**Engineering Mathematics – I**
**Prof. Jitendra Kumar**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture – 41**
**Vector Spaces – Spanning Set**

So, welcome back and this is lecture number 41 will be talking about this Spanning Set. So, to define the Vector Space we need to prepare for many concepts and this is one of them so, called the spanning set.

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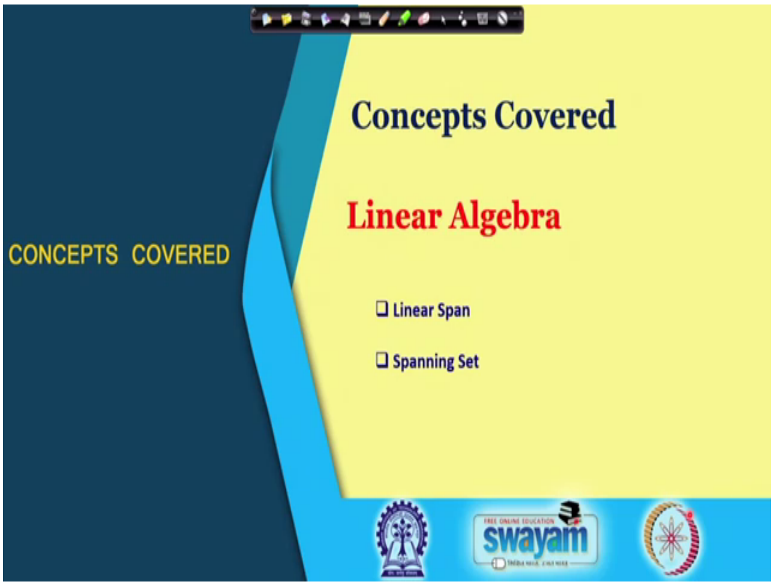


And here we will be talking about the linear span first and then will be talking about spanning set.

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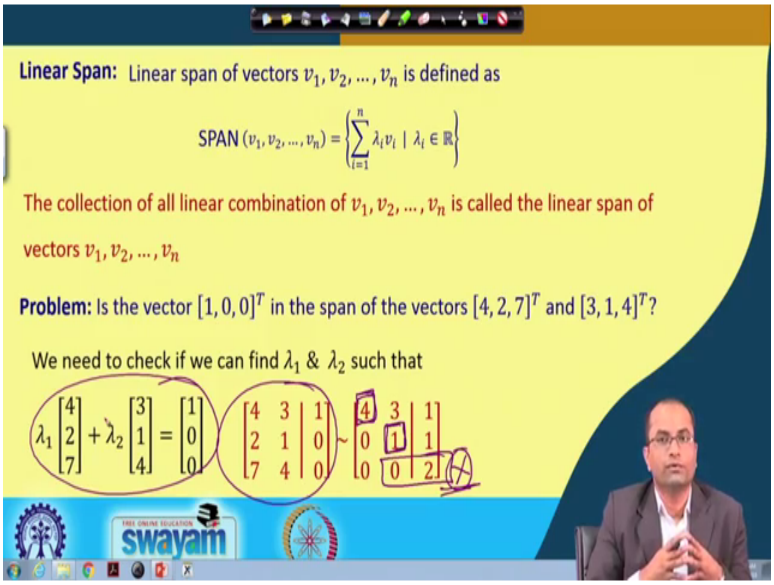


So, what is the linear span? So, linear span of vectors of given vectors v 1, v 2, v 3, v n this is defined as and usually we denote as a span of this v 1, v 2, v 3, v n and we define as the set of all these linear combinations of these vectors v 1, v 2, v 3, v n. So, here exactly this is the linear combination given lambda 1 v 1 plus lambda 2 v 2 and lambda and v n. So, this the linear combination of the vectors and this lambda belongs to the set of real number.

So, this is a set of all linear combinations of the given vectors we call the span of v 1, v 2, v 3, v n. Exactly so, the collection of all linear combinations is called the linear span of v vectors v 1, v 2, v 3, v n and the problem here we will discuss now, is this vector 1, 0, 0 transpose just to tell that this is a column vector, but we can do also with the row vectors. So, here is the vector here in the span of the vectors 4, 2, 7 and 3, 1, 4.

So, what do we need to check basically can we write this vector as a linear combination of these two vectors because this is the question here that is this vector in the span of the vectors of this. So, what is the span of these vectors? All linear combinations of these two vectors and to check whether this belongs to this a span of this these vectors on we will just check whether this vector is a linear combination of these two vectors or not.

So, we need to basically check we need to find the lambda 1 and lambda 2 such that the lambda 1 this vector 4 to 7 plus lambda to the another vector here 3, 1, 4 will give us 1, 0, 0 whether this is possible to find such lambdas or this is not possible the system here is inconsistent or it is a consistent system which can give us the lambda 1 and lambda 2.

