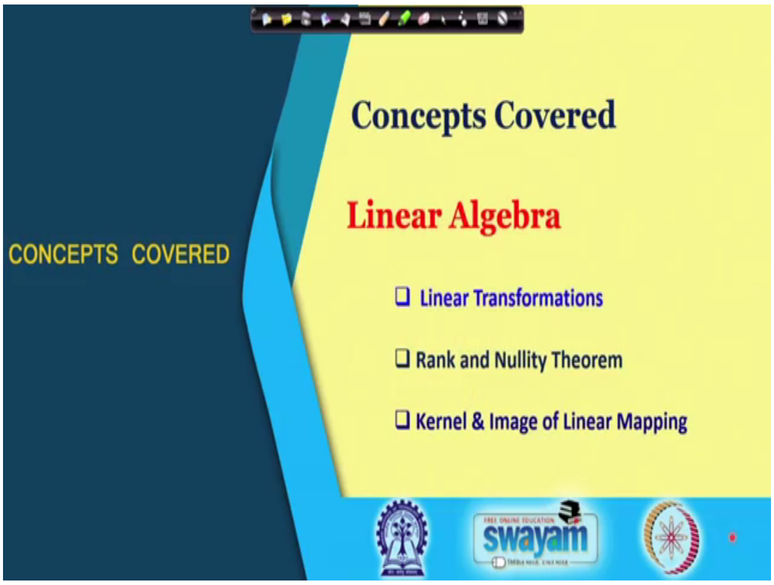
Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 44
Linear Transformations

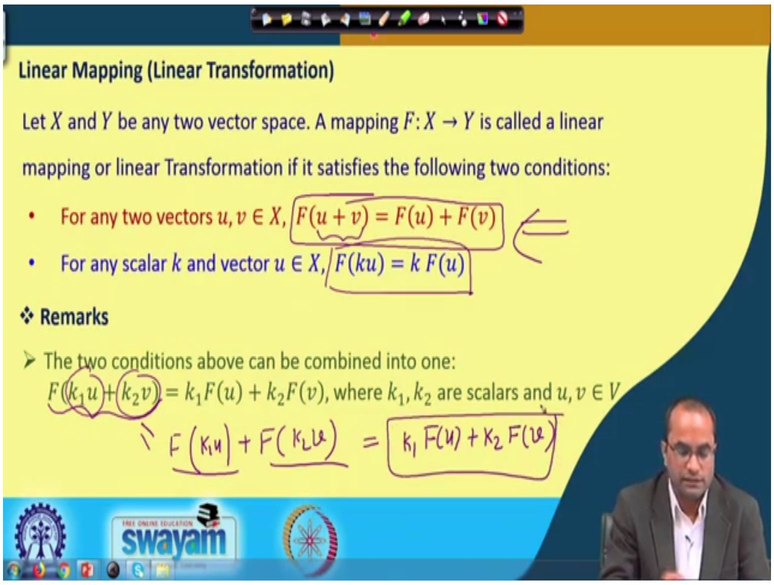
Welcome back and this is lecture number 44 and we will be talking about Linear Transformations.

(Refer Slide Time: 00:22)



And in particular we will also talk about the rank and nullity theorem for these linear transformations and also the kernel and image of linear transformations.

(Refer Slide Time: 00:31)



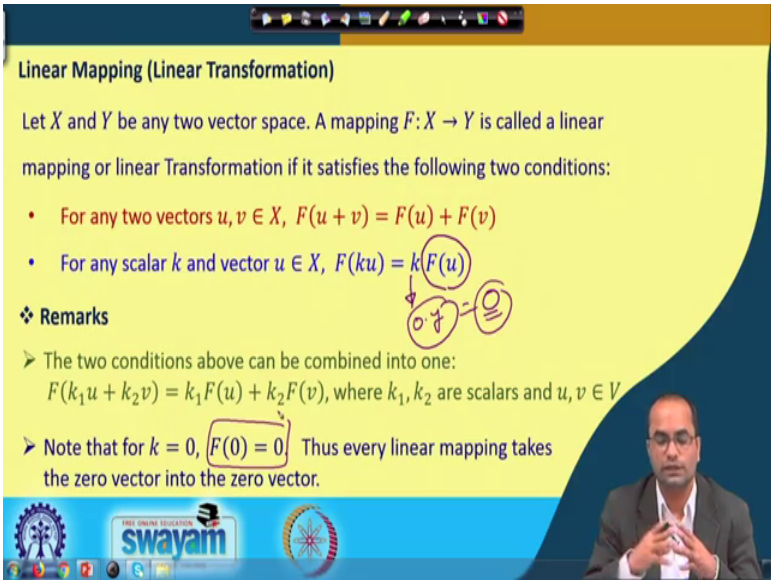
So, let me start with what is linear mapping or linear transformation. So, here we talk about these 2 vector spaces. So, X and Y be two vector spaces and the mapping F from X to Y is called linear transformation or linear mapping if it satisfies the following 2 conditions. So, what are these conditions? For any two vectors u and v from this vector space X, if we apply this transformation F on u plus v this should be equal to F u and plus F v.

So, that is one conditions for out of these 2 conditions this is one which is required to say that this is a linear transformation. The second one for any scalar k here from the set of real numbers and a vector any vector u from this X this F should also satisfies that F of the multiplication of this k scalar multiplication of this k to u. So, F k times u should be equal to k times F u. So, we have these 2 conditions required for the linearity. One is this F u plus v must be equal to F u plus F v and the second condition we have that this map should also satisfies that F k times u k is a scalar number k times u must be equal to k multiplied by F u.

