Engineering Mathematics - I
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Lecture - 43
Rank of a Matrix

So, welcome back and this is lecture number 43. And today we will discuss Rank of a Matrix from this topic of the matrix which are linear algebra.

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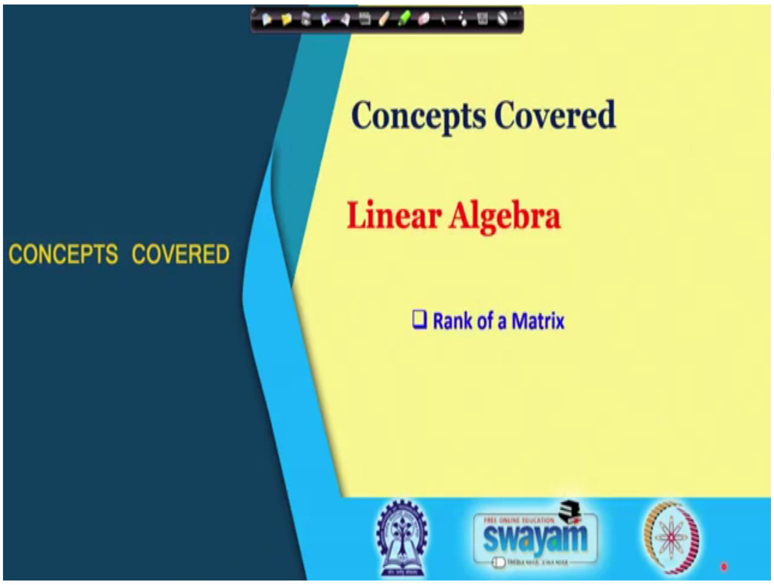

And the definition we start with which we have mentioned in one of our previous lecture that is the rank of a matrix is the number of nonzero rows or the number of pivots in its reduced row echelon forms.

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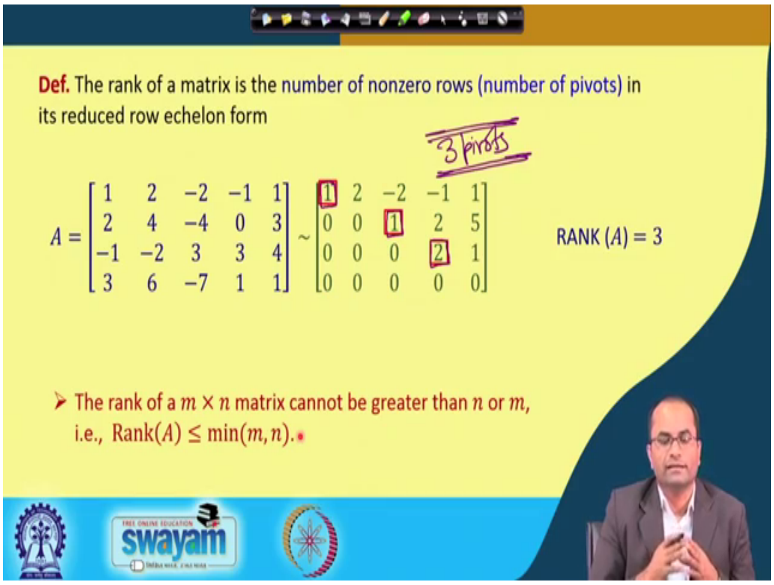

So, this that is that is what this row reduced echelon form is very important, it tells us lot about the matrix and that is one important concept which is used at very applications about this rank. So, rank of the matrix is the number of the pivots which we see in our reduced a row echelon forms. So, for instance we take this A, which was already used in previous lectures we have also seen its row reduced echelon form which is given here again.

So, this is the row reduced echelon form of this matrix. And what is now we need to count the pivots here. So, the 1 is the pivot and also this 1 is the pivot and this 2 is the pivot. So, we have 1, 2, 3, there are 3 pivots here in this row reduced echelon form and therefore, the rank of this matrix is 3. So, there are 3 pivots and the rank as per the definition that rank of the matrix is nothing but it is a number of pivots in reduced row echelon form which is 3 in this example.

Just a consequence of this definition we can easily make it out that the rank of an m cross n matrix cannot be greater than n or m. So, m is the number of rows and number of columns, and the rank cannot be greater than m or n it has to be the less than the minimum of m and n. The reason is clear because our each row or each column cannot have more than one pivot, the column can have at most one pivot or the row also it can have at most one pivot, one row cannot have a two pivots or more, column cannot have a two pivots or more. So, we are talking about the number of pivots. So, it cannot be the rank, cannot be greater than m the minimum it has to be less than or equal to the minimum of this m and n this number.

