Engineering Mathematics – I

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Lecture – 39
Linear Algebra – Vector Spaces

So, welcome back and this is lecture number 39 and we will be talking about a very important topic in Linear Algebra that is a Vector Spaces.

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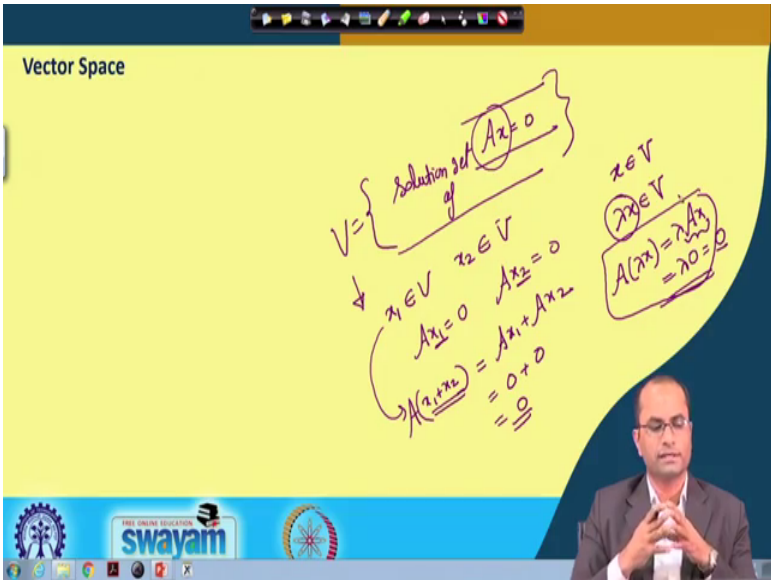

And, just to recall from the previous lecture what we have seen for instance this solution set of the homogeneous system of equations Ax is equal to 0, that is a very special set. So, the solution set of solution set of this homogeneous system of equation Ax is equal to 0. Why that is very special? Because in this set; so, let us call this as the set V which contains all the solutions of this equation Ax is equal to 0, then if we take any two elements of this set let us say x 1 and also we take x 2 from this set; that means, this Ax 1 is 0 and Ax 2 is also 0 because the x 1 this vector is also a solution and x 2 is also solution.

So, if we take two elements of this set and if we add them; so if we consider now x 1 plus x 2 and that will be also the solution of this system of equations Ax is equal to 0, because of this linear property here we have Ax 1 and then Ax 2 that is a matrix vector product and this is 0 and this is 0. So, we have this x 1 plus x 2 is also a solution and not only this if we take any element of this set and if we multiply by a real number, so, let us say lambda. So, this lambda x will also belong to V because this is again a solution if x x is a solution and then this lambda x because A lambda x will be lambda Ax and this Ax is will give 0. So, lambda into 0 and then you will get the zero vector.

So, here what we have seen in this set which is a kind of a special set if we take any two elements of the set and add them that is also an element of the set. And if you multiply by a real number to this element of this set this new element is also an element of this set V. So, we are going into this direction that what do we call these such special sets and that is exactly the vector spaces coming into the picture.

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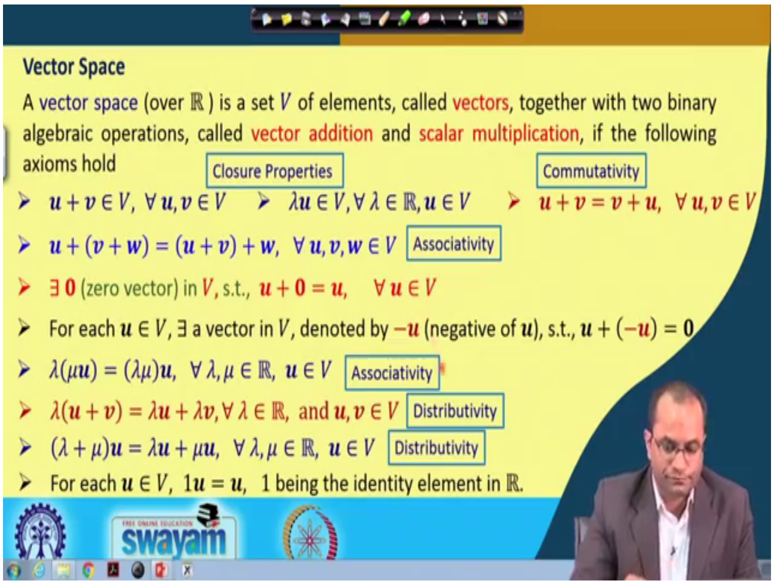

So, here this vector spaces they are the special set here. So, a vector space again here we need to define the vector space over R for instance and this R can be more general like here we have considered this set of real numbers this can be a set of complex numbers and in more general cases what we call the field. So, it can be any field, but we are not going to discuss that into details. So, most of the time we will take this vector space over this R because with this set V 1 another set must be associated which we have taken here this R set of real numbers. So, this vector space is a set V of elements and called vectors.