So, here 2 remarks, one the 2 conditions given above. So, these 2 conditions which we have just discussed this F u plus v is equal to F u plus F v and F k u is equal to this k times this F u. And these 2 conditions can be combined into one and as follows that F times this k 1 plus k 2 v is equal to k 1 F u plus k 2 F v. So, here we are combining basically the two. So, one was this with the addition of this u plus v which is already here that F on k 1 u plus k 2 v must be equal to F k 1 u and plus F k 2 v. So, in the first step for example, we will think it as k 1 u and then plus this k 2 v, this is exactly the condition number 1 and the condition number 2 gives now that this should be equal to k 1 F u and plus this k 2 F v.

So, both the conditions are satisfied. Once this condition here that k u plus k 2 v on this F and we apply if it is equal to k 1 F u and k 2 F v then we can call that the given transformation is a linear transformation.

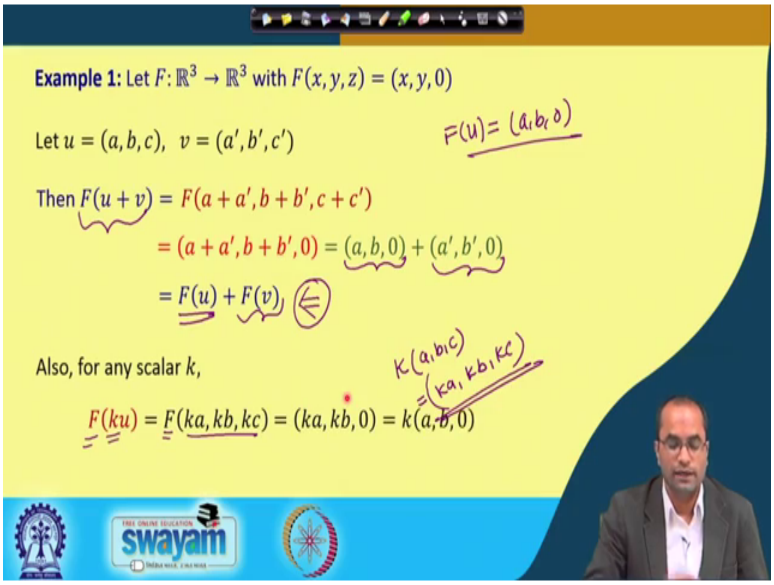
(Refer Slide Time: 03:41)



So, again note that that for k is equal to 0, so, in the second condition for instance if we put k is equal to 0 that will give us that F. So, 0 into u that is a property of the vector space this should be 0 vector then. So, F of the 0 vector must be equal to. So, here again we have this 0 into this F u. So, when we multiply this 0 to F u, F u is an element in this vector space X and again this 0 into this element F u must give 0 element. So, here we are getting again this 0 F 0 must be equal to 0. So, this F u is a element of Y. So, again the same thing should hold when we have this 0 and something from this Y. So, this should give again the 0 vector in this Y. So, here F 0 so; that means, that every linear map takes the 0 vector from this domain X to the 0 vector in this range Y.

So, that is another important property which we can quickly look for the vector spaces; that means, this F 0 must be equal to 0. So, 0 should map to the 0 vector in the vector space Y.

(Refer Slide Time: 05:05)



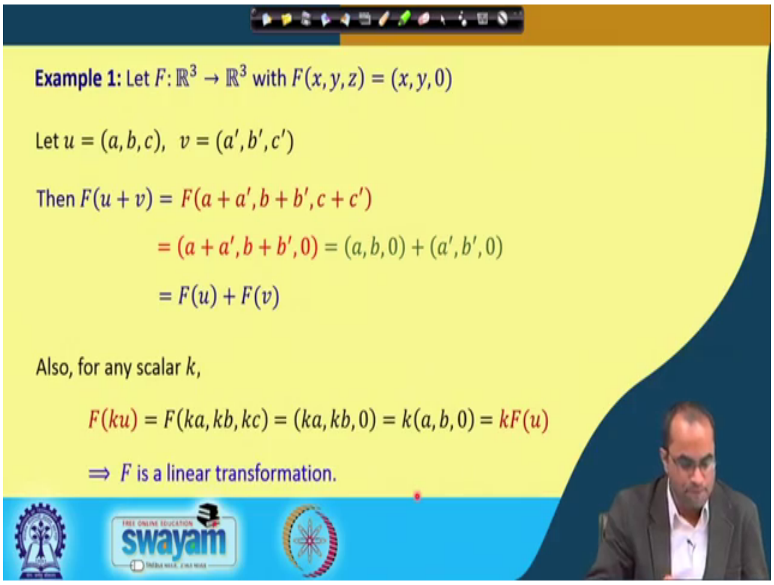
So, we go through some of the examples where we do see this linear maps. So, first here let F is a linear map here R 3 to R 3 with F x y z is equal to x y 0. So, here every element of this R 3 which is denoted by this x y z and it maps to this x, y and the z becomes 0. So, it is a projection of of that point x y z to this x y plain. So, whether this map is a linear map or not we can verify this. So, we take 2 elements from this R 3. So, 2 members of this R 3; one is a b c and other one we have taken this v a dash b dash and c dash. And now these 2 properties of the first property is this that when we apply F on this u plus v they should give us F u plus F v and that we will check here.

So, the F of u plus v; that means, here the F the u plus v will be the sum of these 2 elements here 2 vectors. So, a plus a dash b plus b dash c plus c dash that is exactly u plus v here and now we want to see what is the F of this. So, again the definition as per the definition this third coordinate will be set to 0; that means, this will be equal to a plus a dash and this b plus b dash and the third argument will be set to 0. So, that is what the how this mapping is defined. So, now, we have this vector here again from R 3 because the map is from R 3 to R 3 and now we can rewrite this vector as the sum of these 2 vectors. So, first one a b 0 and the second one a dash b dash and 0 because the sum of these 2 vectors is nothing but this a plus a dash and b plus b dash and the third element is 0.