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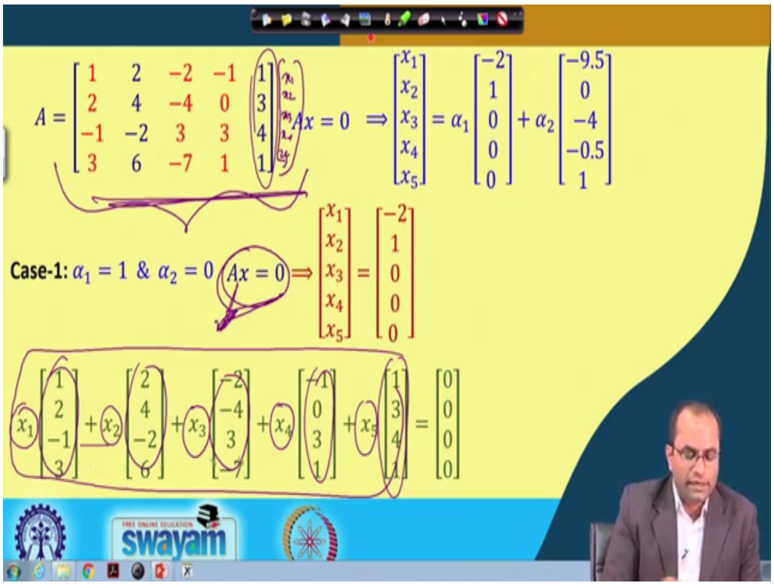

Write down the solution by taking these free variables that is the x 2 here and also this x 5. So, taking this free variables we can write down this Ax is equal to 0, the solution of this Ax is equal to 0 in this form which we have already observed because now what we are going to do we are actually we are looking for the other definitions of the rank because that is not the only definition which we have given in terms of the number of pivots. So, for that we just need some preparations here.

So, if we notice now the case 1, if we take this alpha 1 is equal to 1 and alpha 2 is equal to 0. So, that will be one solution of this system by choosing this alphas basically we are looking for different different solutions. So, for this particular case we are taking choosing this alpha 1 as alpha 2 as 0. So, our solution of this Ax is equal to 0 will become minus 2, 1, 0, 0, 0. So, this is the solution of our Ax is equal to 0 of the system.

And then this Ax is equal to 0, this equation we can write down in this form. This we have already talked before that this product of this matrix A with this x here, the x 1, x 2, x 3, x 4 and x 5. So, this product A and with this x we can also write down as this x 1 and then the first column plus this x 2, the second column x 3, the second third column x 4, this forth column x 5 and this fifth column 1, 3 ,4, 1 and this is equal to 0. So, that is another way of looking at this matrix vector product we can write down in this vector forms. So, having this now and what we have observed that this is one solution of this system of equations with minus 2 4 x 1 and this here 1 4. So, this is minus 2 and this x 2 is 1 and this we can take 0 all these 0s.

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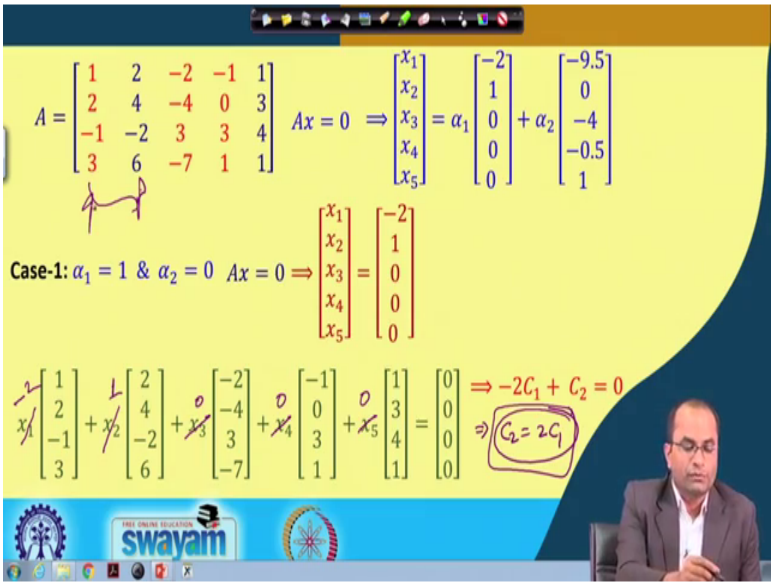

