Engineering Mathematics – I
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Lecture – 40
Linear Algebra – Linear Independence of Vector

So, welcome back and this is lecture number 40, and we will be talking about the Linear Independence of Vectors. In particular, we will cover today the linear combinations of the vectors and also this linear independence of vectors.

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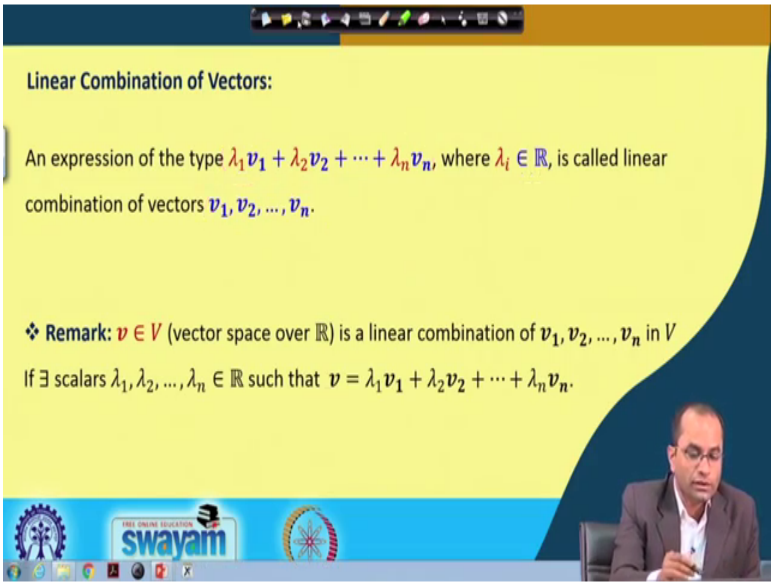

So, what is linear combination? So, an expression of this type lambda 1 of v 1 plus lambda 2 v 2 plus lambda n v n, where these lambdas belongs to the set of real numbers it is called linear combination of vectors v 1, v 2, v 3 and v n

So, here for given vectors this v 1 v 2 v n this expression here lambda 1 v 1 plus lambda 2 v 2 and so on lambda n v n this is called the linear combination of these vectors where these lambdas are belongs to the set of real numbers and just a remark short remarks here. So, v belongs to a vector space over R which is V is a vector space and v small v is an event of this vector space V this is a linear combination of v 1, v 2, v 3, v n in V. So, these all are vectors in V and we call that this V is a linear combination of v 1, v 2, v 3, v n if there exists scalars lambda 1, lambda 2, lambda 3, lambda n and they belongs to R because V these as a as the scalars as such that we can write down this v here the given v as a linear combination of the others.

Then we call that this v is a linear combination of the vectors v 1, v 2, v 3, v n. If we can write down this v as lambda 1 v 1 plus lambda 2 v 2 plus lambda n v n then we call this vector v which is again this vector is a more general term we are talking about from the vector space they can be matrices, they can be polynomial etcetera. So, here if this given v here we can write down as a linear combination of these other vectors v then we call that this is a linear combination of the vectors v 1, v 2, v 3, v n.

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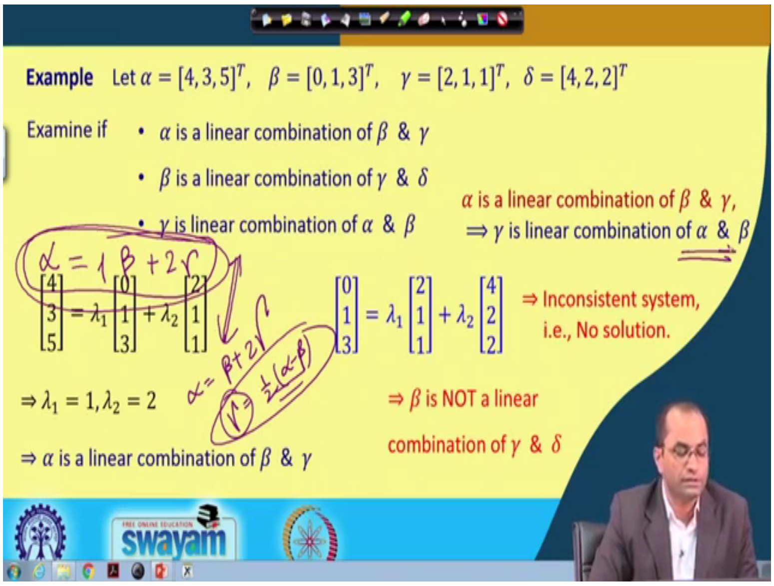