So, that is another point here I would like to mention that the elements of this vector space V will be called vectors. So, they can be matrices they can be functions depending on what type of elements this V contain, but the elements of this V we will call vector, so, a little more general definition here and together with two binary operations. So, we need other than this set here we which we will call vector space we need this set R which we have taken and then two binary operations which we call this vector addition and this is scalar multiplication.

So, what do we have? We have this set of V, we have another set which we have taken here the set of real numbers and then we will have this vector addition which we have to define with this set V and we have scalar multiplication. So, we need to define these four four things to define this vector space. So, the mean set V here one another set of real numbers or the set of complex number and two algebraic operations which we call vector addition and scalar multiplication.

So, having all these here; now when do we call this set V a vector space when the following axioms hold. So, what are these axioms? The first one is that if we take two elements u and v from this set V and if we add them the new elements should also belong to this set V. So, that is the additive closure property of this set that we take any two elements and do this vector addition which has to be defined for this given set V then this addition of this u and v must belong to the set V.

Another property that this lambda times u, so, the lambda which is called is scalar because lambda belongs to this set of real numbers or little more general this set of complex numbers we can talk about, but we are restricting to the set of real numbers here. So, this lambda times u that is the scalar multiplication. So, here this scalar lambda is multiplied to this vector u. So, this lambda u must also belong to V that is another closure property which we call with respect to this is scalar multiplication. So, for any u, the lambda u must also be there in the set for V and these are called the closure properties of the set.

And, there are many more properties which are not so important in our discussion, but we need to just state them for completeness because most of our examples all these properties will trivially a follow, but these two properties are the most important properties which we call the closure properties. So, here the third one that u plus v must be equal to u v plus u, for any element of this set V and this is the commutativity property of this these vectors.

The another one that here u plus first we add v plus w or first we add this u plus v and then w the result must be the same and this is called the associativity property. Again, so, we have this existence of the 0 vector. So, what how do we define the zero vector. So, there must be a zero vector in this set V which we are concerned now. So, here this 0 must be there and what is the property of 0, that if we add this 0 to any element of this set V then that element will remain as it is. So, this is called the zero vector and this must be there in the set V.

And, another one; so, for each u so, for each element of this set V; there must exist a vector V which is denoted by minus u. So, this is just a notation here minus u the negative of u such that when we add this u and this negative of u we must get the zero vector which already exists there in the set. The another one again this is a property with respect to this scalar multiplication that we are multiplying here mu with u or we first multiply lambda and mu in this set of real numbers and then we multiply to this u the result must be same for all lambda and mu from the set of real numbers and this u from the set V and this is again this associativity property with respect to this scalar multiplication.

And, further we have this lambda times v plus u must be equal to lambda u plus lambda v and for all lambda and all u v from this set V. This is called the distributive property of this of these scalars and the vectors. And, then we have this lambda plus mu if you multiply to u, then this should be equal to lambda u plus plus mu times u and this should hold for all lambda mu from r and u from V. So, this is again this distributivity property of these scalars and vectors.

And, the last ones that for each u from this set V we should have that 1 into u; 1 is the identity element in R, so, which is the number R in case of this real number set of real numbers. So, this if you multiply u to this 1. So, this is again here the scalar multiplication which must be defined for each this set here and it must be equal to u and this is again kind of a property which all the elements of this u must satisfy.

So, we have so many properties, but as I said at the beginning that these two properties are most important here; these are the closure properties that u plus v must be there in the set and lambda times u must be there in the set, for all u and for all lambda from R. So, with these two properties and there are other properties as well which we have to keep in mind like the existence of the zero vector, existence of this negativity and all other these distributivity property associativity property etcetera must hold for this set V then we call this set V as a vector space.