So, what we observed now with this that minus this two times the C 1 denotes this column 1 of our matrix. So, this C 1 minus 2 C 1 plus C 2 is equal to 0. So, what we observed at least from this relation that this column 1 and column 2 are dependent. We can write down this column 2 here in terms of column 1, so column 2 is nothing but the two times the column 1. So, that is a relation which we can clearly see from directly from the matrix itself because, but sometimes it is not easy to see when we have more vectors into picture here there were two vectors only, so we can see easily. So, that is the one relation we have seen that this C 2 the column 2 is nothing but the two times C 1. So, these two columns are dependent columns.

One more observation quickly we will have now we will take another case where we will choose this alpha 2 v 1 and this alpha 1 to be 0.

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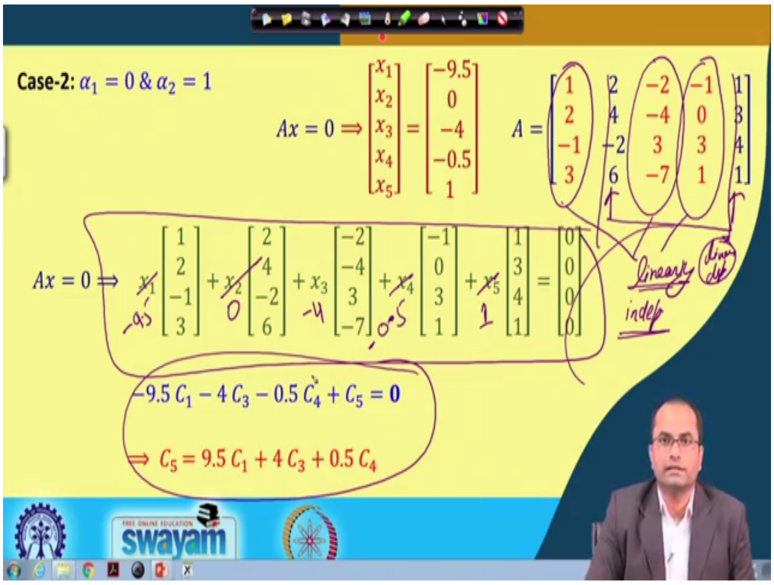

That means, our solution now of this Ax is equal to 0 is given by this minus 9.5, 0, minus 4, minus 0.5, 1 with this vector and that this was given already. So, again this first column third and forth they are the columns which have the pivots in their reduced in its reduced form. So, here Ax is equal to 0. Again, with this observation we will fix the solution now. So, the solution says that this minus 9.5 of this and then 0 here and this x 3 is minus 4 and then x 4 minus 0 0.5 and then this x 5 gives 1.

So, with this relation now what we observe that this is minus 9.5 the column 1, minus 4 times this column 3 and minus 0.5 the column 4 and plus this column 5 must be equal to this 0 vector which has all these component 0 or we get basically this relation that C 5 is nothing but 9.5 C 1 plus 4 C 2 plus 0.5 C 4. So, what we observe again that this C 5 here the column 5 depends on these columns C 1 C 3 and C 4 also this column number 2 from the from the previous case we have observed that this is nothing but the two times of the column 1. So, this column 2 is dependent also we have observed that column 5 is dependent and they depend on these columns which have the pivots.

Another observation which we can find out from the from that reduced echelon form that these vectors with red here the column number 1, column number 3 and column number 4. So, these vectors here one can easily prove directly from the reduced form that these are linearly independent, these are linearly independent that form the reduced form one can talk about this. And we have seen that these columns here corresponding to those free variables, they are linearly dependent. And this is the general result also not just for this particular example that the columns which we have the pivots in its reduced form they are linearly independent always and the other columns which have these free variables or where they do not have the pivots, they actually are linearly dependent.

So, now we can rewrite our definition which says for this rank that this is nothing but a number of pivots, but now we can write down as the number of independent columns. The rank is nothing but the number of independent columns in because this column is dependent and this column also dependent, there are 3 independent columns and they basically correspond to this pivot column. So, now our definition for the rank is that rank is nothing but the number of independent columns in a matrix.

