Engineering Mathematics – I
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Lecture - 04
Indeterminate Forms Part - 2

Hai, welcome to the lectures on Engineering Mathematics and today we will be continuing this Indeterminate Form Part 2 and this is lecture number 4.

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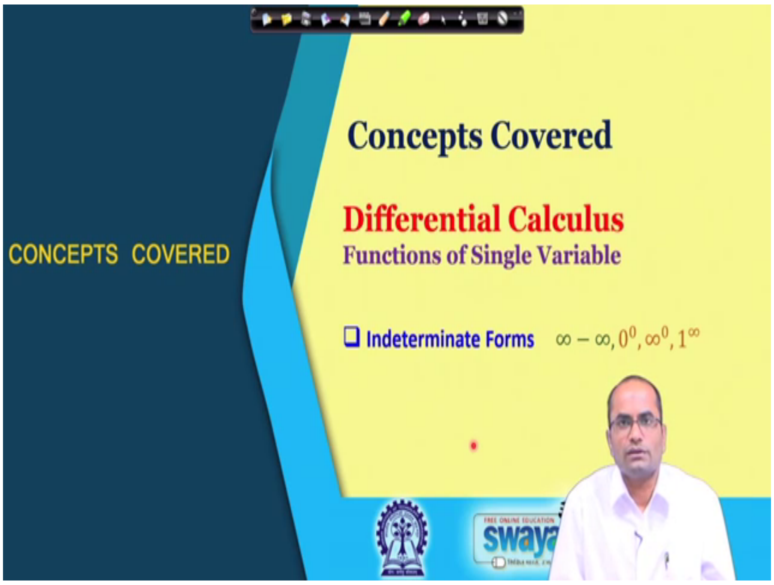

So, we are discussing differential calculus and functions of single variable, and today this indeterminate forms of the type infinity minus infinity 0 power 0 infinity power 0 and 1 power infinity will be discussed.

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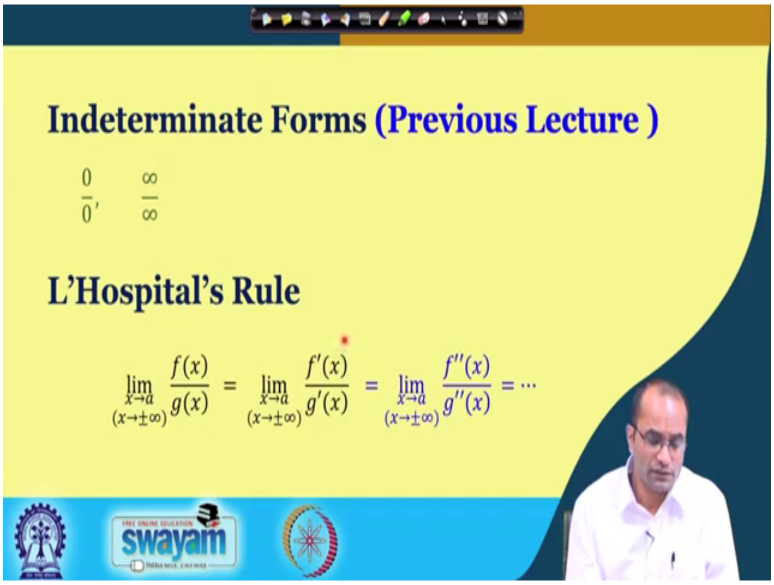

Just to recall from the previous lecture we have discussed already these two forms or rather these two indeterminate forms of the type as 0 by 0 and infinity by infinity. There what we have seen that this L'Hospital rule was very helpful which says that the ratio of this f x and g x where f x and g x both either goes to 0 or they both go to infinity. So, in that case the limit of this ratio will be equal to the limit of the derivatives provided this limit on the right hand this exist.

And we have also seen that we can continue this process provided that all the conditions on this f prime and g prime satisfies and again we have the situation that this f prime and g prime both they go to 0 or they go to infinity. So, in that case the rules says that we can continue this process of these differentiation and again we can differentiate the numerator and again the denominator. We can take the limit till the time we achieve this limit.

The important point was that this limit of the ratio of these two functions exist when the ratio of this derivative of the. So, this limit here exist otherwise for example, this limit does not exist we cannot claim that the original limiter here of the ratio f x over g x does not exist.

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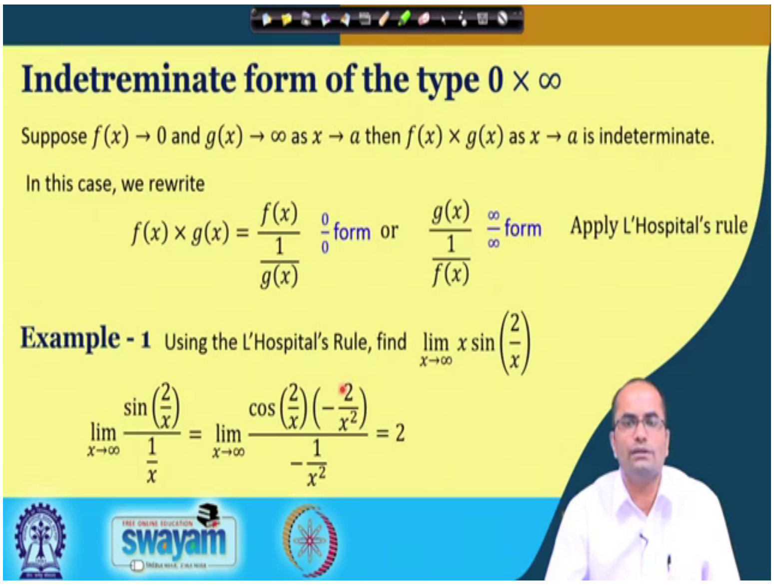