So, here this a b 0 is again as per the definition here is F of u because F of u will be again as per this definition when we take F of u that is u is a b c. So, this will give a b 0. So, this is exactly here a b 0 which we can write down as F u and here this a dash b dash 0 its nothing but the F v. So, what we have seen here that F of u plus v is equal to F u plus F v. So, the first condition of this linearity is satisfied and now we will check for the second one which is also trivial in this case. So, for any scalar number this k, we consider this F of this k into u and as per the definition this k into u will be just multiplied, k will be multiplied to each component of this a b c. So, this will be k a and this k b and k c.

So, now we are getting here the F on this k u; that means, F on this k a k b k c element. And as per the definition of as per the definition of this linear map we will get this as k a, k b and this 0 here. So, again we will use the idea which was used earlier in terms of the addition. Now, we will use that out of this k a k b and 0, we can take this k outside this point a b 0. And by doing so, we are getting here the k times this a, b, comma 0 and this is nothing but again the k times and the function of u.

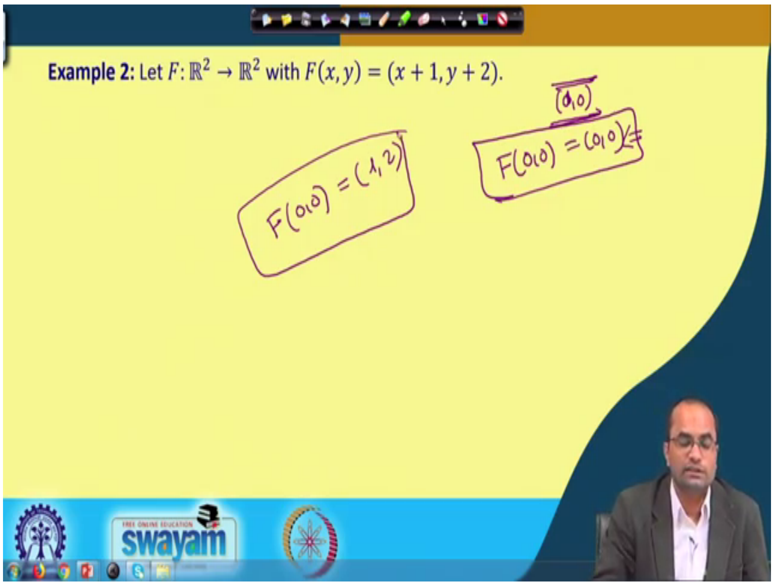
(Refer Slide Time: 09:00)



So, this is k times the function of u. So, what we have checked here again the second property that F when we apply on k u we are getting this k times the function value at u or this map at u.

So, here both the conditions F u plus v is equal to F u plus F u F v and also F times F when we apply on this k u, we are getting k times F u. So, both the properties of the linear map a satisfied are satisfied and therefore, this given map here which maps the point x y z to x y 0; it is a projection map and that is linear map which we have just proved here ok.

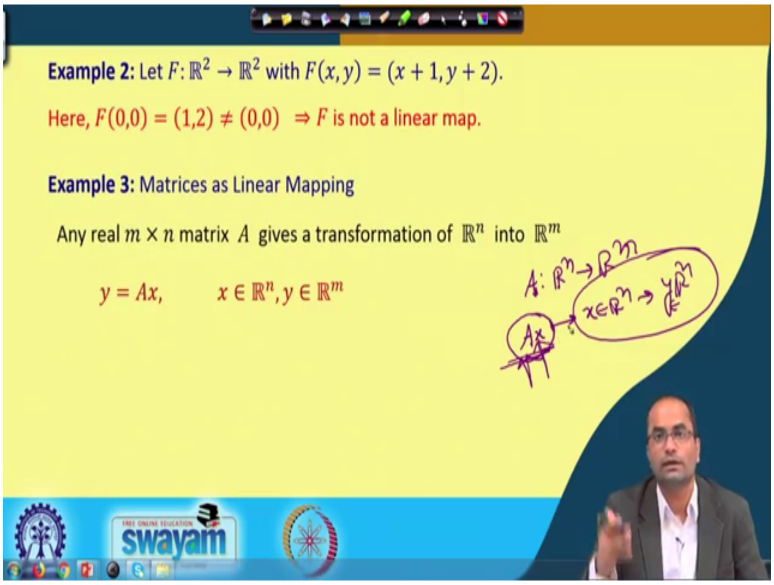
(Refer Slide Time: 09:48)



So, another example we will take that this function F is defined the map is defined as F x y when we apply on this x y we are getting x plus 1 y plus 1. Now the question is whether this is linear map or it is not a linear map? And this is clear from here because one of the properties of the linear map is that it always maps 0 to 0 and we have here in R 2 our 0 element is nothing but 0 comma 0, that is the element in R 2.

So, this F must map the 0 0 element to the 0 0 element if it is a linear map that is a necessary condition of this linear map. But what we observed here then when we apply this F on 0 0, what we are getting? We are getting 1 comma 2.