And again, the same observation which we have done here with the columns we can do with the rows as well. So, those rows where we got these pivots there are there are the same number of rows which are independent in this matrix. So, we can have also this definition that the rank is nothing but the number of independent rows and they are the same the column because the number of pivots are basically fixed in its reduced form.

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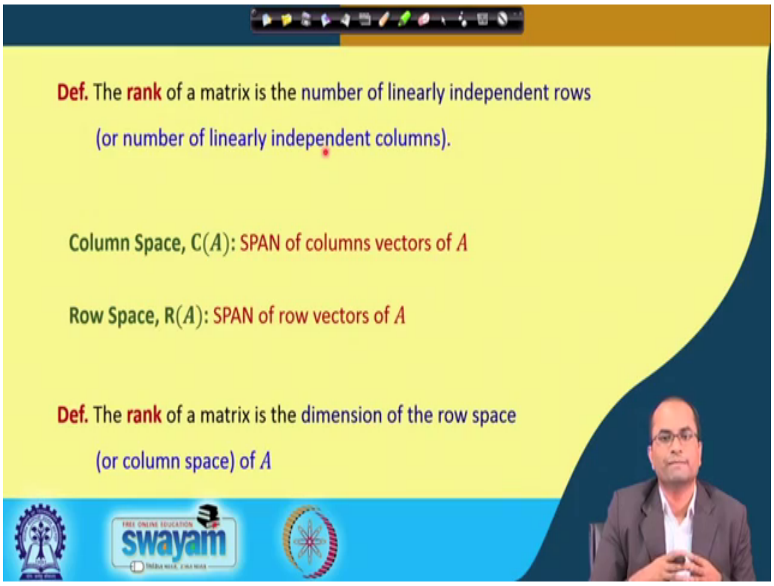

So, what we have, we can now give another definition the rank of a matrix is the number of linearly independent rows or number of linearly independent columns in a matrix that is the rank. One more definition we are going to give in a minutes, before that we just define these two spaces the column space is nothing but the span of the column vectors of A.

So, for a given matrix A we take its column as vectors and then span them, so that is what we call the column space because any span of any vectors that is a vector space which we have already seen in previous lectures. So, here also the span of these vectors of or the columns, column vectors of this a will also form a vector space which we call the column space. So, column space is a vector space that is nothing but the span of the columns of A.

Similarly, we have the rows space row space is nothing but the span of the row vectors of A similar to the column space if we take the rows there and span them, so this space which we call the rows space. And the another definition which we have now for the rank, the rank of the matrix is the dimension of the row space because when we are talking about the rows for instance, so what will be the dimension of these rows space. The dimension will be the number of linearly independent rows in that span in that set here. So, the number of linearly independent rows which is actually the definition of again with the rank; so here, the dimension of the row space is nothing but the number of linearly independent rows in the matrix or the dimension of the column space. So, the dimension of the column space is nothing but the its a number of linearly independent columns in the in the matrix A.

So, we have already 3 definitions so far. The rank is nothing but the number of pivots in a reduced row reduced echelon form. The definition number 2 the rank of a matrix is nothing but the number of linearly independent rows or number of linearly independent columns of the matrix and we in the terms of the dimension we can also say that the rank is nothing but the dimension of the row space or the dimension of the column space of A.

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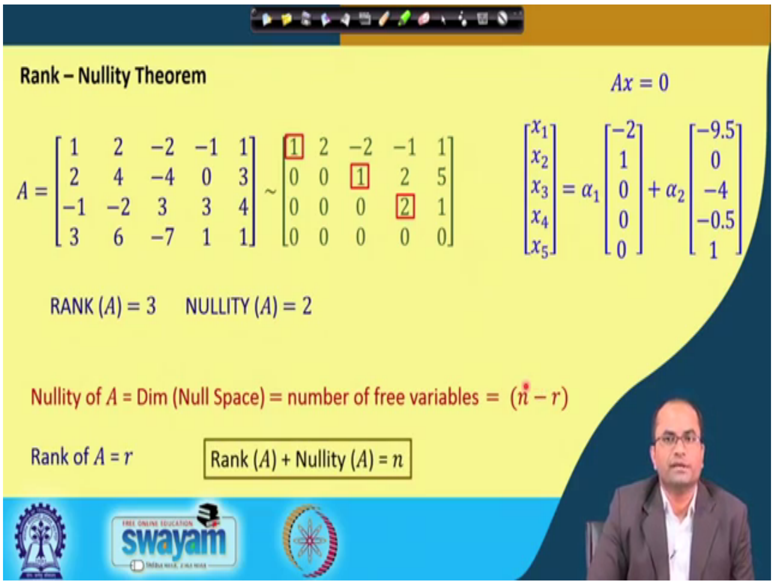

There is a rank nullity theorem which we will just observe with the help of the example. So, again we will continue with the same example. So, we have this A here and its reduced form we have these pivot elements. We had also the solution Ax is equal to 0 which was written in this form x 1, x 2, x 3, x 4 is equal to alpha 1 times this a generator plus alpha 2 times the another generator which generate the solutions.