So, to answer this again we have to get back to this the idea of the augmented matrix and the row reduced or echelon form again just note that this row reduced echelon form is very important almost in our lectures we will be utilizing the idea of this solving the system of equations, using gauss elimination or rather saying this reducing to this row reduced echelon form because once we have this row reduced echelon form, we can tell exactly that what type of solution we are getting out of this given system.

So, here we have this is the augmented matrix so, our equations are like 4 lambda 1 plus 3 lambda 2 is equal to 1. The second equation 2 lambda 1 plus lambda 2 is equal to 0, the third equation 7 lambda 1 plus 4 lambda 2 is equal to 0. So, out of it we have written this a augmented matrix, the coefficient matrix here 4 3 2 1 7 4 and then this right hand side vector which is 1 0 0.

Now we need to reduce it to the row reduced echelon form and that idea is also we have explained several times we need to make this elements 0, then this element 0 with the help of this equation number 1 and then we need to make this element 0 with the help of the equation number 2. So, that doing so, I am writing directly here what will be the reduced echelon form.

So, 4 3 1 and then 0 here 1 1 and 0 0 2 so, this is the row reduce echelon form and from here now we can conclude because this is now our the first column has a pivot element the second column has also the pivot element and, but the problem here is with the consistency of the equation. The last equation says that 0 is equal to 2 which is not possible which is not possible here; that means, we cannot solve this system we cannot solve this system for 4 lambda 1 and lambda 2 there is no solution because our equations are inconsistent.

So, we cannot get any solution of this given a system of equations and therefore, the answer to this question is that this vector 1, 0, 0 or does not lie in this span of the vectors 4, 2, 7 and 3, 2, 4. So, here the system is inconsistent and we cannot write down this as linear combination given there.

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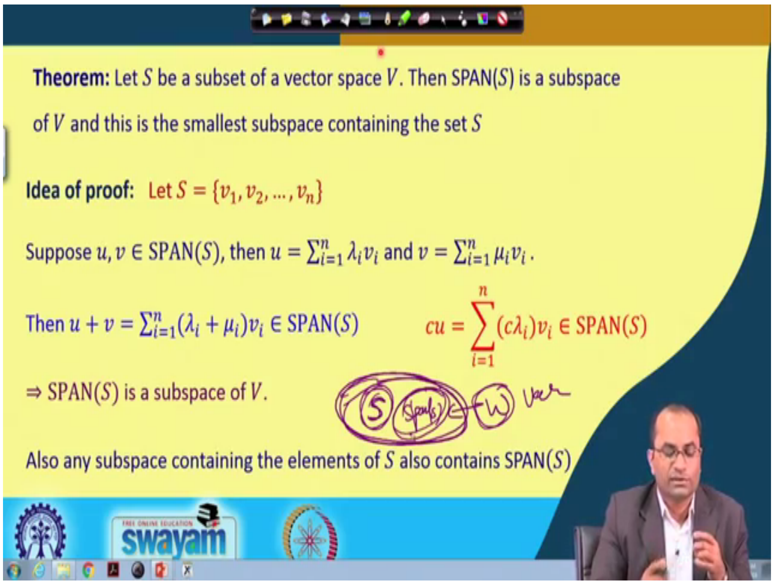


So, there is another nice result here we will not go through the formal proof, but we will see at least some intuition how this is happening. So, if as is a subset of a vector space V then this is span S yes so, what is span S? All the linear combinations of the vectors that is the span S, is a subspace of V and what is the nice property? This span has S so, we take any two any vectors collection of vectors and then the span of this will give you the vector space.

So, this is span S is the subspace meaning this a vector space of this V the subspace of V whose elements we have taken as three subsets and this is the smallest another important information that this is the smallest subspace which contains this set S. So, there cannot be any smaller subset of this which contain s and is a vector space. So, this is the smallest vector space containing the set of vectors.

So, just to see the idea of the proof here so, we take let us say this is our set S which contains v 1, v 2, v 3, v n these are the elements of this S and what we will show that the span of this S is a is a vector space is a vector space of V. So, for showing the vector space we need to show those two properties because we are talking about the subspace here we need to absolutely show those two properties that closure properties; that means, you take any two elements of this set and their vector addition should also belong to the same set and also when we multiply this set by a number a scalar number that the new element should also belong to this set S.

So, here let us suppose this u and v they belong to this span S and now because u belongs to the span S and v belongs to the span S so, we can write down this u and v as a linear combination of the vectors v's. So, here the u is written as a linear combination of the v i's vector lambda is v i so, this is a linear combination. Here also the v we have written as a linear combination of the vector v's.