So, the example here we take these four vectors alpha, beta, gamma and delta. So, here 4, 3, 5 this transpose. So, we have this column vector here also we have 0, 1, 3 and 2, 1, 1 and 4, 2, 2. Now, we have the following questions now we want to examine whether this alpha is a linear combination of beta and gamma we will also see that if this beta is a linear combination of gamma and this delta and this gamma is a linear combination of alpha and beta.

So, we will answer these questions here. To answer the first one if this alpha is a linear combination of beta and gamma. So, what we want to show that this alpha here this is alpha is a linear combination of the beta and gamma. So, this is beta and then here we have gamma.

So, if we can write down this alpha as lambda 1 beta plus lambda 2 gamma for some lambdas then we will call yes, this alpha is a linear combination of beta and gamma. So, now, to do so, we will check whether it is possible or not and eventually we end up usually with the system of linear equations to answer all these questions. And we have to solve this system of linear equations which we can write down like again we can simplify this little bit.

So, left hand side we have 4, we have 3, we have 5 that is a vector; the right hand side the first equation if you write down lambda 1 times 0 plus this lambda 2 times 2. So, we have basically 2 times lambda 2, the second equation will be lambda 1 plus this lambda 2 and then we have 3 lambda 1 and plus lambda 2; so these are the equation. The first equation says that 2 lambda 2 is equal to 4. So, where we get the lambda 2 is equal to 2 from this second equation then because they should add to this 3 and lambda 2 is already 2. So, from here we will get that lambda 1 must be 1 and this 3 lambda 1 plus lambda 2 is equal to 5.

So, whether now we have to check whether it is also satisfied or not. So, 3 times this lambda 1 and plus this lambda 2 and this will give us 5. So, the third equation is also satisfied. So, for this lambda 1 is equal to 1 and lambda lambda 2 is equal to 2 this relation is satisfied; that means, we have this alpha is equal to lambda 1 is 1. So, beta and plus this lambda 2 is 2 2 gamma. So, we have this relation between the alpha, beta and gamma; alpha is equal to beta plus 2 times the gamma. So, naturally this alpha is a linear combination of is a linear combination of beta and gamma which we have seen and in this example.

Now, we will check for the second one that if this beta is a linear combination of gamma and delta. So, what is beta 0, 1, 3 this is the vector beta and then lambda 1 times 2, 1, 1 and lambda 2 times 4, 2, 2. So, we want to check whether such lambdas exist so that we can write down this beta as a linear combination of gamma and delta. And now to check this is a quite obvious here if we just again write down in terms of the equation. So, the first equation is like 2 lambda 1 plus this 4 times this lambda 2 is equal to 0. The second equation is lambda 1 2 lambda 2 must be equal to 1 and the third equation is lambda 1 plus 2 lambda 2 is equal to 3.

So, this equation 2 and 3 they tells us that lambda 1 plus 2 lambda 2 is equal to 1 and lambda 1 plus 2 lambda 2 is equal to 3. So, here there is a inconsistency because lambda 1 plus 2 lambda 2 this equation tells must be 1, but this tells that this must be 3. So, it is not possible to find such lambdas which satisfy these two equations here, forget about the first one.

So, here definitely then we cannot find such lambdas which can satisfy this relation here that beta is equal to this lambda 1 gamma plus lambda 2 gamma lambda 2 delta. So, what do we get now, so; that means, this beta is not a linear combination of these two elements here gamma and delta. So, this is a inconsistent system and it has no solution and therefore, beta is not a linear combination of gamma and delta.