So, here the rank of A is 3 because it s a number of linearly independent rows or number of linearly independent columns or the number of pivots we have this rank 3. What is the nullity of A? Nullity remember the nullity was the dimension of the null space and the null space here is the solutions set of this Ax is equal to 0. So, the solution set of Ax is equal to 0 we have the dimension 2, because these are the two vectors which can generate any solution of this set and these two vectors are linearly independent. So, here the nullity of A is 2 because of this is a precisely the number of this free variables in row reduced echelon form.

In general, or first let us talk about this example once again. So, here we have rank 3 and this nullity is 2 when we add these two, so 3 plus 2 will we will give we will get the number of these variables here number of unknowns in Ax is equal to 0. The reason is clear because the rank defines exactly the number of pivots here and this nullity is nothing but the number of this known pivots. So, we are covering each column for instance here. Here we have pivot so that will go count to the rank here. So, 1 here 1 plus

1 and 1 this plus 1. So, we have the 3 rank because these 3 columns have pivots and these two columns do not have pivot. So, they will be free variables and that is nothing but the nullity of A. So, this will add to the end here which are the number of columns or the number of unknowns in Ax is equal to b.

So, this two 3 plus 2 will add to that 5 in this case we have 5 columns. In general, how do we read this? The nullity of A is nothing but the dimension of the null space which was 2 in the this previous case, meaning the number of free variables or in our notation we also take this number of free variables as n minus r, because we have n columns and the r are the rows where we have the pivots. So, this n minus r or this r is the rank basically of the matrix because these are the rows where the pivot is sitting. So, the rank of A is r that is the number of the of the pivot elements. So, here the rank is r and this nullity is n minus r. So, this rank nullity theorem is nothing but it says that the rank of a matrix plus nullity of the matrix. So, r plus this n minus r must add to n, the number of the columns in the matrix. So, this is called the rank nullity theorem, rank of a plus nullity of A is equal to n.

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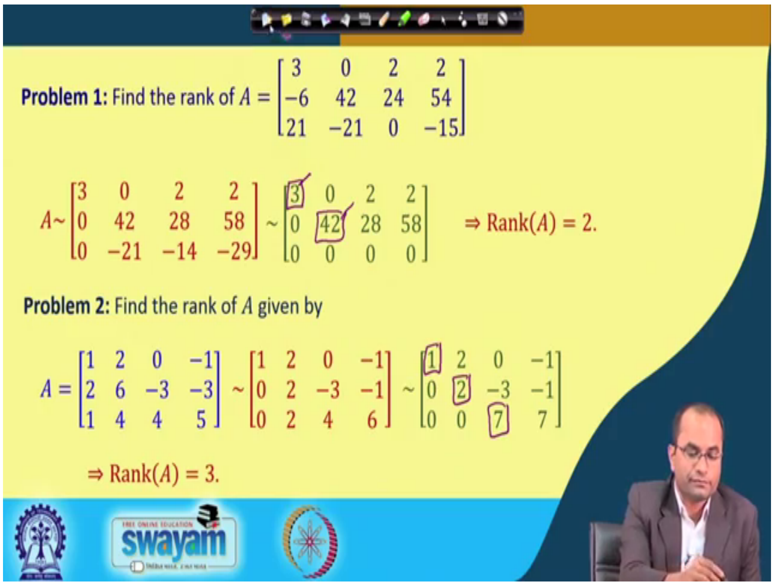

Now, here we will find the rank of the matrix. So, rank of this matrix is here 3 0 2 2 and then. So, what do we need to do? For finding the rank we have to convert to the row reduced echelon form. So, in this case we will try to set this to 0 here with two times the row 1 we will add and that will give us 0 here. This 42 will remain as it is and here the

24 and minus this 2, so this will be 28 and then we have 58 there. So, we are adding actually it is a two times of two times of this row one we are adding to this one So, this was 24 and then to this is 28, here also 54 plus 2 plus 4 it is a 58 and similarly here we will be multiplying here by 7 and then subtracting, so we will get this row.