Note that these lambdas and this mus will be different and the other scalars, but this u is another vector so, this is a linear combination, a different linear combination here v is another vector so, we have a different linear combination there. So, these two vectors which we have taken from the span S they must be the linear combination of the v's.

And now if we add them so, u plus v if we add them so, what will happen when we add them? This will be lambda i's plus mu i and this the v i. So, again this is a linear combination of the v i is because when lambda i's and mu i's are real their sum is also real number. So, we have again this as a linear combination so, this will also belong to span S because this span S contains all linear combination of the v i's and this is a linear combination so, definitely this will also belong to the span S.

And another property the closure property what we check when we multiply by any constant any scalar like here we have taken the c as a scalar. So, if we multiply are this c to this vector u then what we will get? We will get so, here this c not lambda i. So, when we write down this u because u is a linear combination well correct. So, this u is a linear combination so, which we have already written here u is written as the lambda i v i and when we multiply by a constant here c by a scalar c then what will happen the c lambda into; into v i.

So, this c lambda i when we have multiplied this scalar c here two lambda i that is another real number and this is another linear combination of v i. So, once we have this linear combination and all the linear combinations belongs to this span S so, this element will also belong to span S. So, what we have seen here the closure properties are satisfied; that means, this is span S is a vector subspace so, span S is a vector subspace and what else we need to show that that any subspace containing the element of S is also that set will also contain a span S and the reason is clear because this is span S contains all the linear combinations.

So, if you have a suppose you consider a vector space you we have a vector space for instance which contains this set S here , the definitely because if this is a vector space that say w is a vector space which contains S. So, definitely because this is a vector space all the all linear combinations because when we add two elements of this must be there. So, eventually all the linear combinations must be there in this set w; that means, this will also have i span S.

And that itself tell us that span S is the smallest subspace which contains this S because anything else which contains s will also contain this span S. So, it cannot be that this is not the smallest subspace. So, what we have learned here? That this is span S; the span S you take any vectors, from a vector space and having the span of this these vectors that will form a vector subspace of that vector V.

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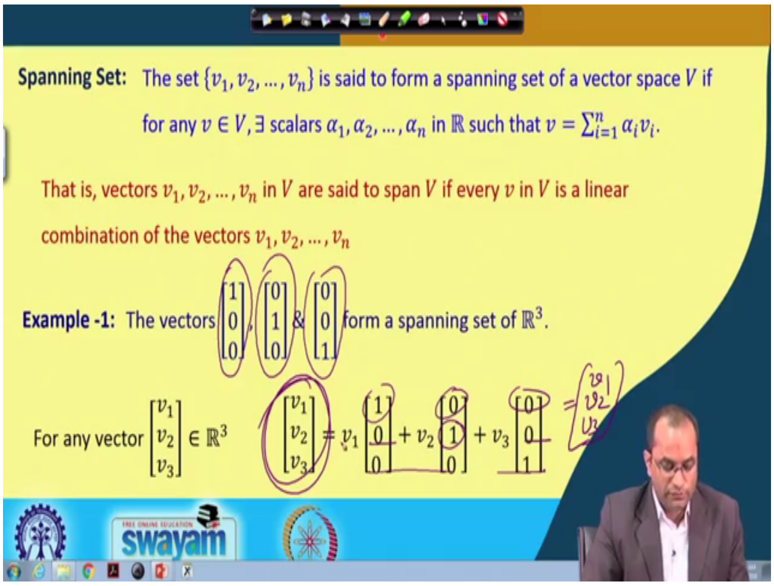


And there is another one this is called a spanning set. So, what is the spanning set? The set here v 1, v 2, v 3, v n is set to form a spanning set of vector space V.

So, we will call that this is a spanning set of this vector v if for any V, if for any v take any element from that vector space V then there exist the scalars this alpha 1, alpha 2, alpha n such that we have v is equal to this linear combination of these vs here. So, basically we call that this is a spanning set of a vector space V if any element of this vector space, we can write down as a linear combination of this given set V or in other words which we have written here the vectors v 1, v 2, v n in V are said to a span V if every v in V is a linear combination of the vectors v 1, v 2, v 3 v n.

So, this is spanning set is very important once we have this spanning set basically we can write down any vector of that vector space in terms of these given or as a linear combination of these given vectors. So, the example here we are considering the vectors 1 0 0, 0 1 0 and 0 0 1 this form a spanning set of R 3 meaning what we have to show? That if we take any element of this R 3 any element of this R 3, then we must be able to write down as a linear combination of these three vectors and that is what we mean this spanning set.