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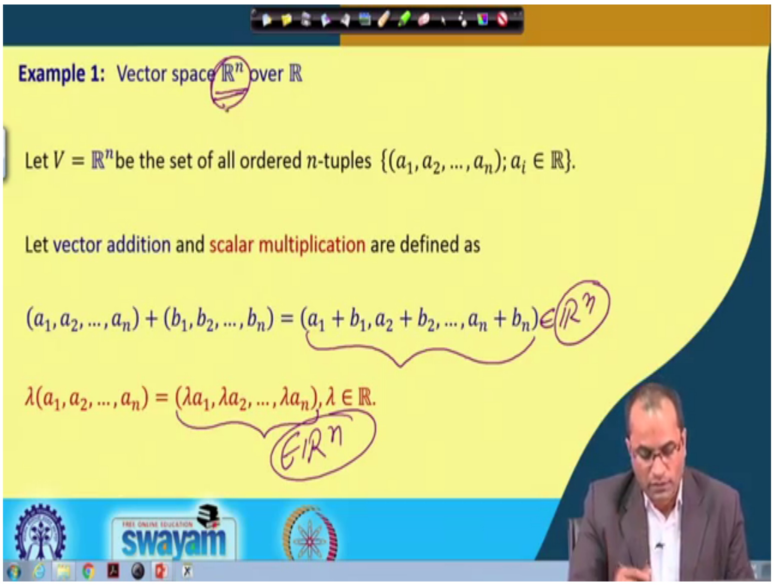

So, now we go through some of the examples where we will understand now better what is this vector space. So, here the first example we are considering that is the vector space R n over R. So, what is the R n? How the elements of this R n looks like? So, this R n is this set of all this ordered n-tuples means we have the elements of this type a 1, a 2, a 3, a n and all these as are from the real number. So, this R and the elements of R n will have these n-elements the n-components in each element of this of this set.

And, here we need to define again this vector addition and scalar multiplication then only we will say that this is a vector space with respect to this scalar multiplication, this vector addition and this set of real number R which we are considering. So, here the vector addition and this is scalar multiplication is defined as usual. So, when we take these two elements here a 1, a 2, a n and b 1, b 2, b n from this set V and when we add them, so, the new element will be just the component wise addition here.

So, first a 1 plus b 1 will be the first element a first component of this new element here a 2 plus b 2 and so on, a n plus this b n. So, that is the how the addition works in this particular case. And, for the multiplication the scalar multiplication so, when we multiply by this scalar lambda to this a 1, a 2, a 3, a n in that case we should get here now the new element will be just the multiplication of this lambda to each of the element of this element or this member of this set V and for all lambda from R.

So, this is how these vector addition and the scalar multiplication is defined for this set and what we can easily now observe those two closure properties at least because all others are trivial in this case again like for instance the zero element will be when all these components are zero and so on, the negative elements will be when we put the minus sign in front of each so, that will be the negative element of a given element. So, all those existence all this commutativity property etcetera one can easily verify.

So, again what is also important the closure properties which are again trivial here. So, when we add two elements here we are getting a new element and that also belongs to this R n because this is again a element of element in this R n. So, this closure property with respect to vector addition is satisfied and also for the scalar multiplication. So, this new element here lambda a 1, lambda a 2 and so on lambda a n this also belongs to this set here R n.

So, the both the closure properties are satisfied, all other properties one can easily check that they are also satisfied. So, this R n this set here R n is a vector space and we can talk about and we can take any integers here. So, the R over R is a vector space R 2 over R is a vector space etcetera it is it is clear now, whatever and we take your R 3 for instance it is also a vector space over R.

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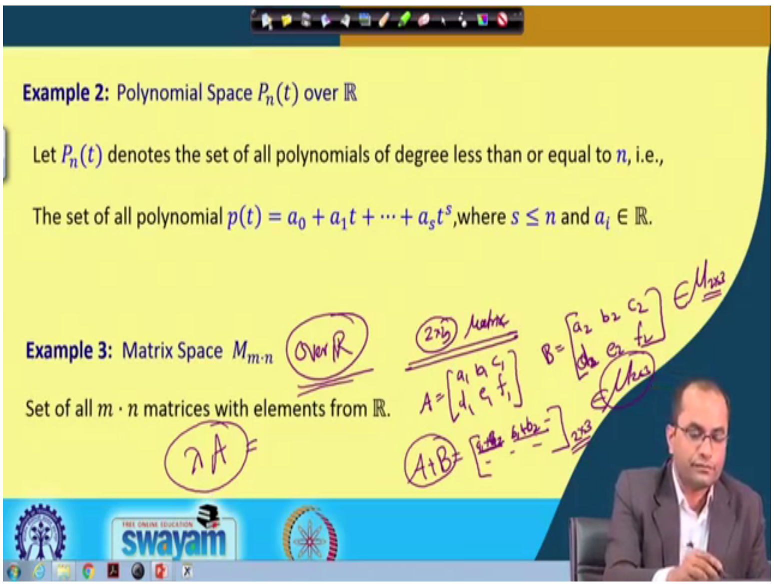

Well, so, now going to the next example here we will consider the polynomial space. So, here we are considering this P n t again over R as I said we will be restricting to a set of real number here. So, this P n t; so, how the elements of P n t look like? So, P n t denotes the set of all polynomials of degree less than or equal to n. So, all polynomial this is a set of all polynomials of degree less than or equal to n and how these polynomials look like they are of this kind a 0 plus a 1 t plus a 2 t plus and so on a s t power s and this s is less than or equal to n because we are talking about the set of all such polynomials.