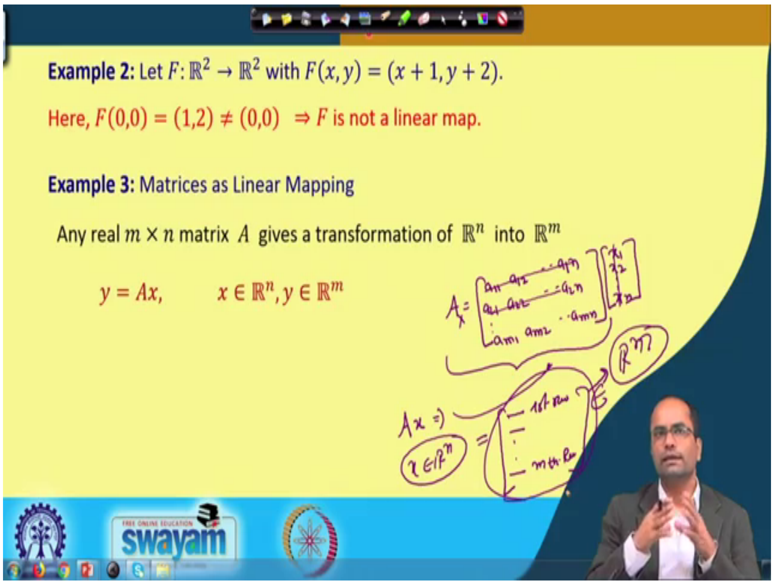
(Refer Slide Time: 10:45)



So, in this way this cannot be a linear map because it is not mapping 0 0 element to 0 0 element, but it is mapping 0 0 element to 1 2 element which cannot be a linear map. So, another very important example we will see that these matrices are also like linear maps because of the reason we take any m cross n matrix and A is the transformation from this R n to X m by this rule here that if we multiply A to this x; x is an element from this R n then we will get element a in R m.

So, here the A is like map from this R n to R m because it is taking by this definition here Ax. So, our function is like Ax. So, this Ax is mapping from this x which is from R n to y which is from R n R m. So, if this A is m cross n matrix then it maps element form R n to one element in R m and this one can see easily.

(Refer Slide Time: 12:06)

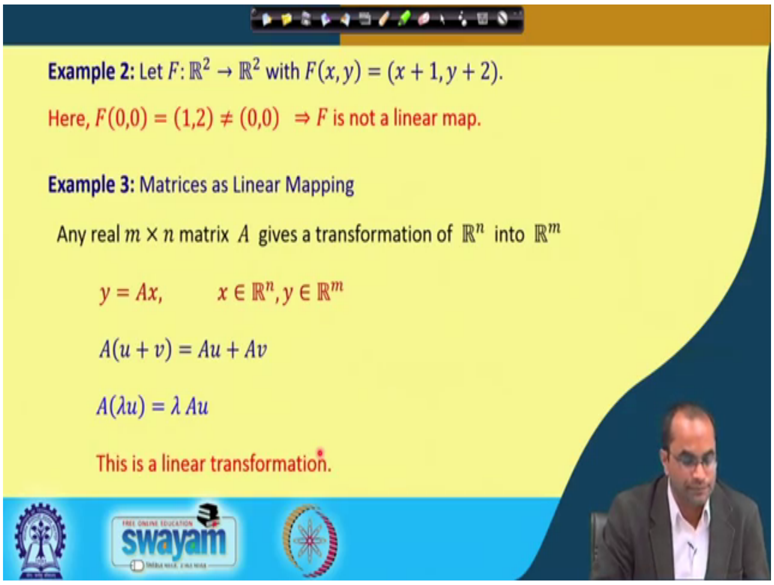


Because if we consider here that is A one matrix here of order this m cross n. So, this is like a 1 1 a 1 2 and a 1 n a 2 1 a 2 2 and so on a 2 n and then this m th row. So, a m 1 a m 2 and we have a m n at the last element. So, this A is a matrix of order this m cross n and if we consider this Ax; that means, this A will be multiplied now by this x here. So, x 1 x 2 and x is from R n that means it has n components; so, x 1 x 2 x 3 x n. And now if we discuss this multiplication, so, what will happen? This row will be multiplied to this column. So, we will get here then this will be multiplied to this and so on.

So, we will get these m rows. So, this is the mth mth row this is the first row. So, as a product here we will get a vector from this R m because this will have m component and so, this vector will belong to R m. So, when we have a matrix of m cross n order and if we multiply vector from R n, so, this Ax will be a vector in this R m space.

So, all matrices of order this m cross n we can think as linear map. Why linear map? Because of the nice properties of such of matrices. So, first what we have seen that this Ax is nothing but it is a mapping now the element this x which was from R n and it is giving us an element in this y in this R m.

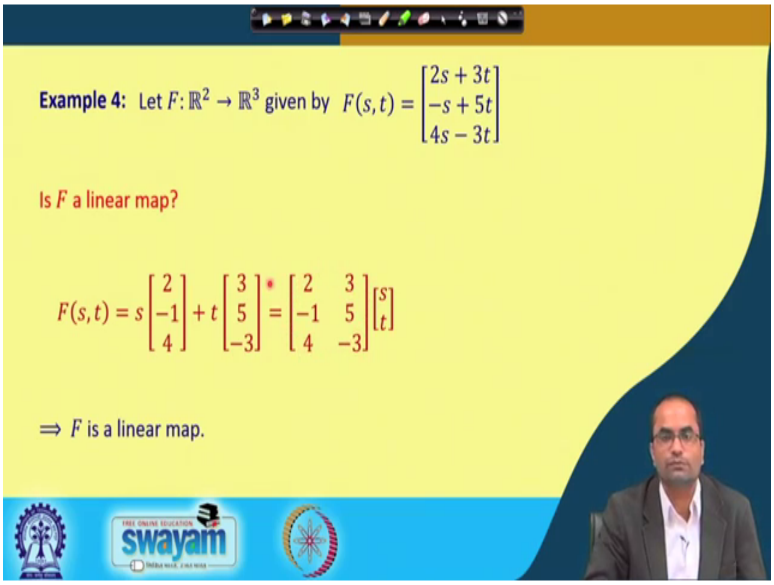
(Refer Slide Time: 14:05)



So, it maps from R n to R m that is clear and these linear property, those are 2 properties when we apply this a on u plus v. So, u and v those are the elements from this R n. So, when we apply A on this u plus v what we will get? This is the property of the matrix itself. So, a u plus a v and that is what we were looking for the linear map. This is one of the properties which must satisfy for the linear map and the second one that the A times lambda u. So, A times lambda u should be equal to the lambda times a u and that is another property of the matrix which we can easily verify if we want.