So, the today we will continue with the indeterminate form of the type 0 into infinity. So, we have the situation suppose f x goes to 0 and this g x goes to 0 goes to infinity sorry. As x goes to a then this product of f x into this g x as x goes to is a is indeterminate, because this is precisely 0 and into infinity which we have to evaluate using the L'Hospital rule discussed in the last lecture. So, in this case when we have such a product where one f x this function goes to 0 and the other one goes to infinity.

In that case we can rewrite this expression here f x into g x as f x divided by 1 over g x. So, in this case here f x goes to 0 and then this 1 over g x will also go to 0. So, we have here 0 by 0 form we can also rewrite in this form that we keep this g x in the numerator and in the denominator we take this f x as 1 over f x. So, in this case this g x goes to infinity and 1 over f x since f x goes to 0. So, this term here 1 over f x also goes to infinity. So, we have infinity by infinity form. And in either case we know that we can apply the L'Hospital rule. So, applying the L'Hospital rule we can get the limit because we have already learnt in the last lecture how to deal search forms 0 by 0 or infinity by infinity.

So, let us take a simple example here, we want to apply this L'Hospital rule to this problem x sin 2 over x and x goes to infinity. So, here we have x goes to infinity and sin 2 over x so, 2 over x goes to 0. So, this is infinity into 0 form. So, in this case we will use this idea which is discussed above that we can make either 0 by 0 or infinity by infinity form. So, in this case we can bring this x to the denominator as 1 over x and then sin 2 over x.

So, in this case we have the 0 by 0 form. So, sin 2 x 2 over x goes to 0 and 1 over x also goes to 0. So, in this case we can now apply the L'Hospital rule because we have the 0 by 0 form and then the derivative of the sin 2 x will be the cos 2 x and the derivative of 2 over x will be minus 2 over x square and here also we have minus 1 over x square. So, this is the situation now this 2 over x square and this 1 over x square.

So, the x square terms get cancelled and then we have here cos 2 over x as x goes to infinity. So, x goes to infinity 2 over x goes to 0 and then cos 0 goes to 1 and we have this 2 here so, the limit is 2.

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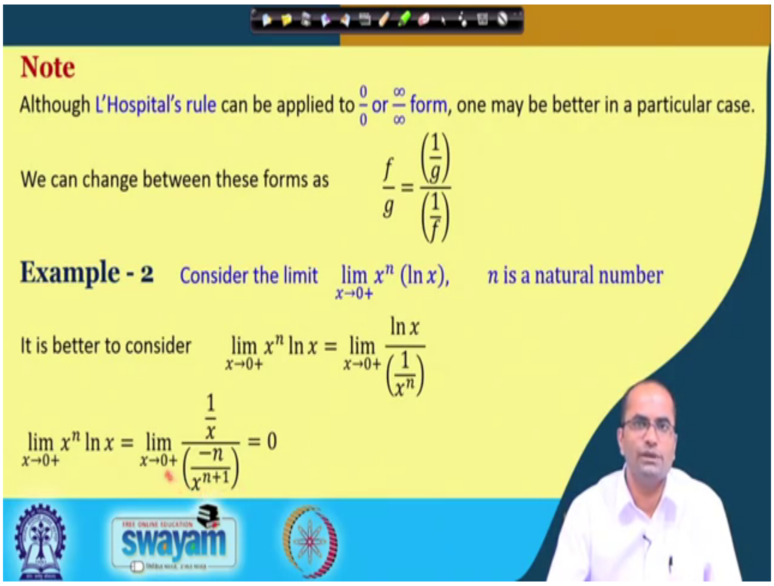

Now, as we have seen in the last slide that although this L'Hospital rule can be apply to 0 by 0 or infinity by infinity form, but 1 may be better in a particular case this we will see with the help of example in a minute.

So, the point is that we can change between these form. So, for example, this f over 0 is of the form 0 by 0 then we can rewrite it as 1 over g divided by 1 over f and in this case if this f and g both goes to 0 here we have the infinity by infinity form. Or other way around if f and g both goes to infinity in that case 1 over g divided by 1 over f we will have form of 0 by 0. So, we can interchange these two forms 0 by 0 and infinity by infinity depending on the convenience of the derivative there. So, for instance we consider this example the limit x goes to 0 from the right side x power n and ln x the natural logarithmic n so, here n is a natural number.

So, in this case it is much convenient to consider this is 0 by 0 form sorry infinity by infinity form. So, we keep this ln x in the numerator and bring this x power n as 1 over x power n in the denominator. So, in this case when x goes to 0 the numerator goes to infinity and also here the denominator goes to infinity.

So, we have infinity by infinity form. And in this case the derivative of this ln x is simply 1 over x and then the derivative of 1 over x power n is minus n over x power n and then we can simplify this. So, we have the limit 0 because here we have the x power n plus 1 and n is a natural number. So, 1 2 3 so, here whatever n is you will get some x power something there and x goes to 0 it will become 0.