So, let us check whether how this 1 0 0, 0 1 0 and 0 0 1 form a spanning set for the vector space R 3. So, for any vector so we take a general vector this v 1, v 2, v 3 which we have denoted by this v 1, v 2, v 3 that can be any vector from R 3, we are not restricting anything on v 1, v 2, v 3. They can be any real number meaning that this vector belongs to this R 3 and it's a general vector yet can it can be any vector from this R 3 and what we will, what we are supposed to do now? We will show that this vector which is a general vector from this R 3 without any restriction. So, any vector from R 3 we can write down as a linear combination of these given vectors and why this is trivial in this case? This is v 1 v 2 v 3.

And the structure this is called the later on we will come up with another name this called the standard vectors or rather standard basis which you will refer later on this these terminologies. So, here we have 1 0 0, 0 1 0 and 0 0 1, we what we need to do? We need to just multiply this first vector by the number v 1, the second one v 2 and the third one v 3 and as we see clearly this is the first component here v 1 here we have 0 0 so, basically this will add to v 1 the second one here 0 and there 0 only the v 2 will be coming up and in the third one when we add only the v 3 will be coming up.

So, this any vector of this vector space R 3 we can write down as a linear combination of this a these vectors 1 0 0, 0 1 0 and 0 0 1 which a was trivial at to see in this particular case.

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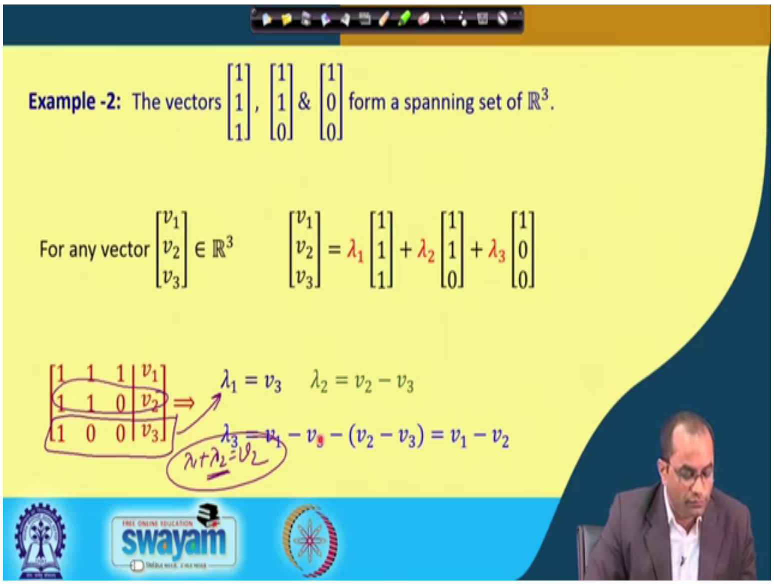


Another example we will be talking about these vectors here 1 1 0 and 1 1 are all 1 here so, 1 1 1 and we have 1 1 0 and we have taken here 1 0 0 and this also form a spanning set of this R 3, how this forms a spanning set? So, we have seen already one example where the vectors were taken as 1 0 0 and 0 1 0 and the 0 0 1. This is another set and then again we are claiming that this form is spending set of this R 3.

So, naturally this spanning set is not unique, we can have we can have a different set of vectors and that spanned the same vector space. So, here this is another example where we will see that this set can also span R 3; that means, any element of this R 3 we can write down as a linear combination of these three vectors. To see this we will take a general vector here v 1 v 2 v 3 from this R 3 and we will try to write down this v 1 v 2 v 3 as a linear combination of the given vectors meaning that lambda 1 times this first vector lambda 2, second vector lambda 3, the third vector.

Now the question is can we write down as a linear combination here because here is not as trivial to see as the earlier case though this is also not difficult. So, what do we do? We again we will solve these three equations the system of equations the first equation is lambda 1 plus lambda 2 plus lambda 3, the second equation is lambda 1 plus lambda 2, third equation is just the lambda 1 is equal to v 3.

So, without indeed reducing to this row reduce echelon form we can directly also solve this equation, we do not have to do that because that this equation the third equation tells that lambda 1 is equal to v 3. So, from here itself we get direct answer that lambda 1 is equal to v 3. Now you will take the second equation which tells us that is lambda 1 and plus lambda 2 is equal to v 2 so, from here we will get this lambda 2 as v 2 minus v 3. So, the second equation will give lambda 2 as v 2 minus v 3 and from this equation number 1 which is lambda 1 plus lambda 2 plus lambda 3 is equal to v 1.