So, both the properties of the linear mapping satisfied for matrix for a matrix of order m cross n. So, every m cross n matrix maps and maps and these entries of these matrices are real of course then it maps an element from R n to an element in R m. So, this is another important point which we will be exploring further in this lecture that this matrices are nothing but they are linear map.

(Refer Slide Time: 15:21)



So, the example here for instance we see this F which maps the R 2 to R 3 and given by this the relation that F 1 it takes this s t s t is a point in R 2 and it provides here 2 s plus 3 t minus s plus 5 t and 4 s minus 3 t an element in this R 3. So, this F is a linear map and that we have to check it and one way of checking would be that we take the 2 elements form R 2 and then see whether this F when applied on this u plus v gives us F u plus F v and the second condition that F 1 lambda u is equal to lambda times F u.

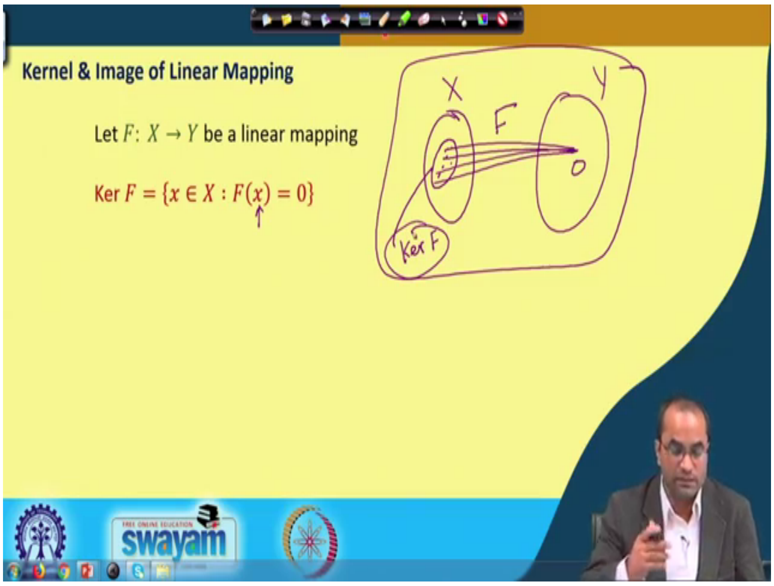
So, we can check those 2 conditions separately on this map. The other possibility what we will in fact look here that this F s t we can also write down here that s from this first component we can take this out s and then we have 2 minus 1 and 4, 2 minus 1 and 4 and plus this t we can take common then we have 3 5 minus 3. So, this here addition of these 2 vectors is exactly the given vector here which this F maps this s t to this given vector. So, here we have exactly the same vector, but we have made into 2 2 vectors.

And now this if we if we recall the properties of the matrix which we can put this in the form of matrix vector multiplication because s is multiplied to this column here, t is multiplied to this column here and that is exactly the property of the matrix vector multiplication. So, if we put these 2 columns as the first and the second column of a matrix and then this s t again column vector there, so that multiplication which exactly give this s times this vector plus t times this vector. Meaning that we can write down this as this matrix whose columns are these given vectors are 2 minus 1 4 and 3 5 minus 3 and this s t we can put again as a column vector there.

So, this is exactly when we look into this product here, 2 times s plus 3 t that is the first member here minus s plus 5 t that is the second member and 4 s plus minus this 3 t that is the third one. So, this given mapping is defined exactly by this matrix are 2 minus 2 3 minus 1 5 4 minus 3. So, this F, the given map is nothing but this matrix here and then what we have seen that these matrices are linear map.

So, we do not have to check anything else because we can write down this as this in terms of the matrix and therefore, this has to be a linear maps. So, without that fundamental derivation of this to show that this is a linear map we have taken this way that this we can represent as a matrix map and therefore, this F is a linear map because we have written in terms of the matrix and matrixes are linear map.

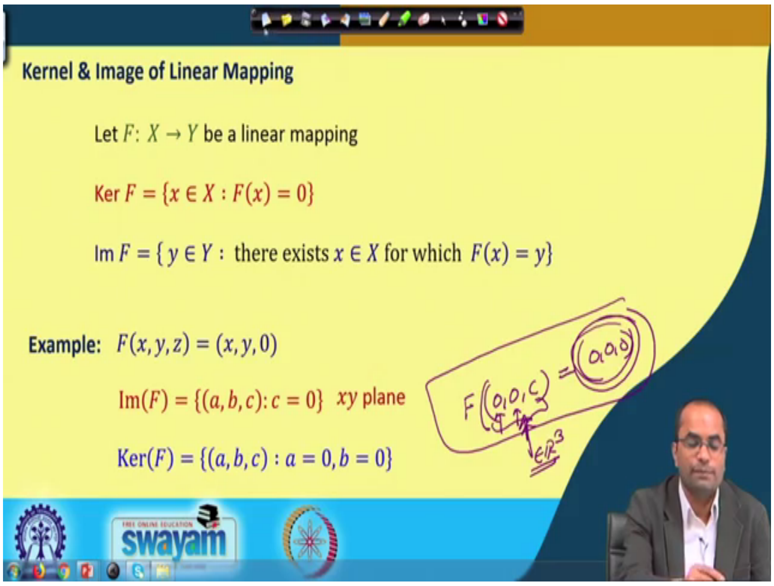
(Refer Slide Time: 18:56)



So, another point here to be discussed that is called the kernel and the image of the linear mapping. And this let F again be a linear map which maps the elements from X to the elements in Y. And the kernel F is defined as that is the definition of the kernel F is the elements from X elements from the vector space X such that they map to the 0 element in Y. So, this is the kernel here. So, we have this vector space X we have this vector space Y and what is the kernel?