But, in the same example if you would have consider x power n and over ln x then the derivative of this 1 over ln x would have created a problem there and it was certainly it would have not been such a simple calculation. So, we have to observe that which form whether 0 by 0 or infinity by infinity is convenient in a particular example.

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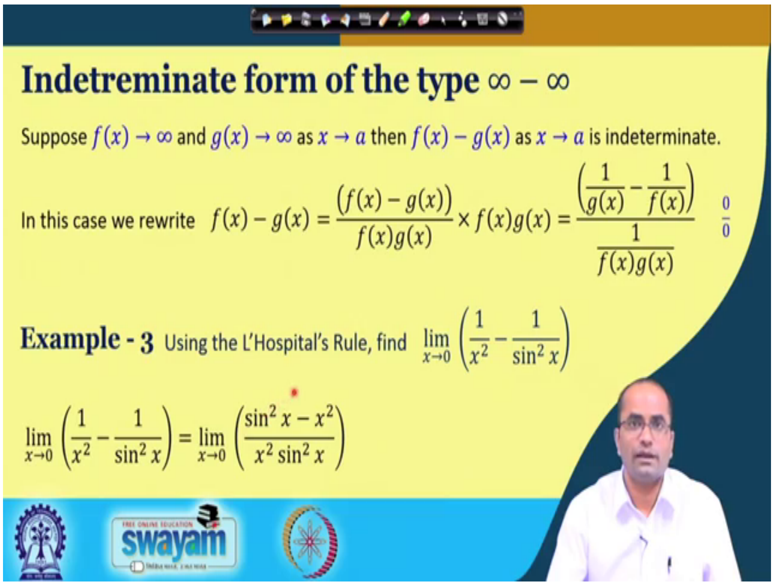

So, now we will discuss the indeterminate form of the type infinity minus infinity. So, suppose f x goes to infinity and g x goes to infinity as x goes to a then this f x minus g x this difference here as x goes to a is indeterminate because, we have infinity minus infinity form. And we have already discussed in the last lecture that this infinity minus infinity form is an indeterminate form and we have to be careful to evaluate such limits.

So, we can again the rewrite this f x minus g x term; if we divide here f x minus g x by f x into g x and multiply by f x into g x and then this term here we have the f x over g x and minus g x over f x and multiplied by this f x over g x. So, in this case we have the situation and then since f x and g x both goes to infinity. So, we have this 0 and then again minus 0. So, 0 here and 1 over infinity into infinity will be infinity it is not an indeterminate form so, 1 over infinity this is 0 again. So, we have here 0 by 0 form which we can easily handle with the help of L'Hospital rule.

So, in this case let us take the simple example again if we have this limit x goes to 0 1 over x square minus 1 over sin square x. So, here since x goes to 0 so, we have this 1 over 0 this which is which is going to infinity and minus again here 1 by 0 it is also going to infinity. So, we have infinity minus infinity form in this problem.

So, as discussed above here we can rewrite this as sin square x and then this minus x square divided by x square sin square x. So, in this case now the sin square x goes to 0 and minus the 0 so, we have the 0 divided by 0. So, this infinity minus infinity form changes to 0 by 0 form which using L'Hospital rule we can get the limit.

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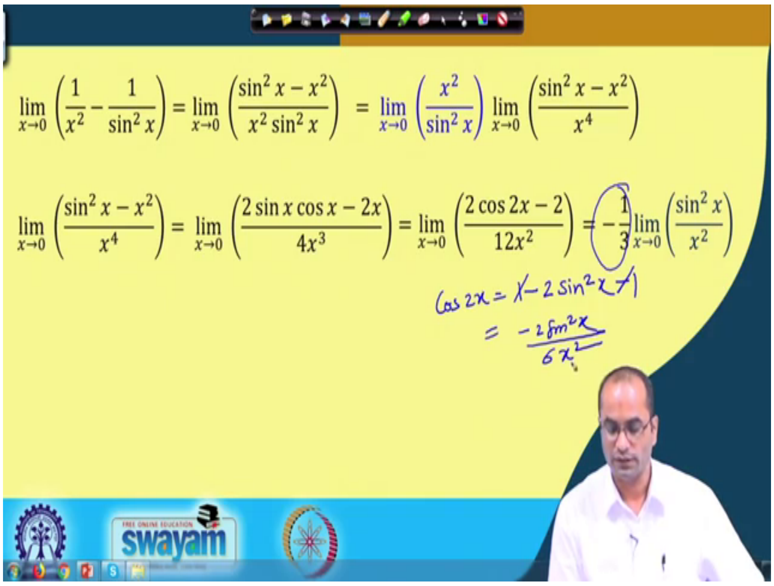

So, here moving further. So, when we take this when before we applying the L'Hospital rule we just rewrite this expression in this form. So, we take 1 x square there and divided by 1 x square here. So, this x square and sin square term together and then the rest here sin square x minus x square there was x square already and 1 we have divided so, we have x power 4. So, this limit sin square x minus x square divided by x square sin square x we have rewritten in this form the limit of x square sin square x and the limit of sin square x minus x square over x 4.