So, here the constants also belong to this, the linear polynomial belongs to this, quadratic at so and so on depending on this n here. So, this is a set of all polynomials of degree less than or equal to n for given n and all these a's here are just the real numbers.

So, in this case when we have polynomials now so, our vectors are these polynomials and now, again if we look at the closer properties here, but the closure properties are satisfied because if we take any two elements from this set. So, for example, we have taken like one element here which I am calling p 1 t and which we can denote again here this with a's.

So, this is one element so, let me take this n. So, t power n this is one element of this set here the polynomial space which you are talking about. Now let me take the another ones which I can denote by this b 0. So, b 1 t and here the degree can be n or it can be less than n, so, b and t n. So, these are the two elements from the same set and when we add these two so, p 1 t plus this p 2 t how the addition works in this set it is just we have to add the corresponding coefficients here.

So, a 0 plus b 0 these were the constant terms and with the t we have this a 1 plus this b 1 with this t square we will have this a 2 plus b 2 and with this t n we will have a n plus b n. So, this is the new polynomial now after this addition, but naturally when a 0 and this b 0 R in R the sum is also n R similarly this a 1 b 1 will be in R and a n plus b n will be also in R.

So, here this is a new polynomial which belongs to again this set this P n t. So, how this is I mean now one can see this closure property with respect to the addition here. Similarly when we take this p t from this from this set P n t and if you multiply by any scalar here the lambda times this p t. So, what will happen? The lambda will be multiplied to this a 0, lambda will be multiplied to a 1, the lambda will be multiplied to this to this a s here and this new polynomial will be again an element of this P n t.

So, again we have the closure property with respect to this scalar multiplication, we have the closure property with respect to the addition and this addition is defined in this way which we have explained the multiplication is defined in this way for this set. So, this polynomial space P n t over R is a vector space which we have seen in this example.

The next one we are talking about the matrix space; here the M the set of all matrices of order this m cross n we are talking about. So, why in this case it is a matrix space again this is a vector space because if we take two matrices of what are m and n. So, let us consider this 2 by 3 matrices for instance just for simplicity.

So, for 2 by 3 matrices so, we have taken one element like a b c and then the d e and f this is the one element we have taken from this set. We have taken another element of this set which we are denoting by a 2 b 2 c 2 and a 3 d 3 d 2 and e 2 and f 2 these are the two elements for example, from this set here M 2 by 3 are all matrices of order a 2 by 3.

So, when we add these two matrix a and b and we can take any two matrices the result would be again this matrix of order 2 by 3 and we will be just adding these components here a 1 plus b 2 this b 1 a 1 plus a 2 b 1 plus b 2 and so on. So, this will be again a matrix of order 2 by 3 and this is also over this R set of real numbers we are talking about. So, here these are all real numbers when we add them they would be also real number and the new matrix will also belongs to this set here m 2 by 3.

And, also the scalar multiplication; so, when we multiply any member of this set here by lambda the new matrix will be just multiplied by lambda each component will be multiplied by lambda and again, this new matrix will be also a member of this set here M 2 3. So, here what we have seen that this matrix spaces of order this m cross n is also a vector space.

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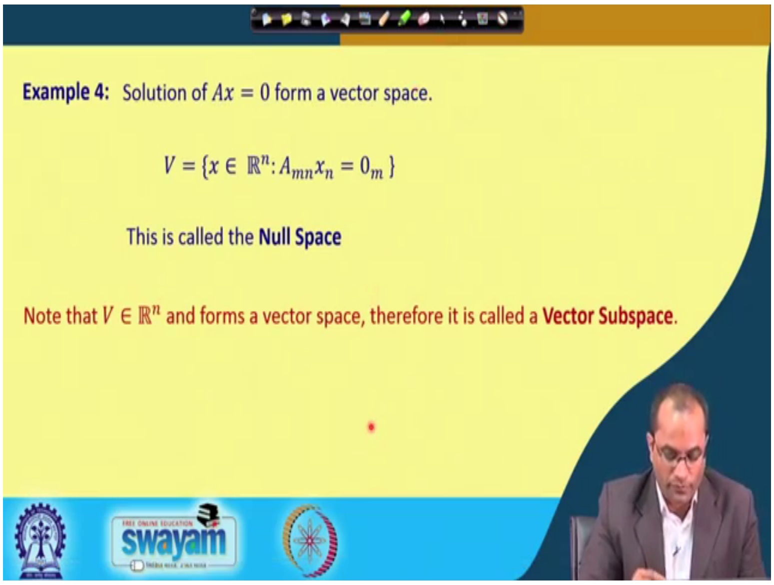