So, this is the first iteration of the direction that we have set these elements here below this 3 to 0, now we will take this element and try to get eliminate now this here minus 21 with the help of this row 2. So, that will be the next step. So, the first row and the second row will remain as it is and now we will try to put this to 0 so that means, the half of this we are adding we are adding to this row here. So, this becomes 0, this also becomes 0 and this also becomes 0.

So, this is the row reduced echelon form, and in this row reduced echelon form we have to count now the number of the pivots. So, we have one pivot here, we have another pivot as 42, so, the total pivots here we have 2 and that is what the rank here is of this matrix is 2. So, this is a simplest way of getting the rank. We just take the matrix and try to reduce it to the row reduced form and then count the number of pivots and that is precisely the rank of the matrix.

Let us take the another example where we will find the rank of a a is given by this 1 2 0 minus 1, and 2 6 minus 3 and minus 3, 1 4 4 5. So, with this matrix again we will reduce it to the row reduced echelon forms. So, the first we will set these elements to 0, so 2 and 1, so we get 0 0 here. In the second step now, we will try to set them to 0 below this 2, so it is said to be 0 and now we can count the pivots here. So, this is the pivot and this is pivot and this is also pivot here. So, we have 3 pivots in this case and therefore, the rank of this matrix A is 3.

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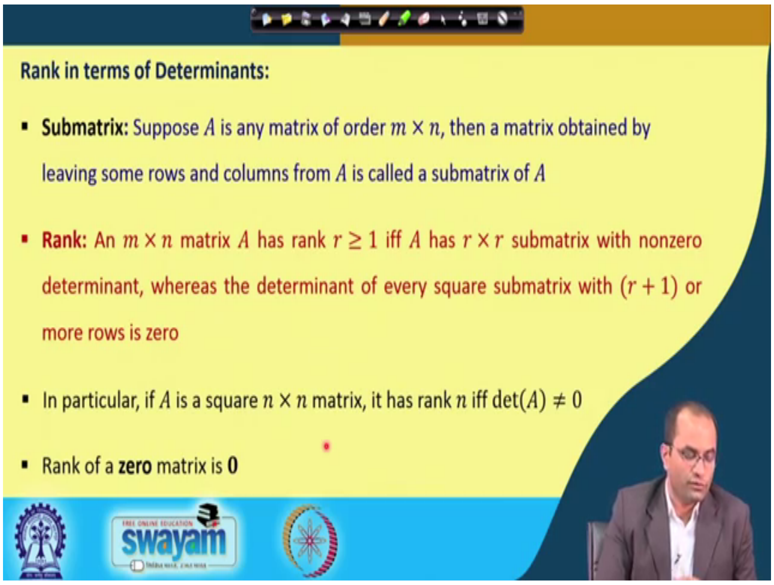

There is another way of determining this rank in terms of the determinant. So, that we will shortly now give. So, first the definition here of the submatrix. Suppose, this A is any matrix of order m cross n, then a matrix obtained by leaving some rows and some columns from A is called a submatrix. So, matrix is given we remove let us say first row, the third column then whatever left this matrix is a submatrix or we can remove more rows more columns. So, whatever this smaller matrix by removing some rows and some columns, that is called the submatrix.

So, here the rank is defined can be also defined from this determinant as the n m cross n matrix A has rank r, this will have rank r when if and only if a has r cross r submatrix, so the square, a square are submatrix with nonzero determinant. So, we have to now get out of these given matrix r cross r submatrix which has the nonzero determinant whereas, the determinant of every square determinant or every square submatrix of r plus 1 or more rows is 0. So, we have to start with the whole matrix if for example, it is 2 by 3 matrix then 2 by 2 submatrix is we have to look and see if they all are 0 then we have to go to the one level and so on. So, we have to see now this submatrix with nonzero determinant.

In a particular case when A is a square n cross n matrix it has the rank n, if the determinant A is not equal to 0 say if we have a square matrix and its determinant is not equal to 0 then it has a full rank, so the rank is n because determinant itself is not 0. If the determinant is 0 then we have to reduce to this different submatrices of order n minus 1 we have to check all the submatrices of order n and their determinant if they all are 0 then we have to reduce to n minus 2, so on. But for instance, this n minus one submatrices even one has nonzero determinant then the rank will be that n minus 1, and rank of a 0 matrix is 0 because we do not have for example, here this any submatrix with nonzero determinant any submatrix we take, even up to level 1 also because all the entries are 0. So, the all the submatrices you take whatever order the determinant is going to be 0. So, this is what we call the rank of 0 matrix is 0.