And now let us consider the 2nd one 1st so, sin square x minus x square over x 4. So, in this case we have again this 0 by 0 form. So, sin square x goes to 0 minus 0 and then divided by 0. So, we can apply the L'Hospital rule here which says that the derivative of the numerator will be 2 time sin x and the derivative of sin x will be cos x and minus the derivative of x square will be 2 x divided by the derivative of x power 4 which is four x power 3 and the limit x goes to 0. So, again here we have the sin 2 x 2 sin x cos x is sin 2 x minus 2 x divided by 4 x cube.

So, sin 2 x goes to 0 minus this x goes to 0. So, we again end up with 0 0 0 by 0 form which we have to differentiate again. So, here we have the sin 2 x which will become 2 cos 2 x and minus the derivative of this 2 x will be just 2 and the 4 x cube which will become the 12 x square. So now, here when we check this limit again so, 2 times the cos 2 x and x goes to 0.

So, this is 1 so, 2 minus 2 which is 0 so, again we have 0 by 0 form. But, what we can do now we can simplify a little bit so, which is cos 2 x we can write down as so, let me just workout here. So, this cos 2 x term we can write down as 1 minus 2 sin a square x and then you have minus 1 there. So, the 2 we can take common and we will divide to this 12 here.

So, this is 1 minus 1 we will get cancel and we have minus 2 sin square x and divided by 6 x square. So, this 1 by 3 so, minus 1 by 3 and then here we have the limit x goes to 0 sin square x over x square. And now this limit here is x goes to 0 sin square x over x square is similar to what we have their x square over sin square x. And we will see in a minute there these both limit can be handle in a similar fashion.

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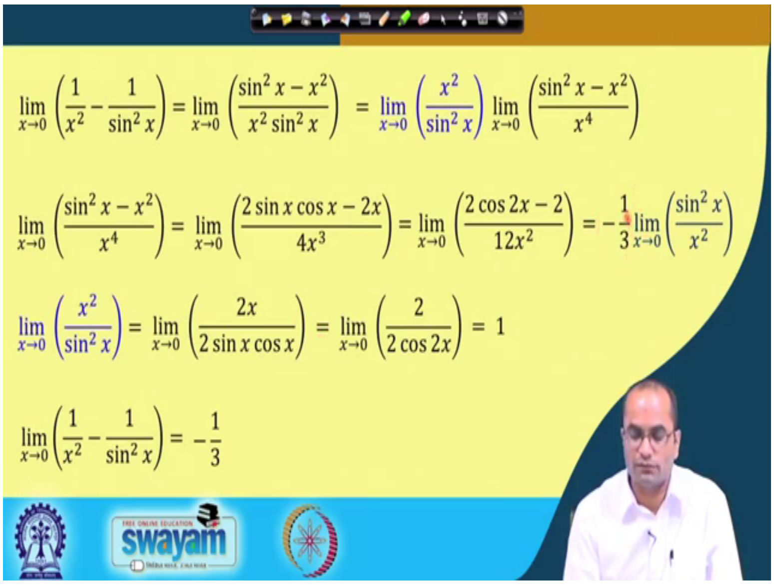

So, what we have let us consider this 1 we could have considered sin square x over x square both will result to the same limit. So, you just consider limit x goes to 0 x square over sin square x. And in this case we can apply again a L'Hospital rule because this is 0 by 0 form and then we have limit here. So, the derivative of x square will become 2 x and then the sin square x will give us 2 sin x cos x. So, here we have 0 by 0 form again and then so, the derivative of the numerator will give us 2 and the derivative of the denominator which is sin 2 x will give us 2 times the cos 2 x.

So, now if we take the limit here x goes to 0 so, the cos 0 is 1 so, 2 by 2 the limit is 1. The same thing if we consider here sin square x over x it will result exactly in the same form and will lead to the limit 1 because, we will have just the reverse the denominator will become numerator and numerator will become denominator and we will end up with limit 1. So, here now getting back to the original limit. So, this limit x goes to 0 sin x square over sin square x is 1 and the limit here for the second term which we have evaluated which was minus 3 and this one becomes 1.

So, the limit there 1 over x square and minus 1 over sin square x is minus 1 by 3. So, minus 1 by 3 this is 1 and this is minus 1 by 3 so, the limit is minus 1 by 3.

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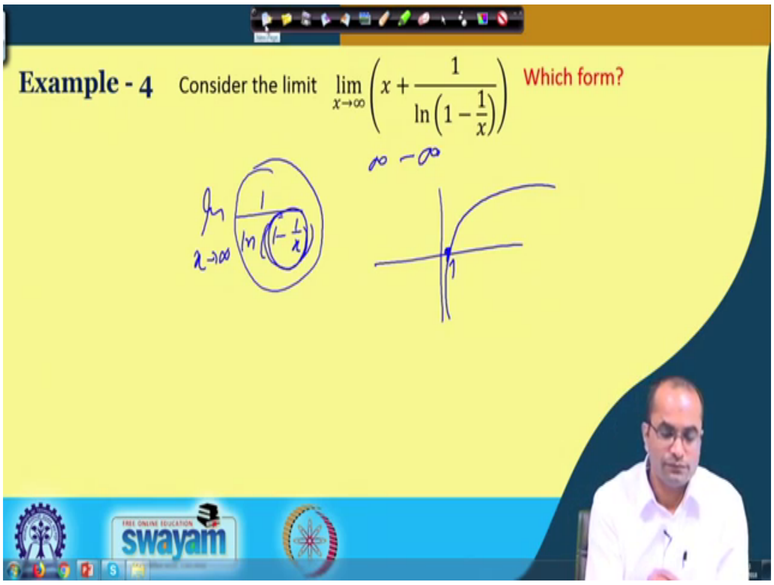

Now, the next example again of this nature is like x goes to minus infinity x plus 1 over logarithmic 1 minus 1 over x.

