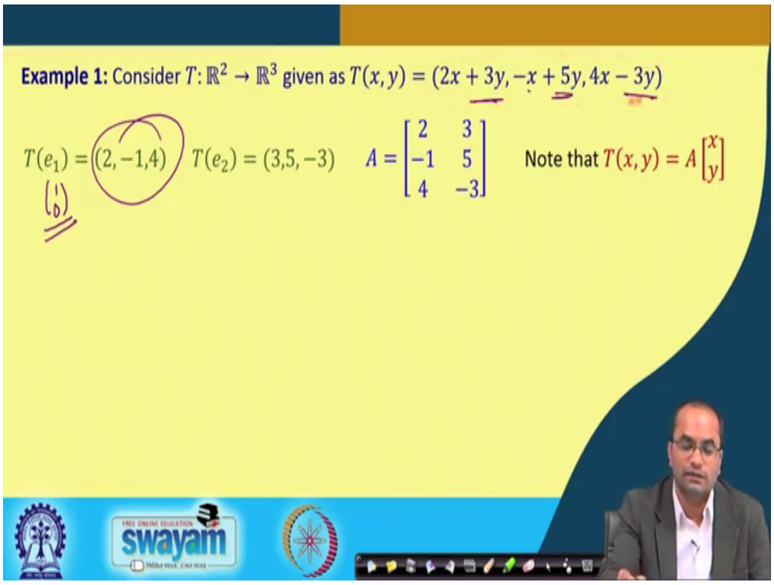

So, now let us just quickly go through some examples here. So, this was the one example which we have already discussed before. So, this T x y is given as this is from R 2 to R 3. So, this is an element from R 2 and then we are getting this element R 3. So, now, how to get this corresponding matrix A? The idea was that we compute this T e 1. So, T e 1 means the e 1 here is nothing but this 1 and 0.

So, the T e 1, so 1 and 0, so y will be set to 0. So, we will get 2 minus 1 4, 2 minus 1 4. Similarly, we can get the T e 2 and that will be just the 0 and 1 there. So, 0 and 1, so we will get 3 here 5 and minus 3. So, that will be the second element. And then in A we will place them, so 2 minus 1, 4 has the first column and 3 5 and minus 3 has the second column of this matrix A. And we note that now that this T x the given linear map here which was defined in this way we can also defined as A and this element of this R 2 because this a into this x y will exactly give us this linear map because if we multiply A with this x y. So, here we have 2 3 minus 1 5 and 4 minus 3 and with this x y what do we get here 2 times x and plus 3 times y the second component minus x plus 5 y then we have 4 x and minus 3 y.

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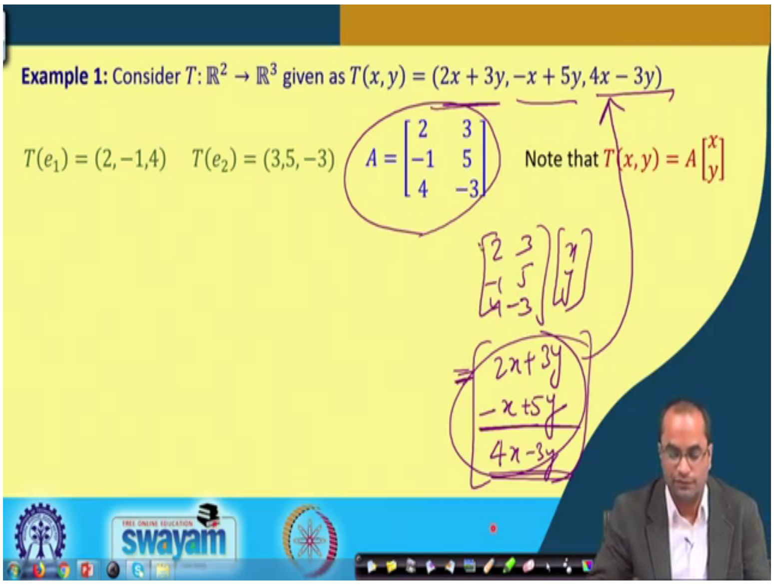

So, we are getting exactly the same mapping which was defined there with the help of this T, the first component this 2 x plus 3 y then minus x plus 5 y 4 x minus 3 y. So, the given linear map we have defined with the help of this matrix A.