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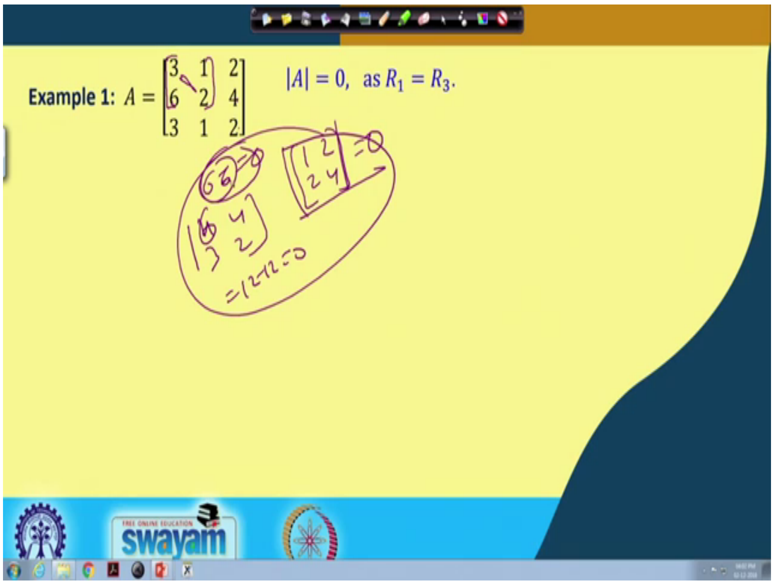

Here we will take this simple example a which 3 1 2 and the 6 2 4, 3 1 2. So, in this example we can observe that this determinant of this A is 0 which is clearly visible because here 3 1 2 and this row is also 3 1 2. So, the determinant of this will be 0. So, the determinant of A is 0; that means, now the rank cannot be 3 because we are not getting this 3 by 3 matrix, it should be nonzero the determinant should be nonzero then the rank will be 3

So, now the rank will be less than 3. So, we have to check now the submatrices the determinant of submatrices of order 2 by 2. So, if we take any this 2 by 2 submatrix for example, we take this one by leaving this second row third column and the third row. So, we have this determinant. And now the determinant here will be the 6 minus 6, so again this 0. We take any other 1 2, 2 4, one more submatrix here also this is 0 when we take the determinant.

Any determinant we take this 6 4, 6 and this 4 and 3 and 2. So, here also 12 minus 12, 0. So, any determinant we take of order 2 by 2 out of this matrix the answer is 0.

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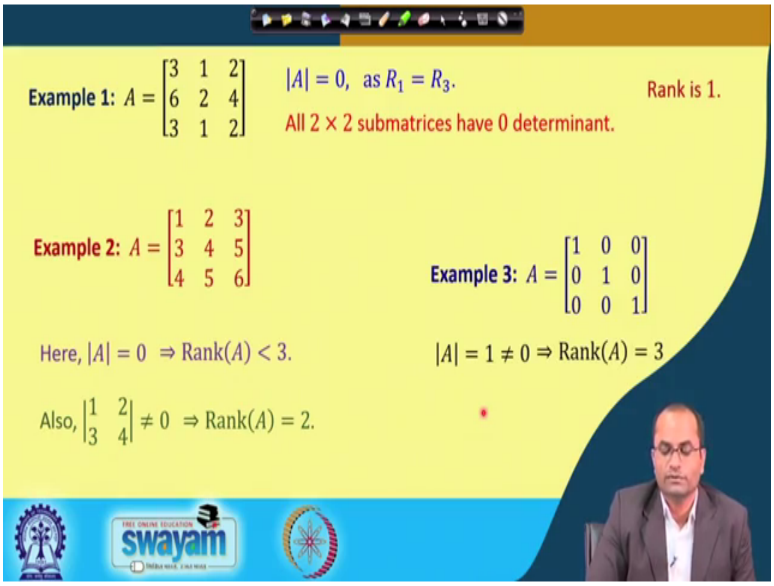

So, all 2 by 2 submatrices have 0 determinant and that is the reason here we now have to look for this determinant of order 1, but that is a nonzero matrix anyway. So, whatever value we have, so the rank has to be 1 in the that case because it is not the 0 matrix.