So, in this case the question is what is the form of this expression when x goes to infinity. So, let us just check so, here x goes to infinity. So, we have the infinity here now this limit 1 over 1 over ln 1 minus 1 over x as x goes to infinity. So, here we have to be careful when x goes to infinity. So, this is 1 minus something this expression here is 1 minus and 1 over x. So, this is less than 1 and the logarithmic here ln.

So, if we just draw the graph so, here it is 1. So, less than 1 the logarithmic take the negative value. So, here in the denominator ln 1 minus 1 over x are taking all negative value. So, when x goes to infinity this is going to 1, but 1 over and so, all these values were negative. So, 1 over 0 this is actually tending to minus infinity. So, what we have there this form is minus infinity minus infinity form.

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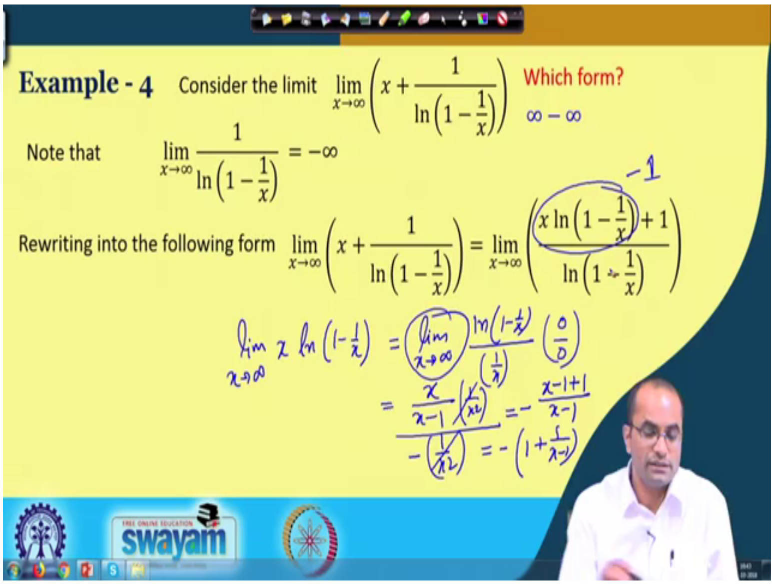

And therefore, we can apply so, again have written down. So, that this term here 1 over ln 1 minus 1 over x is minus infinity so; we have infinity minus infinity form.

And now we will tell exactly as done in the previous example. So, we have x over ln 1 minus 1 over x we can rewrite this x into ln 1 minus 1 over x plus this 1 and divided by ln 1 minus 1 over x. So, we need to check again what is the limit now here. So, when x goes to infinity so, this 1 ln 1 goes to 0. So, here we have 0 now this 1 this x into x into the ln and 1 minus 1 over x as limit x goes to infinity.

So, in this case we have the infinity here and then ln 1 so 0. So, we have to rewrite this as limit x goes to infinity and ln 1 over x 1 minus 1 over x divided by 1 over x. So, now, this is go going to l n 1 that is 0 and here also 0. So, we have 0 by 0 form and we can apply the L'Hospital rule. So, here ln 1 minus 1 over x will become the derivative x over x minus 1 and the derivative of this term which is 1 over x square and then we have here also 1 over x square with minus sign.

So, this gets cancelled and then we have with minus sign. So, x and then minus 1 plus 1 divided by x minus 1. So, which we can write down as 1 plus 1 over x minus 1 and when limit x goes to infinity. So, this goes to 0 and we get this limit as here minus 1. So, in this case we have 1 minus 1 0 and divided by 0. So, this is basically 0 by 0 form.

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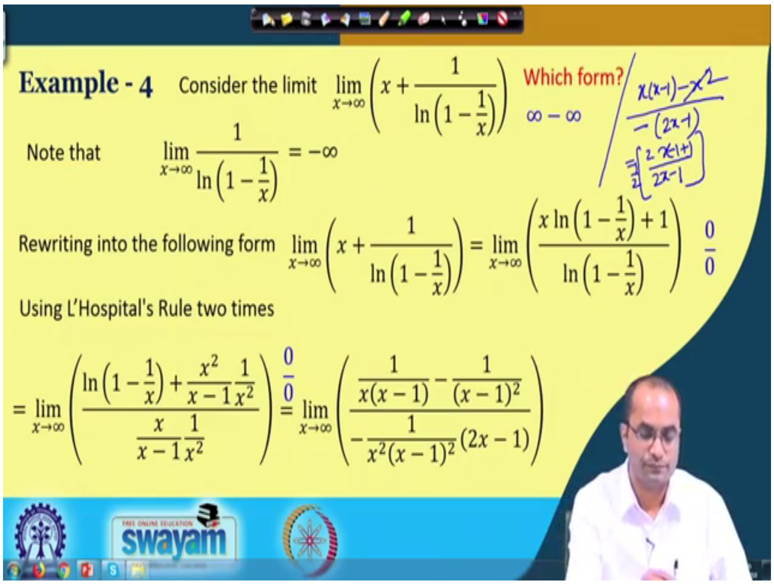

And now we can apply the L'Hospital rule. So, we have to get the derivative of this term and the derivative of this term. So, the derivative of this term will be the product rule will be applicable here. So, derivative of x is 1 and then we have exactly this term ln 1 minus 1 over x then plus x will remain the derivative of this which we have just evaluated before that was x over x minus 1 and 1 over x square and this x makes this x square and again here this is x over x minus 1 and 1 over x square. So, in this case it is a 0 and then here it is again 0.