We will collect all these points here in X and if they map to this 0 element in Y. So, then this set will be called the kernel of this mapping F. So, this is the definition of the kernel of F; all the elements in X which map to the 0 element in Y. Naturally, the 0 element must be there because the 0 must map to the 0 for the linear map. So, definitely the 0 will be there in this kernel, but there can be many other elements than the 0 elements in this kernel here.

(Refer Slide Time: 20:22)



There is another one that is called the image, image of F is defined as that the y the elements from the vector space Y those elements where there exists x in X for which F x is y. So, we are exactly this is the image which we usually discuss which we considered here. So, we will collect all only those elements of y which are the map of some elements from this X ok.

So, with this 2 definitions the kernel F and the image F, we let us just look at this simple example with F x y z. This again its a projection map. So, this any element here in R 3 it maps to this point in x y plain that is x y 0. So, what is the image F?

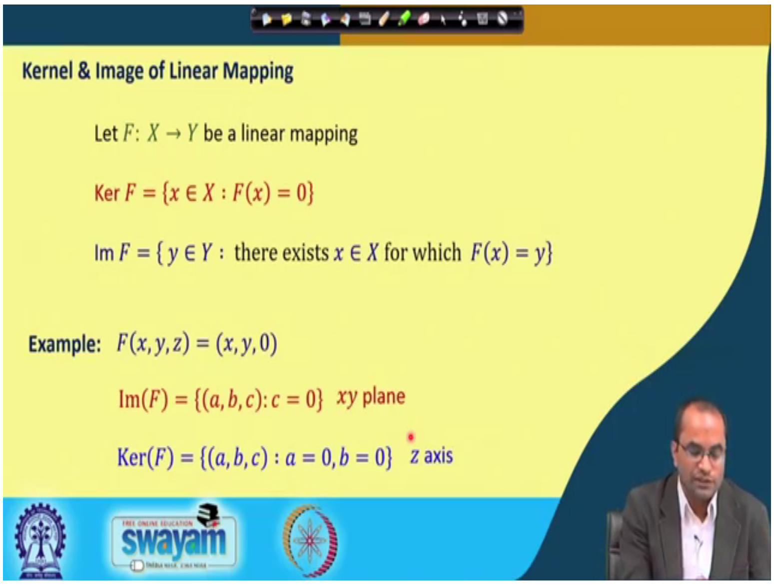
So, image F what we are looking for the all the elements in y whose pre image is there in X. That means, all the points is a b c in R 3 whose the third component is 0 because they are the candidates of the Y and corresponding to when the c is 0 the corresponding element is also there in X and that is nothing but this a and b. So, here the image of this mapping is nothing but all the points a b c whose third component is 0; that means, all the points a b 0; for a and b this belongs to again the real number.

So, this is the image here and when we have set here all the points where this third component is 0 this is nothing but the xy plane. So, the image as we discussed already that this mapping is nothing but it maps the point in this R 3 to the to the x y plane and that that is exactly the projection map we are talking about. And therefore, this image is nothing but this x y plane because this maps the point to the x y plane only. So, the image is there in the x y plane.

The kernel all are those points from X whose map is 0, they map to the 0. So, here as per the definition of this linear map, any point here x y z it maps to x y and 0. So, the third will be set to 0. So, we want all those elements whose who those elements in X whose map is as a 0 element. So, if we take a point here with a and b 0, so, a and b 0 then and when we apply the; so, the point is here that if we apply this map where or to any element where we have taken this first 0 and the second component 0. So, this will be nothing but this will be the 0 0 0 element. So, that is a 0 element in this R 3.

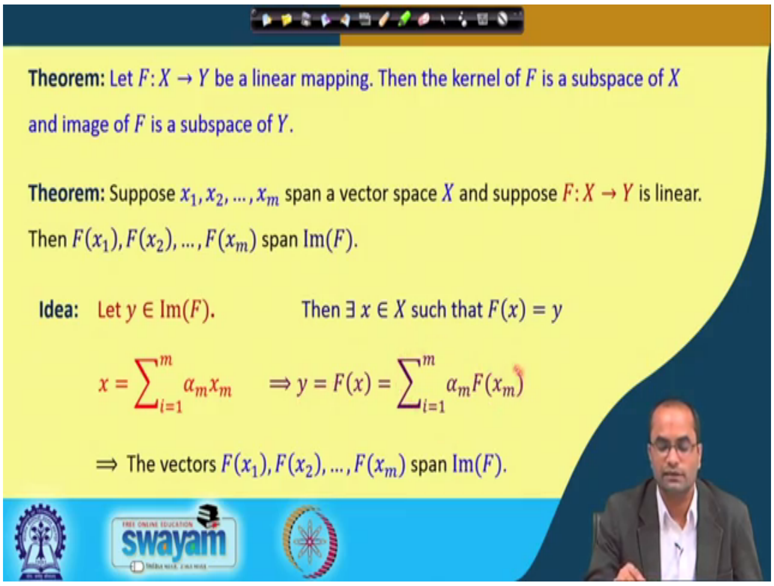
So, we are looking for those points in this R 3 because their element their image is nothing but the 0 element in y. So, this will form the kernel of this mapping.

(Refer Slide Time: 23:53)



And this is exactly the z axis because on the z axis the x is 0 and y is 0. So, the whole z axis will be mapped to this 0 element in R 3 and that is natural here because the z axis when we take the projection of the z axis is nothing but the origin that is means a 0 element in R 3. So, here the image is the xy plane and the kernel of F is nothing but the z axis because the whole z axis is mapping to this origin and the image means here that all the points in xy plane that is the image of this mapping.

(Refer Slide Time: 24:34)



So, there is a nice theorem here that F X to Y be a linear mapping. Then the kernel of F, so, this kernel of F and is a subspace of X and also image of F is a subspace of X.