So, here the rank is 1 and in this example 2, we take another example where we check that again this determinant of A is 0. So, when determinant A is 0 the rank has to be less than or equal to the order of the matrix here it is a square matrix. So, the rank of A has to be less than 3 and in this case we also observe for instance this determinant itself we take 1 3 2 4, and the value is not equal to 0 the value if 4 minus 6 minus 2. So, this is not equal to 0. So, we have a submatrix whose determinant the submatrix of order 2 whose determinant is not 0 and there we can decide now the rank has to be 2.

Though it is little bit difficult than the earlier process of getting the rank where we reduced to the row reduced echelon form and then we can easily identify what is the rank. Here you have to look for these different orders of determinants and check whether something is nonzero. For example, this was a easy we have easily seen that this submatrix has determinant nonzero. So, the rank is 2.

If we take the identity matrix here the 1 1 1, we know that the determinant here is one which is not equal to 0 and therefore, the rank of this matrix is 3.

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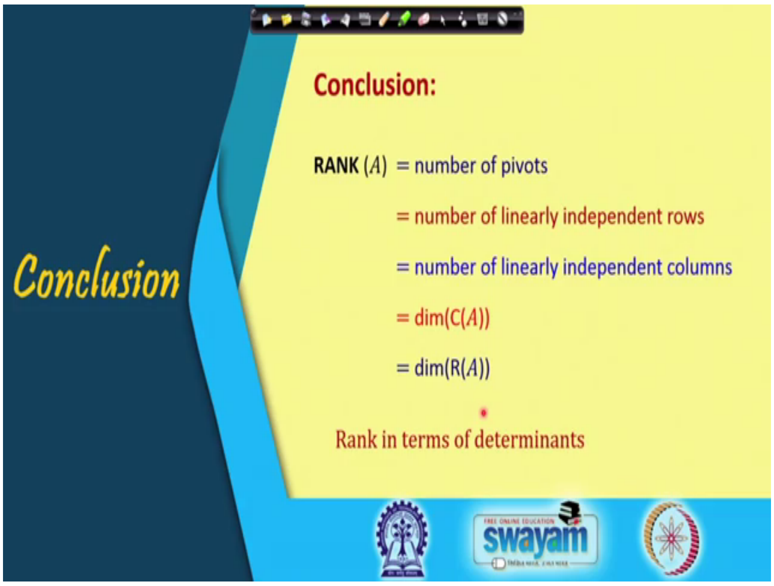

Coming to the conclusion what we have seen the various definitions of the rank, one was the number of pivots in the in the row reduced echelon form, the other one was the number of linearly independent rows. There was a number of linearly independent columns, and we can also talk about the dimension of the column space that is nothing but the rank dimension of the row space that also is the rank and we have also discussed the rank in terms of the determinants.

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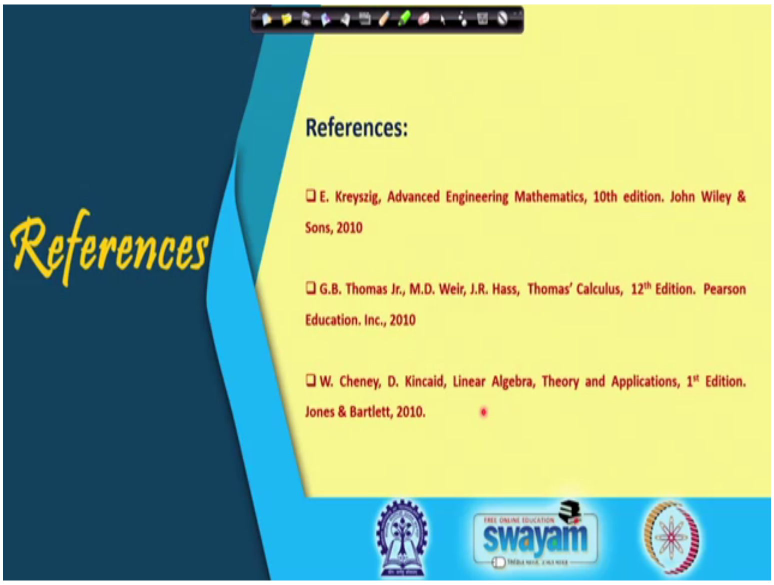

So, these are the reference used here.

And, thank you for your attention.