Now, coming to the third one here gamma is a linear combination of alpha and beta we have just seen in this case 1 here that this was alpha here and then we had a beta and we have gamma that alpha is a linear combination of this beta and gamma and this was the relation alpha is equal to beta plus 2 gamma and now, we are asking whether gamma is a linear combination of alpha and beta. So, naturally gamma is a linear combination of alpha and beta and we do not have to prove anything because in the first part we have already shown this relation that alpha is equal to beta plus this 2 gamma and then we have naturally that gamma is equal to one half of alpha minus beta.

So, this gamma is a linear combination of alpha and beta which is trivial from this part 1 of this problem, ok.

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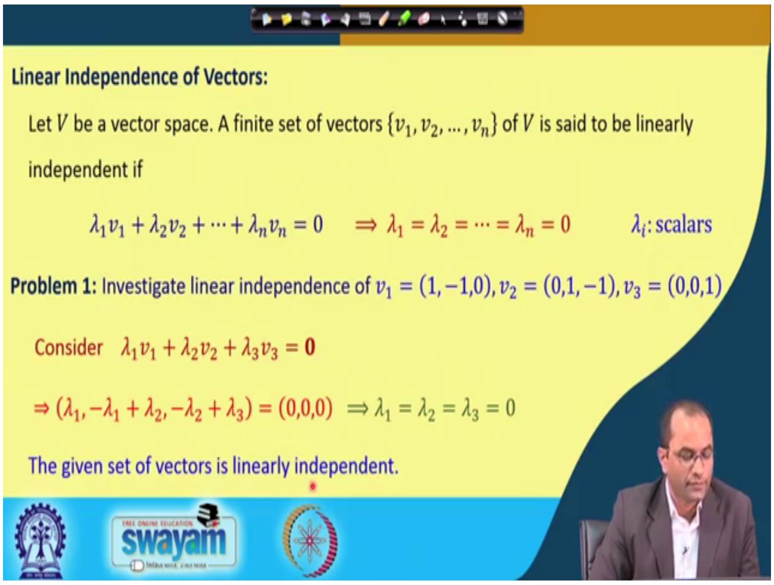

Well, so, now going to the next topic here we will be talking about linear independence of vectors and what do we have here. So, V is a vector space we take and then a finite set of vectors. So, v 1, v 2, v 3, v n of this vector space V this is said to be linearly independent. So, the set here of vectors is said to be linearly independent if this relation here lambda 1 v 1 plus lambda 2 v 2 and so on lambda n v n is equal to 0, implies that this is only possible when all these lambdas are 0; otherwise we do not have any other lambdas which can make this linear combination to 0; so these lambdas are scalars.

So, now we can investigate again here the linear independence of these given vectors which are given here. So, what do we have to check for this linear independence that if this linear combination which is given lambda 1 v 1 plus lambda 2 v 2 lambda n v n is equal to 0, after solving these equations if we get the only solution as lambda 1 is equal to lambda 2 is equal to lambda n is equal to 0, then we call that these this set is linearly independent, but if we get some other solution of the lambdas nonzero solution then they will be dependent here. So, they are not independent in that case.

So, let us consider this example. So, here we have taken these three vectors 1, minus 1, 0; 0, 1, minus 1 and this third is 0, 0, 1. So, we will consider now this expression lambda 1 v 1 plus lambda 2 v 2 plus lambda 3 v 3 is equal to this zero vector. And now, from here we will get basically the equations because we have this lambda 1 times this v 1. So, 1 minus 1 0 here we have lambda 2 the v 2 minus 1 and then we have also this lambda 3 0, 0, 1 and the right hand side the zero vector.

So, now this comparing the each component, so, the first equation we will get as lambda 1 is equal to 0 which is already lambda 1 here when we add these three we will get lambda 1, we will get minus lambda 1 plus lambda 2 and the third 1 we will get here minus lambda 2 and plus lambda 3 minus lambda 2 plus lambda 3 and is equal to the right hand side is the zero vector. So, we have the 0 comma 0 comma 0.