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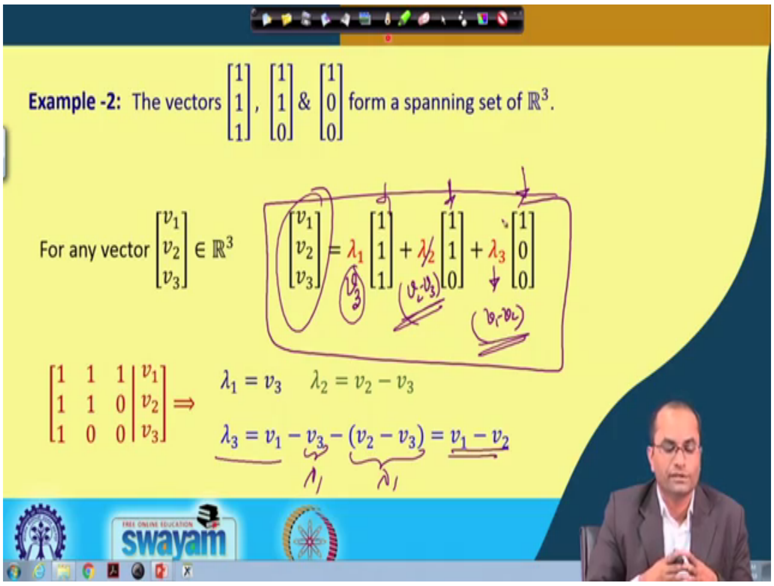


So, lambda 3 will be v 1 minus this lambda 1 which is v 3 minus lambda 2 which is here given as this is lambda 2 and this is lambda 1 and this was the right hand side. So, from this first equation we got that lambda 3 is equal to v 1 and minus this v 2. So, here also we got the v 1 so, here the lambda 1 is just the v 3, the lambda 2 we need to take v 2 a minus this v 3 and this lambda 3 we need to take v 1 minus v 2.

So, we got this linear combination here that this any vector v 1 v 2 v 3 we can write down as a linear combination of these given vectors in the form that v 3 the first vector, v 2 minus v 3 the second vector and v 1 minus v 2 this third vector. So, again we have seen in this example that this was a different set here from than the earlier example, but again this is also a spanning set of this R 3.

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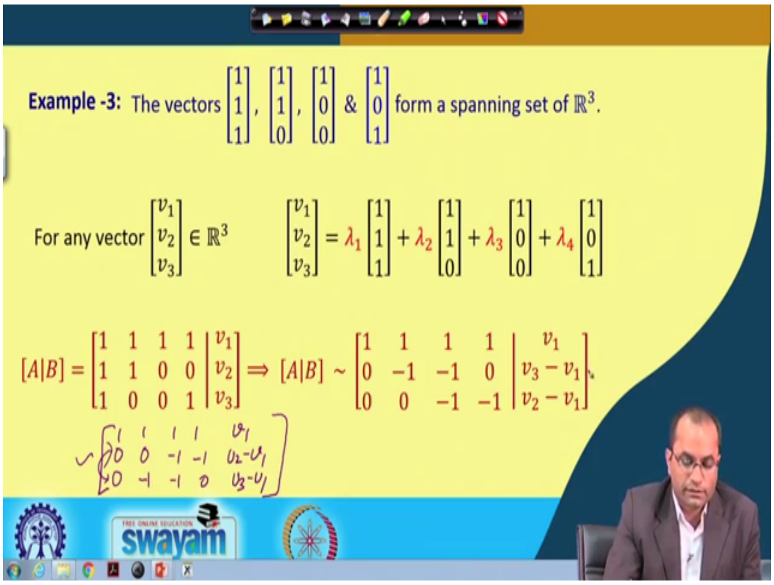


So, what we have now? We have this example number 3, where we will show now we have taken from the previous example yeah so, these vectors 1 1 1, 1 1 0 and 101 these are from the previous example and we have taken one more that 1 0 0 1 0 1. So, we have added one more vector here now our set here is bigger and now again this also forms a spanning set of R 3 that is what we will observe now.

So, the existing set and we have added one more this vector 1 0 1 and this is also forming a spanning set. So, you can have really a different number of elements in this spanning set exactly, I mean quite different elements also in the in the in the set. So, here we will check how this forms a spanning set.

So, for any general vector v 1 v 2 v 3 what we have to again consider this linear combination that whether we can write down this v 1 v 2 v 3 as a linear combination of these four vectors. So, we have these 4 lambdas now lambda 1 plus lambda 2 plus lambda 3 plus lambda 4. So, we have again three questions, but there are 4 unknowns; unknowns are these lambdas and we have the three equations a the first equation is this sum of all these lambdas is equal to v 1, the second one lambda 1 plus lambda 2 is equal to v 2 and from the third equation we have lambda 1 plus lambda 4 is equal to v 3.

So, having these three questions and then four unknowns we will again set up this augmented matrix. So, our augmented matrix now, here these columns will be these vectors here 1 1 1, 1 0 1, 1 0 0 and 1 0 1 with the right hand side v 1 v 2 and this v 3. So, this augmented matrix we can reduce it to this echelon form which is I believe very simple again. So, here this is the first will remain as it is v 1 and then here we will make it 0 so, subtract row 1.