Similarly, here also 0 so, we have again 0 by 0 form and we need to apply the L'Hospital rule to this expression once again. So, the derivative here is 1 over x minus 1 as done before. So, and the derivative of this term which is 1 over x minus 1 will become minus 1 over x minus 1 whole square here we have 1 over x into x minus 1. So, the derivative will be minus x square x minus 1 whole square and the derivative of this x into x minus 1 which will be 2 x minus 1.

So, the now again we need to simplify this term a little bit and what we will get. So, if multiply here in the numerator by x square and x minus 1. So, what we will get; we will get x into x minus 1 and then minus x square here with the negative sign and then this will become 2 x minus 1. So, this x with x square will get cancel we have this one minus x and this minus will make it plus. So, x over 2 x minus 1 and then again here the 2 so, we can divided by 2 and here minus 1 plus 1. So, we have this 1 plus 1 over 2 x minus 1 and when x goes to infinity. So, it will become 1 and the half is sitting there so, this value will be half.

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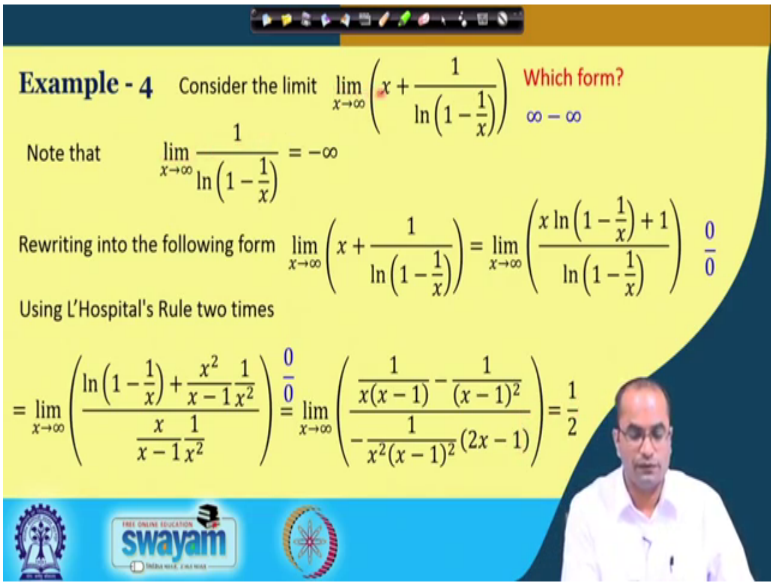

So, this limit of x minus 1 over ln 1 minus x as half.

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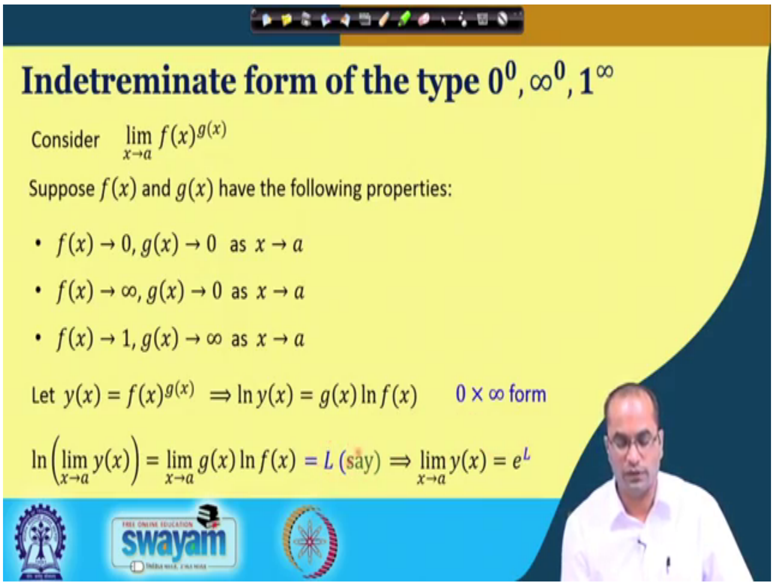

Now, we will move to the indeterminate form of the type 0 power 0 infinity power 0 and 1 power infinity. So, all these type of limits we can handle all together. So, we consider this f x power g x as x goes to a and suppose this f x and g x have the following properties.

So, the 1st one f x goes to 0 and g x goes to 0 so, we have this limit here as 0 power 0. In the 2nd case we will consider that f x goes to 0 and this g x goes to 0. So, in this case we have infinity by infinity power 0 form and in the 3rd case we take as f x goes to 1 so, we have 1 here and this g x goes to infinity so, 1 power infinity the 3rd form. In all these cases we consider a new function here y as f x power g x this 1.