So, we are not going to formally prove this here, but this is very simple as per the definition of the kernel and this image. We can think that they will form a subspace because the kernel was nothing but all the elements here which maps to 0. So, we have to see basically those closer properties of the subspace. So, if we take any 2 elements and they map to the 0, so, their sum will also map to the 0. So, the sum is addition will be also belong to the same kernel here and similarly for the image also we can consider exactly the exactly the same consideration.

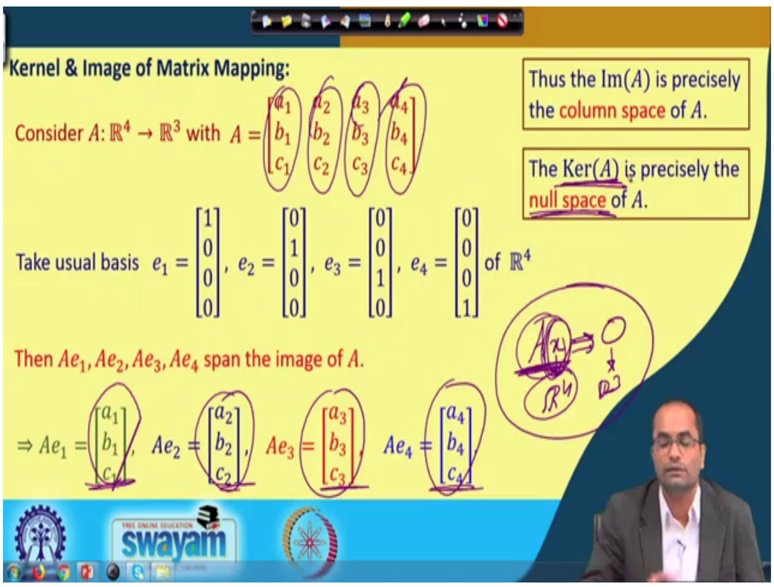
So, here the point is that these kernel and the image they form subspace and there is another nice theorem which will be used later. That suppose this x 1 x 2 x 3 x m is span a vector space X and suppose this F is a linear map, then their image of these points here what these vectors from x which is span the vector space X then this F x 1 F x 2 F x m will also span will span the image F.

So, that is a very nice point here if we know the vectors from this space x and we know that these vectors span the vector space x and we know the mapping here F then this F x 1 F x 2 F x m they will just span the image F and the idea of the proof is very simple. So, if we take a point in the image F now and there exists a X because this is a point in the image, so, there must exists a X in this vector space X such that this F x is equal to y.

And then we take this x because any element of this vector space x can be represented as a linear combinations of x 1 x 2 x 3 x m because these vectors span the vector space X. So, by taking this we have represented this x in terms of the x i's; that means, the linear combination of these x i's. And now we apply this linear map F on x which will give us the element y naturally. So, this F on x and due to the linearity of the map we can take this F here inside this x i's because these are the constant that is the definition of the linear map.

So, we have the F alpha 1 x 1 plus alpha 2 x 2 and so on will be equal to alpha 1 and F 1 x 1 plus alpha 2 and F 1 x 2 and so on. So, this is because of the linearity and now what we see that this y element we have written as a linear combination of this x F x m and this is exactly that any element y in the vector space y we can write as a linear combination of these vectors F x i's. Therefore, these vectors F x i will span the image F.

(Refer Slide Time: 28:11)



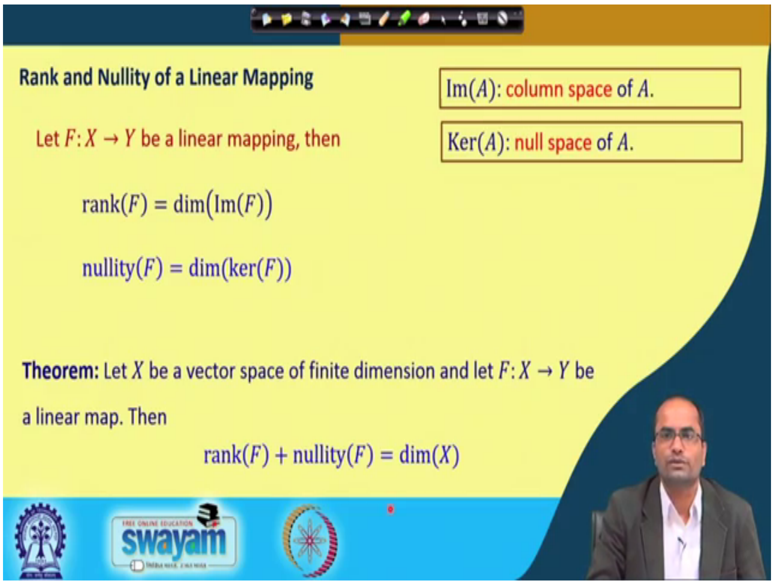
So, here the kernel and image of the matrix mapping; so, because this matrix are also linear maps. So, if you consider just quickly this simple example of this 3 rows and 4 columns and we take for instance the usual basis here e 1 e 2 e 3 e 4 from R 4 these are the standard basis of R 4 which we have already discussed before.

Then as per the previous theorem, we know that Ae 1, Ae 2, Ae 3, Ae 4 will span the image of A. So, now, we compute this Ae 1, Ae 2 what are these numbers here. So, if we take a and this e 1 we will get nothing but a 1 will be the first and then we multiply here will be b 1 and then c 1 that will be the Ae 1, Ae 2, Ae 3 and Ae 4. So, these vectors here a 1 b 1 c 1, a 2 b 2 c 2 and so on a 4 b 4 c 4 in R 3, these are the vectors in R 3, these are the vectors in image and we know from the previous theorem that these vectors here will these vectors will span the image.