And now, from here we get actually three equations and after solving these equations we will see whether we are getting a unique solution as in terms of lambda and the unique solution will be naturally the 0, 0, 0 because that is always the case here when we have such a system with lambda 1 v 1 plus lambda 2 v 2 lambda 3 equal to 0. So, naturally all the trivial solution what we call here is 0, 0, 0 for lambdas, but you will see whether we have some non-trivial solution or not if you have only the trivial solution that is the zero solution then we call that they are linearly independent.

So, here what we will observe because the first equation itself tells you the lambda 1 is equal to 0, when lambda 1 0 the lambda 2 is equal to 0 that is coming from the second equation and from third equation when lambda 2 0 we are getting also lambda 3 is equal to 0. So, that is the only possibility we are getting here that is a unique solution we can also in general we will come right down in the form of this augmented matrix and then reducing to this row echelon form from there also we can observe whether we are getting a unique solution or we are getting infinitely many solutions or this is the this cannot be the case of no solution naturally because this is always a system of this homogeneous equation which always has at least a trivial solution.

So, here we have seen that these vectors are linearly independent.

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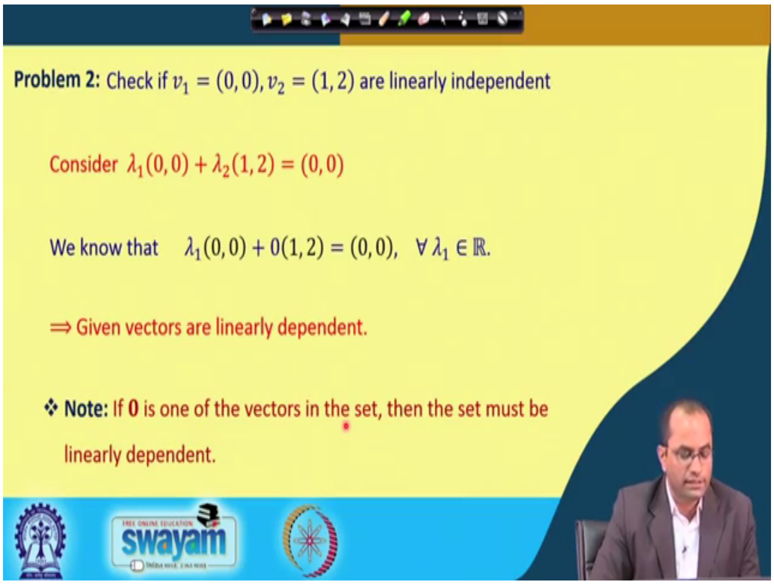

And, we will check now the second example which says that v 1 is equal to 0 comma 0, v 2 1 comma 2 are linearly independent. And this is a trivial task to check whenever we have this zero vector is there in the set because we can always consider this equality lambda 1 0, 0; lambda 2 1, 2 is equal to 0, 0 and since the zero vector is there so, I can take any lambda here it does not where matter I can take any lambda with this and then the 0 value to lambda 2 and the equation will be satisfied because with this 0 we have now the zero vector here and I can choose any lambda still this will be a zero vector and 0 plus zero vector will give us zero vector.

So, for all lambda 1 from this set of real numbers this equation is satisfied and therefore, we are getting a nonzero solution that our lambdas are not zero in this case, but they are we are getting infinitely many possibilities it is which are adding to this zero here and therefore, this set is linearly dependent.

So, whenever we have a zero vector included in the set of these vectors then these vectors are going to be linearly dependent. So, this 0 is one of the vectors in the set and then the set must be linearly dependent.