So, f x power g x in either case when we take the limit as ln y x it will become g x ln f x and now consider for example, the 1st case when f x goes to 0. So, this will go to minus infinity and g x goes to 0 so, 0 into infinity form. In the 2nd case when f x goes to infinity. So, this will become infinity and g x goes to 0. So, again 0 into infinity form in the 3rd case when f x goes to 1 when f x goes to 1 and g x goes to infinity. So, infinity and ln 1 will become 0 so, we will have again the 0 into infinity form.

So, in all these cases whether f x goes to 0 g x goes to 0 means 0 power 0 form infinity power 0 form 1 power infinity form in all these cases we will have 0 into infinity form once we take the logarithmic of this expression and then we take the limit; so while taking the limit here this will become ln and since this is a continuous function. So, we can bring this limit to the argument here l limit x goes to a y x and is equal to the limit x goes to a g x into ln f x. And now here as we have already discussed that these are the forms of 0 into infinity form which we can deal as we have done before and suppose that this limit is L. So, once we compute this limit and we can take this algorithm exponential both the sides to get this limit x goes to a y x because this was the desired limit here this was the y x.

So, limit x goes to a y x will become the exponential of this number L. So, what we have seen that all these types of form 0 power 0 infinity power 0 and 1 power infinity they can be handled by taking the logarithmic of this function y which we have assumed here f x power g x.

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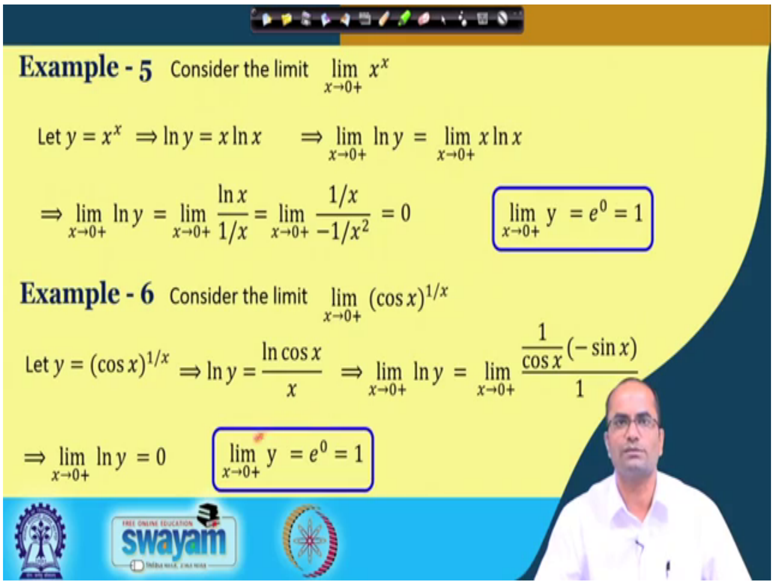

Now, let us take the example here x power x and x goes to 0. So, in this case as discussed before we will take y x is equal to x power x we will assume this x power x as y and then take the logarithmic here. So, log y will become x ln x and in this case. So, we have the limit now of x ln x.

So, we have 0 into infinity form which we have already discuss before so, we will bring into this infinity by infinity form. So, ln x and 1 over x and now apply the L'Hospital rule here. So, ln x will give 1 over x and here minus over 1 square which is simply x here and x goes to 0. So, this will become 0 and then taking the exponential of both the sides. So, we will get the limit y is equal to exponential power 0 here which is 1. Another example if you consider here cos x power 1 over x which is 1 power infinity so, cos x goes to 1 and 1 over x goes to infinity.

So, we have 1 power infinity form and the similar process we assume y as cos x power 1 over x take the logarithmic. So, log y will be ln cos x over x and now this is log 1 and here again goes to 0. So, 0 by 0 form once we take the limit. So, we will apply the L'Hospital rule. So, we will take the derivative here of ln cos x which will be 1 over cos x and this cos x the derivative of cos x will be minus sin x divided by the derivative of 1 which is 1. So, we have here minus 10 x x goes to 0 whether this goes to 0.

So, we have this limit is 0 and then taking this exponential both the sides we will get the limit y which was the desired function here cos x power 1 over x it goes to 1.

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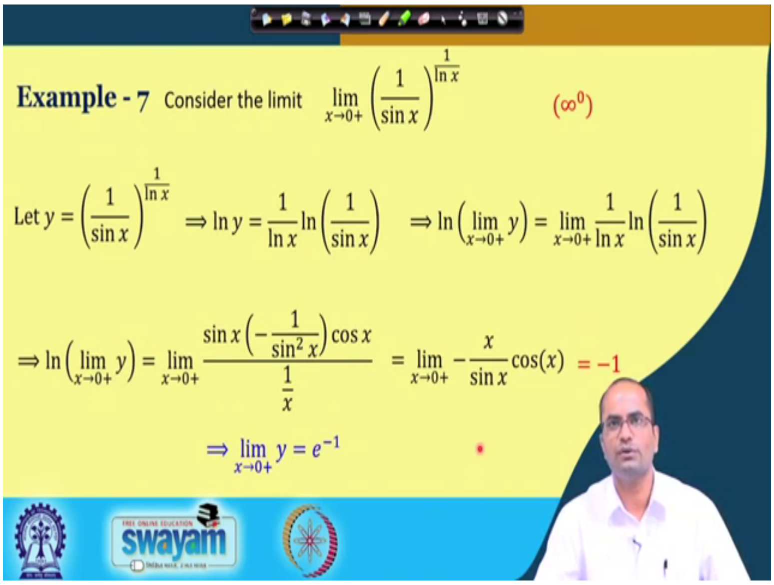