So, this is also 0 this is minus 1 minus 1 and then we have v 2 minus 1 here again we will subtract that so, we will get this 0 and v 3 minus this v 1. So, and now we can exchange these two rows and eventually we will get others this row reduced form. So, we have this reduced form here and now what we will observe out of this reduced form whether we can get such lambdas or we cannot get such lambdas now.

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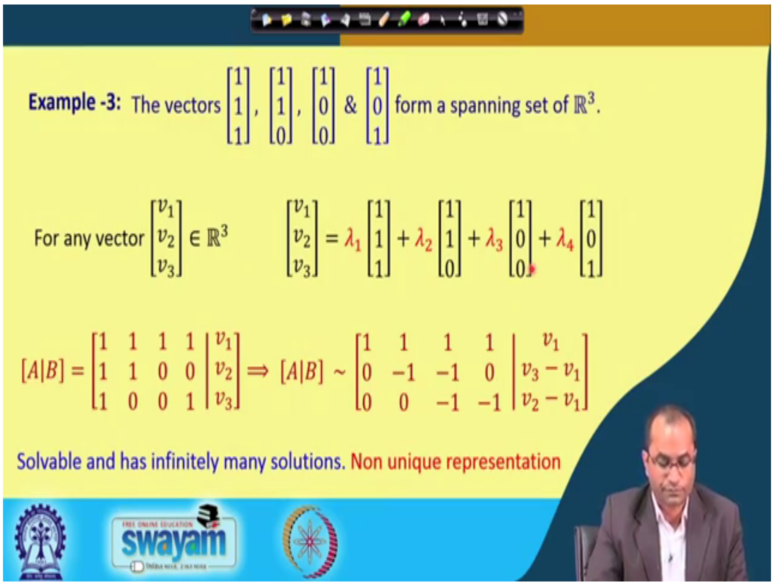


So, getting again back to this pivot elements so, the first column has a pivot, second column has also pivot element and the third column also has a pivot element. So, we have the three pivot elements and here we do not have this pivot element this is not the pivot element. So, this lambda corresponding to this 4 so, lambda four is a free variable right so, lambda 4 is a free variable and all other lambdas will depend on this choice of this lambda 4.

So, what do we get now? So, the question is the answer to this question is that we do have such lambdas which will add up to this v 1 v 2 v 3 and indeed now in this case we have ; we have many choices; we have many choices for these lambdas to give this v 1 v 1 v 1. In fact, this it is not unique now this representation is not unique while in the earlier cases one can closely observe and doing this echelon form. So, what we will get? We will not get this column for instance, if we consider just with these three examples.

So, we will get these every column will have a pivot here and that will be equal to the number of these unknowns the number of lambdas in that case and you will get a unique representation. So, the earlier example when we did not have this vector there, we have the unique representation of the of this v 1 v 2 v 3 in terms of the given vectors.

But what happened now in this case we do not have a unique representation, but we can represented, we can write down any vector from this R 3 in terms of these given four vectors. So, indeed this is a spanning set the only difference from the earlier example to this example here is that that we have a non unique representation, that we have to choose now lambda 4 and then compute lambda 1 lambda 2 lambda 3.

We have basically infinitely many solutions now so, this is a non unique representation in the first two examples we have a unique representation. So, that was the difference here and now in this case we have a non unique representation.

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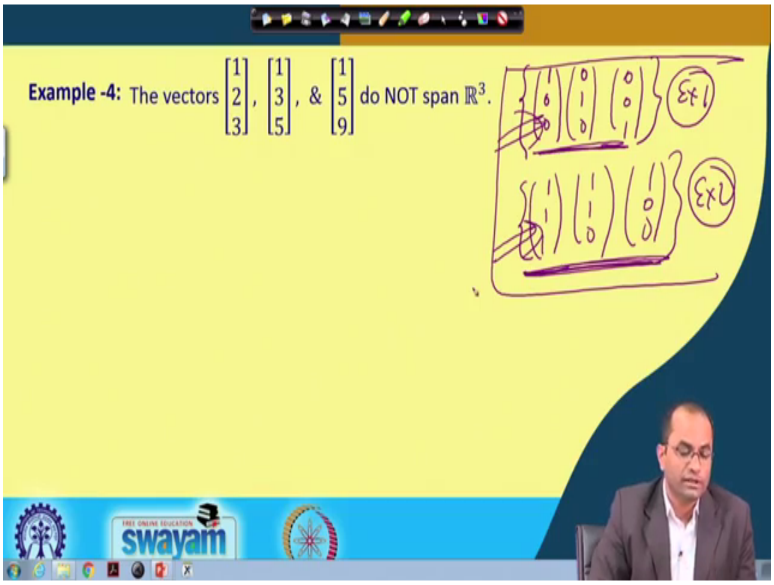


Coming back to this example 4, we will see that again we have three vectors here 1 2 3, 1 3 5, and 1 5 9 these are three different vectors. So, but they do not span R 3 in example 1 in example 2 we have three vectors and they span R 3 means any vector of R 3 we can write down in terms of those given vectors. So, in the example 1 we have this, these set here 1 0 0 and 0 1 0 and then we had 0 0 1. So, these were the three vectors from example 1. Example 2 I guess it was 1 1 1 and then 1 0 1 and then we had 1 0 0 so, this was from example number 2.