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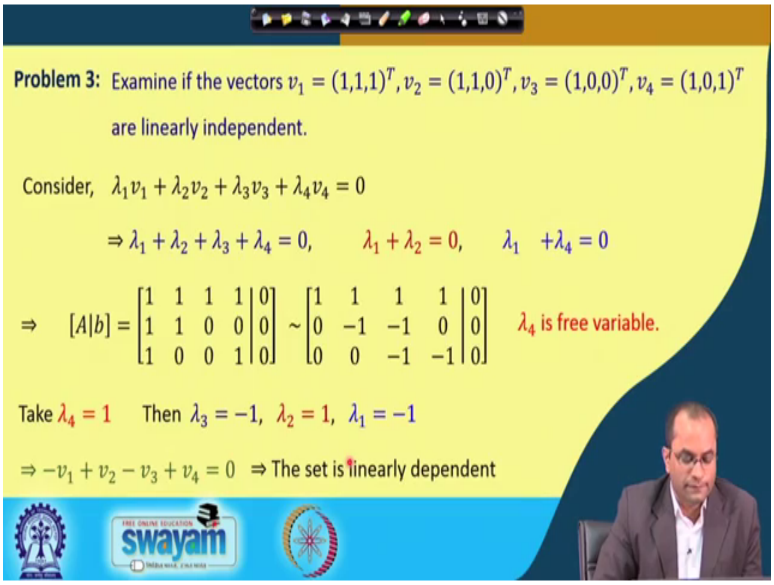

And, this problem 3 we examine these vectors 1, 1, 1 and 1, 1, 0; 1, 0, 0 and we have 1, 0, 1 are linearly independent. So, you have taken these 4 vectors now and we want to check whether they are linearly independent or they are linearly dependent the same process we will continue now. So, we will consider this linear combination here and that will be set to 0. So, we have lambda 1 v 1 plus lambda 2 v 2 plus lambda 3 v 3 plus this lambda 4 v 4 is equal to 0.

And, this will imply; so, we have to now form these equations here lambda 1 is multiplied to the first vector here to the second vector and so on. And, we will now set it to 0. So, out of this we will get the questions here the three equations because here the lambda 1 is multiplied by this 1, 1, 0 and the lambda 2 is multiplied with 1, 0, 0; lambda 3 is multiplied by 1, 0, 0.

Now, this is 1, 1, 1; then the second is 1, 1, 0; third one is 1, 0, 0 and this fourth 1 is 1 0 1. So, these are the and now the right hand side again this 0 vector. So, from here we will get the first equation as lambda 1 plus lambda 2 plus lambda 3 plus lambda 4 is equal to 0, that is the first equation here. The second 1 have lambda 1 plus this lambda 2 and here there is no component, so, this is equal to 0. And, the fourth the third equation will be lambda 1 from here and lambda 4 from here is equal to 0.

So, we have these three equations coming out of this linear combination because there were three components in each vectors. So, having this now, this system of equations which we want to solve for lambdas the general way would be always to write down in the form of this augmented matrix and then do this elimination or reduce to this row echelon form and from there the situation will be clear whether we are getting unique solution or we are getting non unique solution.

So, here for this system of equations we have written here this augmented matrix. So, the coefficient matrix here 1, 1, 1, and 1; from the second equation we will get 1, 1, and 0, 0 from this 1 and this last component 1 in the middle will be having 0. The right hand side the zero vector which we have kept here the zero vector and then we can reduce it to the echelon form which is very simple in this case because this first row will remain as it is.

From second now our aim is to make this 0. So, we subtract the row number 1. So, this R 1 will be R 1 minus sorry R 2 will become now R 2 minus R 1. So, this is 0, here 0 will get minus 1, minus 1 and 0. Here again we will subtract R 3 minus this R 1 again. So, here we will get 0 and this will be minus 1, this will be minus 1, this will be 0, 0.

So, this and now we can interchange the row number 2 and 3 and we will get exactly this echelon form. So, having this echelon form now row reduced echelon form we can now look into it whether what kind of solutions are we getting. So, looking at this so, this is the pivot here. Here also we have the pivot the second column has the pivot, the third column also has the pivot and the fourth column does not have a pivot because this cannot be pivot because there is a nonzero quantity sitting left to this.

So, here that is what we call and if you remember already from our previous lectures this is corresponding to lambda 4 and which we call the free variables; free variable in this case only one and these are the dependent variables lambda 1, lambda 2, lambda 3. So, this having a free variable in the solution in the system here we are naturally getting infinitely many possibilities of the solution now. And once we have infinitely many possibilities of the solution that system cannot be linearly independent because we do not have on the unique solution which is lambda 1 all these lambdas to be equal to 0.