Finally we consider one more example of this limit 1 over sin x and 1 over ln x; x goes to 0. So, we have the infinity power 0 form and again taking the limit here or taking the logarithmic we will get lny is equal to 1 over ln x and ln 1 over sin x. So, here it will become 1 over lnx x goes to 0 so, infinity. So, it is a 0 and then here ln infinity. So, 0 into infinity form and then we can rewrite it as while taking the logarithmic am we taking the limit now.

So, this is 1 over ln x and ln 1 over sin x. So, 1 over sin x is infinity here. So, ln infinity infinity and divided by the ln x. So, this is also going to infinity. So, we have infinity by infinity so, the numerator is ln 1 over sin x and denominator is ln x now we can apply the L'Hospital rule. So, here ln 1 over sin x the derivative of this will be sin x and the derivative of 1 over sin x will be minus 1 over sin square x and the derivative of sin x will be cos x.

So, this is the derivative of the numerator term ln 1 over sin x and then ln x will give 1 over x. So, now, we can simplify this so, we have here the cos x over sin x term and then the x there. So, x cos x over sin x and now we know that this x sin x this limit as x goes to 0 becomes 1 and then cos 0 is also 1. So, we have this minus 1 and then taking the exponential both the sides. So, we will get ln x goes to 0 y as exponential minus 1. So, we have considered at least 1 example of each nature.

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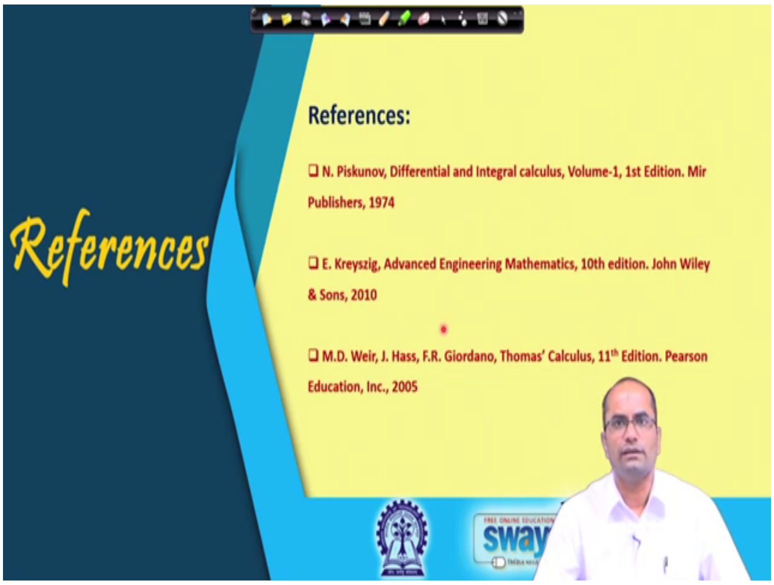

And these are the references used for preparation of this lecture.

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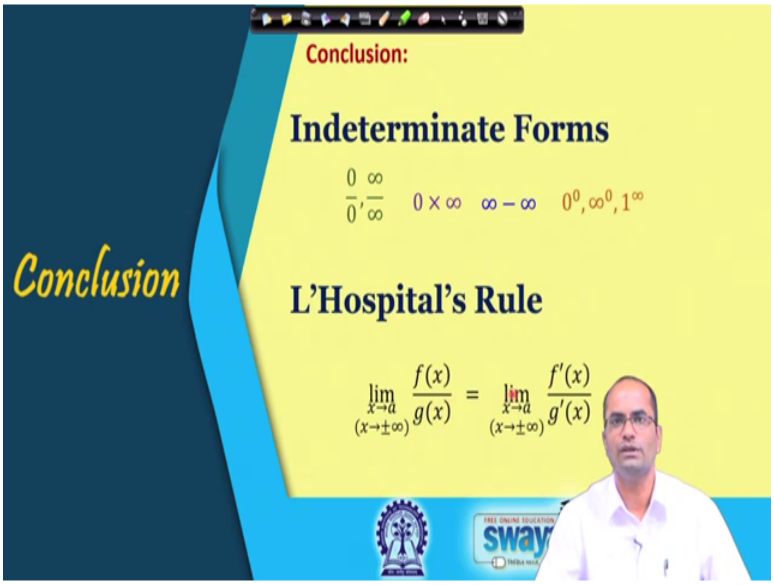

So, what we have learnt today the indeterminate form of 0 into infinity infinity into infinity and these three which were handle using the logarithmic function. In all the cases we have used the L'Hospital rule which says that the limit of the ratio of the two functions when they go to the 0 by 0 or infinity by infinity form will be equal to the limit of the derivatives ratio of the derivatives.

So, that was a very useful rule L'Hospital and we have handled with the help of the single concept all these types of indeterminate form. So, that is the end of the lecture.

Thank you very much.