So, with these though they have the three different elements from R 3, they span; they span R 3 means any element of R 3 we can write down in terms of these vectors or in terms of these vectors, but now in this case what we will observe that this is not possible that you take any three elements from R 3 and that will form this spanning set.

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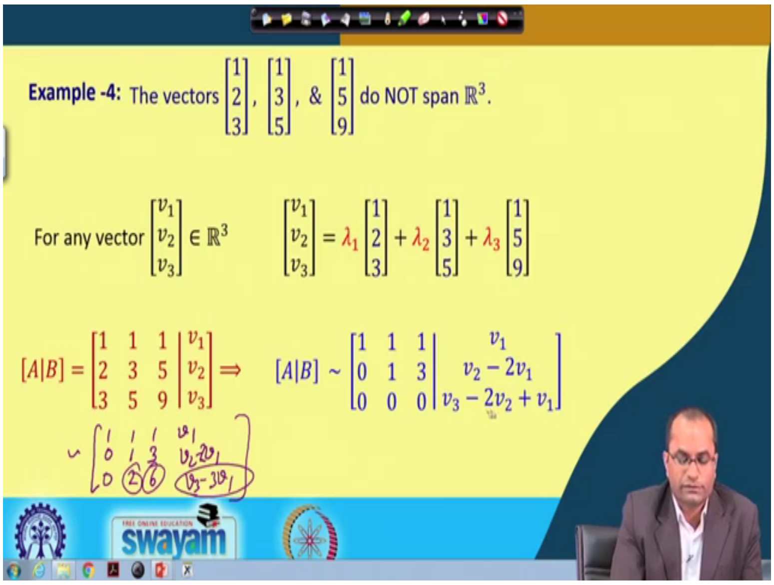


How to check this? We will take a general vector here v 1 v 2 v 3 as usual and we will try to write down this v 1 v 2 v 3 in terms of the lambda 1 lambda 2 and lambda 3. So, lambda 1 with the first vector lambda 2, the second vector in lambda 3 with this third vector 1 5 9. So, writing this in terms of the lambdas, we have the vector the augmented matrix here now with the coefficient matrix.

So, this 1 2 3, 1 3 5 and 1 5 5 the right hand side here v 1 v 2 and v 3, we need to reduce to the echelon form. So, we can make this 0 by multiplying this two times and then subtracting here to 0 and then we will get here 1 and then here we will get 3 and then v 2 minus this 2 v 1 and then again we have to also do this manipulation later for this case. So, just to give you the idea the first, row we have this one, the second one the two times of this we will subtract so, this is 0 this is 1 two times so, here we will get 3 and this will be two times minus v 2 minus 2 v 1.

The third one three times or r 1 we are subtracting so, here the 2 and the 3 so, here we will get 6 and here we 3 minus 3 v 1 and then once again this two times of this we will subtract though this will become 0, this will also become 0 and here this factor will be coming. So, this is the row reduced echelon form which we got here 110 and 013 and 000.

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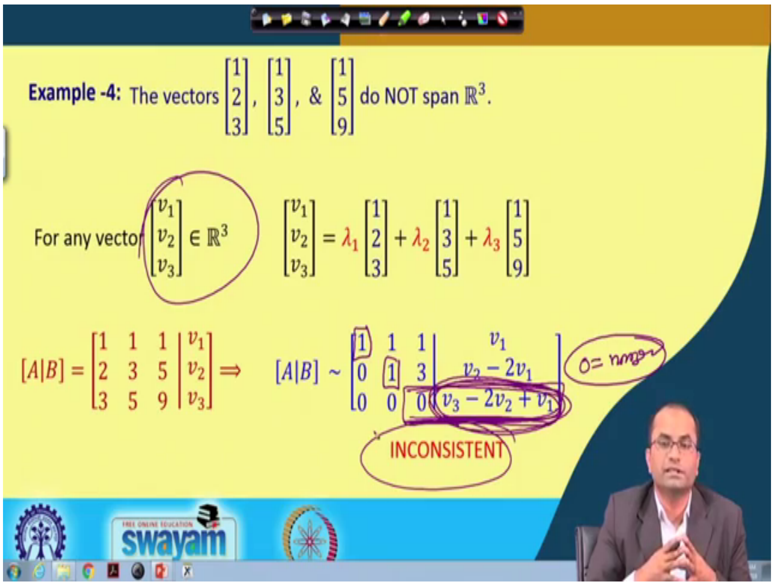