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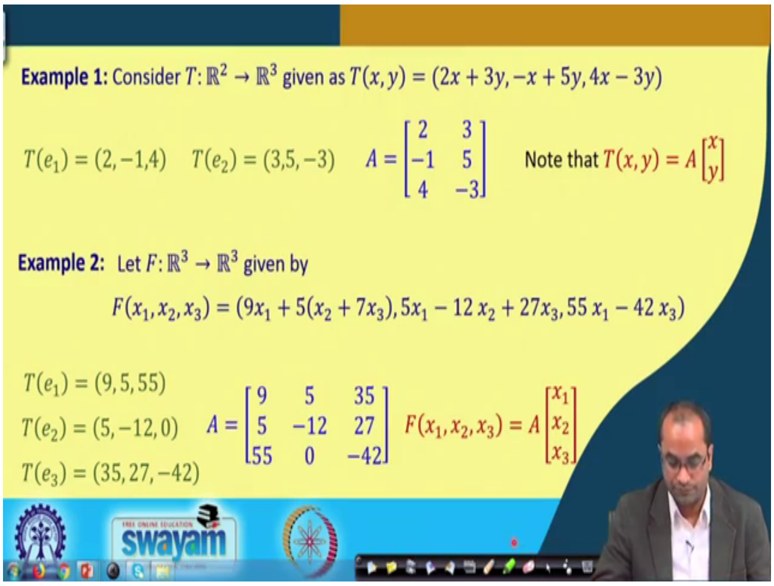

One more example we can look quickly here this R 3 to R 3 which was which is given here by this element from R 3 and then here also we have this element from R 3. This is the first component then we have we have a this first component and then we have the second component, we have the third component. So, this maps R 3 to this R 3 and here the same idea which we can use again, which we have used before that this T of e 1 will compute that will become 9 5, 9 from here, then 5 from here and 55 from there. T e 2 and then T e 3 we can apply on this standard basis and then we can place them in the columns that will be exactly our matrix here 9, 5, 55. Here 5 minus 12, 0 and 35 27 and minus 42.

So, we have this A here, the matrix A and that will define this linear map the given linear map as this matrix vector product. So, again this working with the matrices are much easier. So, this is one way where we can, where we can represent a linear map with the help of this matrices.

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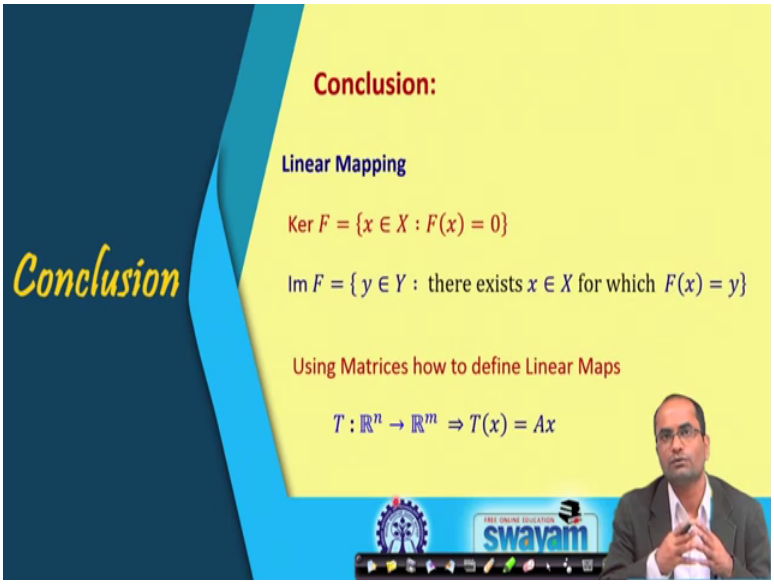

So, here what we have done today, with this from the linear map we have done some examples where we have computed the Kernel F also the dimension of the Kernel F as well as the image F and its dimension. Further, we have also looked at this using matrices how to define the linear maps. Whenever there is a linear map T which maps R n elements to this R m elements, we can always define corresponding matrix A which will work as this Ax instead of this T x. So, that is very useful as I said before this working with matrices are much easier.

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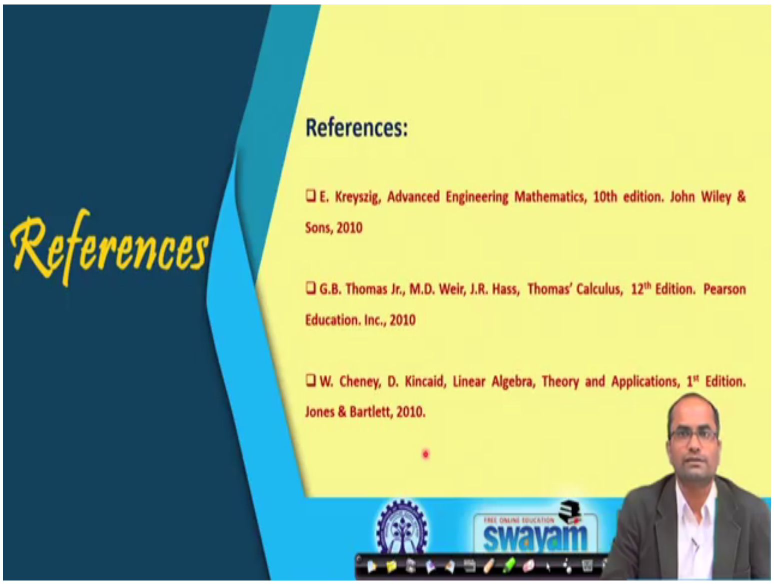

So, here these are the references here we have used for preparing the lectures.

And, thank you very much.