So, in this case we have now the so many solutions infinitely many solutions because this lambda 4 is a free variable and we can choose anything we want. So, lambda 4 for instance we have taken as 1, and then from this equation we can get this lambda 3 as minus 1, from the second equation lambda 2 and from this first equation we can get lambda 1 as minus 1.

So, we are getting for this choice of this lambda 4 all other lambdas here lambda 3 is minus 1 lambda 2 is 1 and lambda 1 is minus 1. So, that means, we are getting this relation here. So, naturally they are dependent, they are not independent and they depend on each other with this relation. So, this given set of vectors here is linearly dependent.

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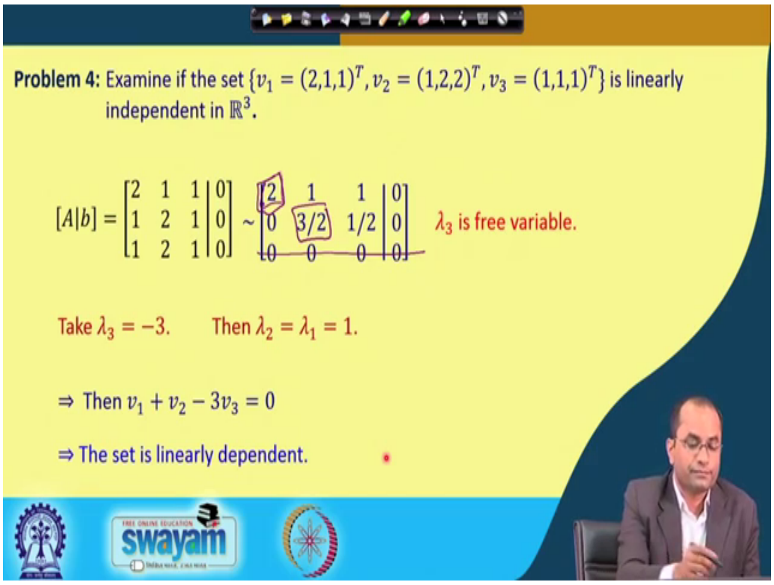

So, the another problem here we will examine now this set 2, 1, 1 and 1, 2, 2 and 1, 1, 1 whether it is linearly independent or it is linearly dependent. So, just getting back to the previous one, what we have it is a similar problem. So, we had the four vectors here and our augmented matrix was having this columns here all these vectors were the columns here 1, 1, 1 this was 1, 1, 0. So, we kept all these in the columns here and the right hand side this b is always 0, we can we do not have to write also this 0, 0, 0 always because that is not going to change. So, we can just work with the A itself and get the row reduced this echelon form from there we can conclude the existence of the solution.

So, here now getting back to this one the new problem we have these three vectors and you want to check the linear independence of these vectors. So, what do we consider? We consider directly the augmented matrix which we have earlier formed through the system of linear equations. So, here the each vector will be a columns 2, 1, 1 and then we have 1, 2, 2 and then the third one we have 1, 1, 1 and the right hand side again 0.

So, now, we will reduce this to this reduced echelon form which is much easier in this case as we see 1, 2, 1 and this 1, 2, 1 these are the same. So, one of them will become immediately 0 here and then from the equation number 1 we can also reduced the equation number 2. So, we will get finally, this row reduced echelon form and where now we have this is the pivot element and this is the pivot element and this third column will not have a pivot element. So, we have these two pivot elements here. This is the 0 rows and what we conclude now?

So, we have the free variable. We have the free variable this lambda 3 corresponding to the column 3 it is a free variable and which we can choose whatever we like. So, just for the sake of simplicity of the calculation we have taken lambda 3 is equal to minus 3. And then this lambda 2 and lambda 1 from this equation number 2 and from the equation number 1 we will get the other 2 once we fix this lambda 3. We can take any other number also here for lambda 3 we have free to choose lambda 3.

So, with this combination we are getting for instance v 1 plus v 2 and minus v 3 times v 3 is equal to 0 and there are there can be so many possibilities here if we choose this lambda 3 as 1 we will get different lambda 1, lambda 2 or any other number for this lambda 3 correspondingly this lambda 1 and lambda 2 will change.