So, now what do we observe in this case? That here we have pivot here also we have a pivot and this forget about the pivot we have the inconsistency here, while inconsistency because this v 1 v 2 v 3 they these are they can be any real number because our vector here is from R 3 and say any arbitrary vector from R 3. So, having so, we cannot; we cannot just observe that this will be 0 this will be a non zero number as well in many situation and then 0 is equal to something nonzero we are getting.

So, this system we cannot solve for general v 1 v 2 v 3 from R 3; that means, the system is inconsistent and hence we cannot write down in this case as a linear combination of these given vectors. So, what we have observed? That just the number is not important that we have taken here three elements from R 3 in this, in the third example we have taken four elements from R 3 in first two examples we have taken again only three elements.

In the first two cases we had a unique representation for a given vector from R 3 in terms of the given vectors, in the third example where we have taken 4 that was also spanning set, but there you are getting non unique representations of a given vector from R 3 and now in this example though we have three vectors from R 3, but they are not sufficient to represent a general element or general vector from R 3. So, this does not form a spanning set of R 3.

So, there is a lot more to discuss on this spanning set and that will follow in the next lectures.

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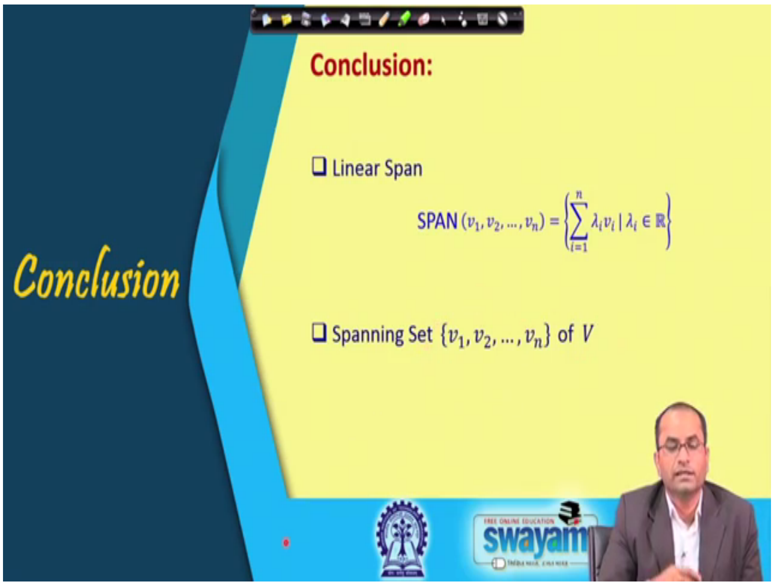


Here the conclusion is that this linear span is nothing, but all the all linear combinations of the given vectors and the spanning set which v 1, v 2, v n of v we will call that this is a spanning set, if any vector of this v, we can represent in the form of these vector these then we call that this is a spanning set.

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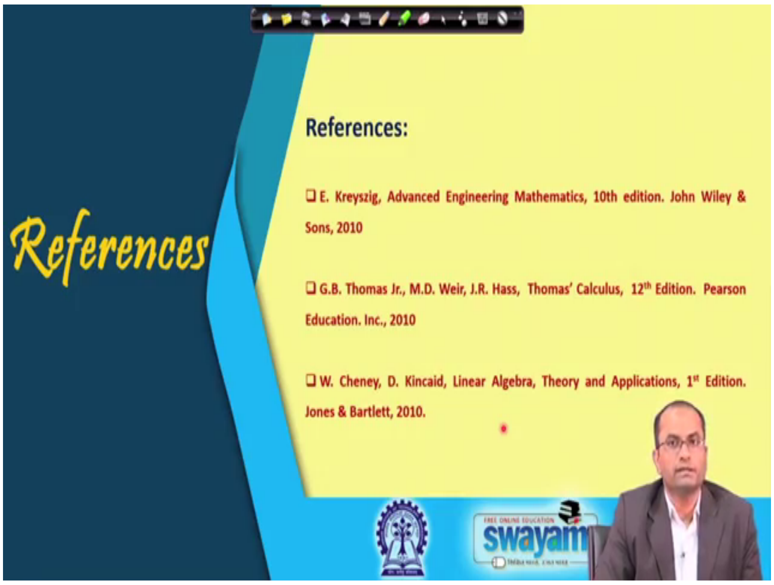


So, these are the references use for preparing these lectures.

And thank you very much for your attention.