So, we know already the spanning vector of this image R 3 which is in R 3. So, image of this A. So, what are these? These are the precisely the columns here. These are the columns of this given matrix. So, the image is nothing but image is nothing but the column the column space the column spaces exactly the span of these columns that is the column space we have already discussed and what we have observed that the image is nothing but the span of these columns.

So, thus the image A is precisely the column space of A. So, when we talk about the matrices here the column space is nothing but they will span the column space is nothing but the image of A. Similarly, the kernel because the kernel will be all the points here whose who map is to 0s; that means, we are looking for all the points x here in R 4 and whose map to the 0 in R 3. So; that means, what we are looking exactly? We are looking for the null space of this null space of this matrix A because that was the definition of the null space that all the I mean the solution of this system of (Refer Time: 30:58) equation gives us the null space. So, here the null space in the form of the kernels is same as the null space of a that is the kernel of the matrix mapping.

(Refer Slide Time: 31:11)



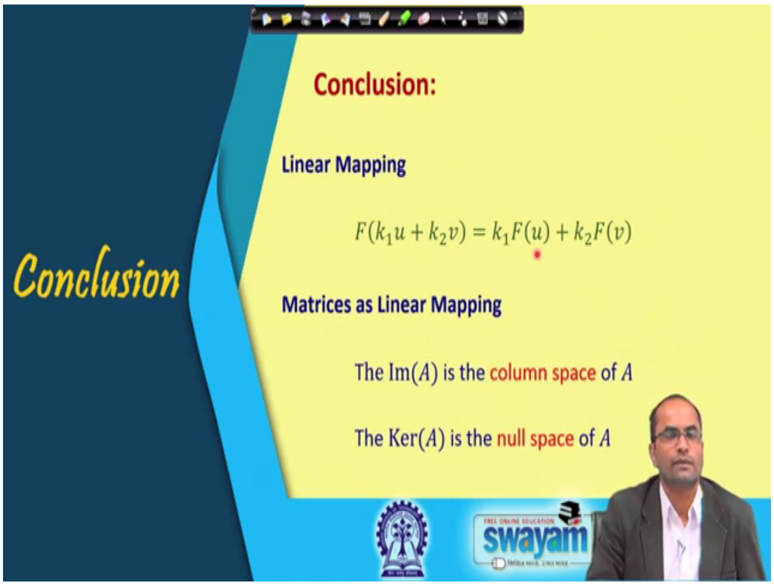
Last one here the rank and the nullity of a linear map. So, if we remember the image A is the column space of A and then the dimension of the column space when we have discussed the rank of the matrix. So, that was the dimension of the column space that was the rank there.

So, here in terms of the linear map, the rank of a linear map will be the dimension of the image of F. So, image of F will be the rank. So, this is a more general definition of the rank which the in case of the matrix that is the special case and we have separately discussed the rank concept of the matrix. So, the rank of the matrix was the dimension of the column space, but here we have the more general terminology that is called the image of the mapping F.

So, this kernel F is nothing but the dimension of the image F. Similarly, the kernel A was null space of A and here again this nullity which we call the dimension of the null space, so that is the nullity which we have discussed already before. So, the nullity is nothing but the dimension of the kernel. So, these 2 again we have the 2 more definitions of about the kernel which is more general than the definition of the rank we have discussed for matrices.

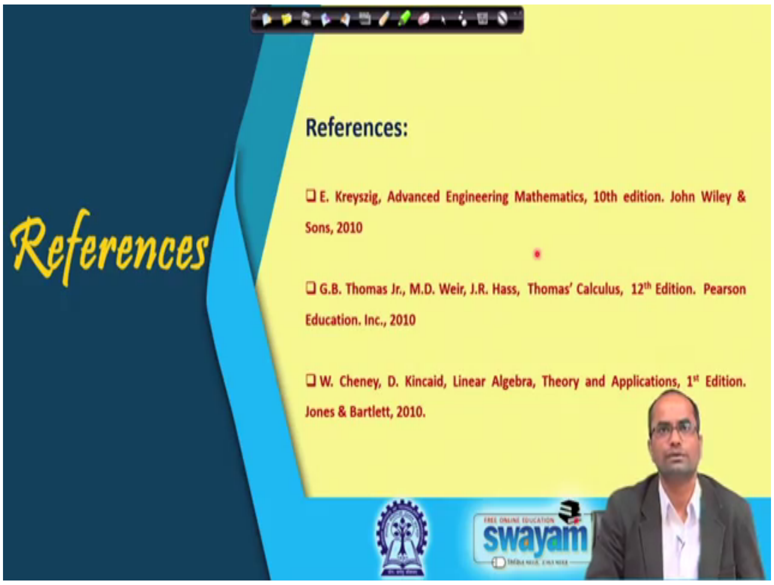
So, here the rank of F is the dimension of the image of F and nullity is nothing but the dimension of the kernel of F. And there is a theorem that let X be a vector space of finite dimension then and let this F be a linear map then this rank plus nullity is equal to dimension of X. So, this rank nullity theorem we have also discussed for matrices where rank plus nullity was the number of variables n which is here in this case that is because this A maps from R n to R m; so that exactly the dimension of this domain here the dimension of X. So, the rank plus nullity is always the dimension of X that is a more general result than what we have for the matrices.

(Refer Slide Time: 33:20)



Coming to the conclusion here; so, we have discussed the linear map and to prove the linear map we just need to apply this F on this k 1 u plus k 2 v and if we get the k 1 F u plus k 2 F v then we call that the given map is the linear map. And what we have also seen that these matrices are also linear map and if matrix is of order m cross n then they map the elements of R n to the elements of R m. And this image of A is nothing but the column space of A and this image the kernel of this matrix this linear mapping A is nothing but the null the null space of A.

(Refer Slide Time: 34:04)



So, these are the references used here and thank you for your attention.