So, this is definitely not a unique representation we can have as many representations as possible here, but what it says that these vectors are linearly dependent and one of the relations here for their dependencies already here v 1 plus v 2 is equal to the 3 times v 3 that is 1 dependency we have shown in this case and this set is linearly dependent.

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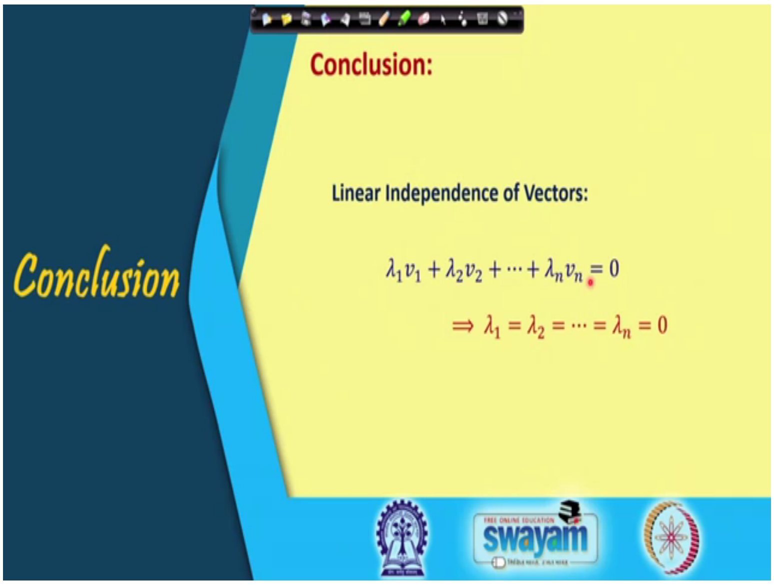

So, what we have learnt today, it is about the linear independence of the vectors. This is again a very important topic when while considering the vector spaces which we will continue our discussion in the next lecture and what was important that this linear combination of the vectors lambda 1 v 1 plus lambda 2 v 2 plus lambda n v n is equal to 0.

If we out of this equation if we get the unique solution as the lambda 1 is equal to lambda 2 is equal to lambda 3 is equal to lambda n is equal to 0, then we call that this set is linearly independent. Because we can there is no dependency on the vectors on each other because that is the only possibility coming out of this relation, so we cannot write any dependency among these vectors and that is the reason we call this linear independence that they are independent.

Whenever we are getting a nonzero solution and that will be the case of infinitely many solutions. So, there we can have now many ways to write down these relations. So, 1 can be written in terms of the others and that is the case where what we call a linear dependence. So, in that case those vectors will be linearly dependent; so, these are the references we have used in this lecture.

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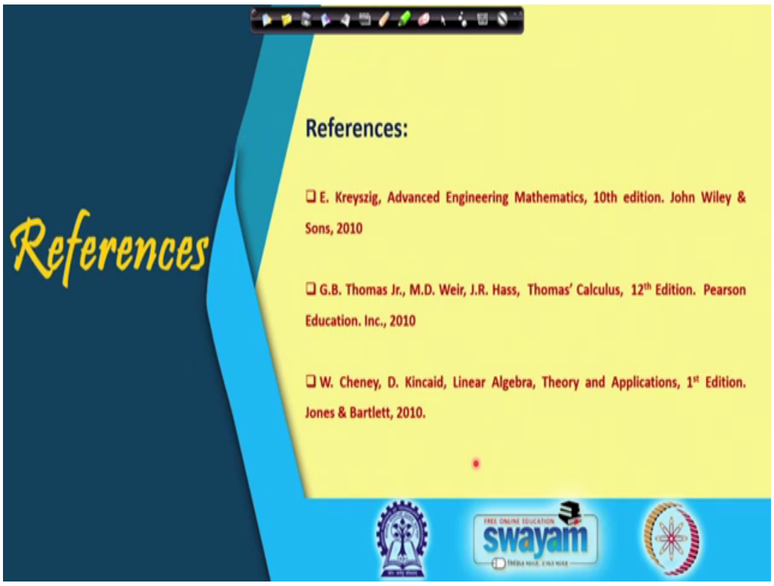

And thank you for your attention.