So, another example which is also a very important example from the system of linear equations; so, the solution of this Ax is equal to 0 this form a vector space and how it forms a vector space? So, we take this set V here x belongs to this R n and it satisfy this Ax is equal to 0; we have discussed at the very beginning of today's lecture about this set that the special set only.

So, this is called null space and has discussed before because here also those closure properties are satisfied and the elements are from R n; R n is a vector space already we have seen. So, this elements of this V belongs to R n and R n is a vector space. So, naturally all the properties which we discussed, they are satisfied automatically about this distributivity, commutativity all are valid here because these elements actually belong to R n; R n is a vector space.

And, about the closure property; so, whenever we have taken the two elements here x 1 and x 2 from this set a set V. So, naturally this the sum x 1 plus x 2 that is the property of this solution here that this a this x 1, x 2 will be also the solution because A times x 1 plus x 2 will be 0 and also A times the lambda x 1 or lambda x 2 with any element you can multiply by lambda that will be also 0.

So, that is special for this homogeneous system when we have the right hand side 0, if we do not have this right hand side 0; if it is a non homogeneous system we do not have this property these two properties for example, valid for the solution of non-homogeneous system of equations. So, for the homogeneous system of equations the solution set is a vector space and this vector space has a special name which we call as null space that is what this space coming into the picture. So, this is called the null space of the matrix A.

Note that this V the set V is a subset of is a subset of this R n and forms a vector space and therefore, this is called also vector subspace. So, we will be talking about that soon that what are these vector subspaces. So, this is for instance vector subspaces because this is a vector space and this is a subset of another vector space R n and therefore, we call this as a vector subspace.

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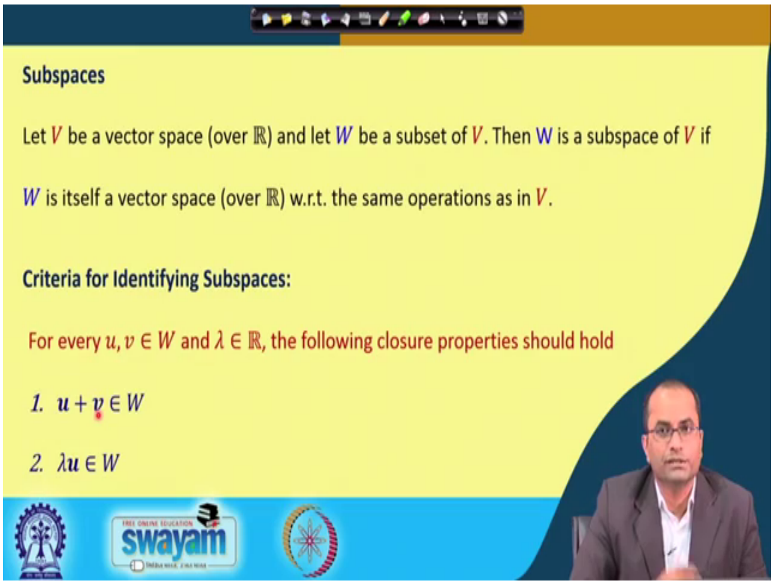

So, here again this what are the subspaces here. So, let V be a vector space and let W be a subset of V. So, V is a vector space and W is another subset of V and if V W is also a sub also a vector space in that case we call that W is a subspace. If this W itself is a vector space over the same the set of these real numbers with respect to the same operations what we are done in the vector space V then this W is called a vector subspace.

The criteria for identifying the subspaces is much easier now and we do not have to worry about all the properties which we have discussed for defining the vector space and here we would be talking about only the closure properties because those are important now all others will be trivially satisfied. So, basically when we check whether a given subset is a vector space or not what we have to check we have to check these closure properties. We take the two elements any two elements the sum the addition of these two must belong to this subset and also the lambda u must belong to this subset and that is enough to check that the given space given vector space is a is a subspace.

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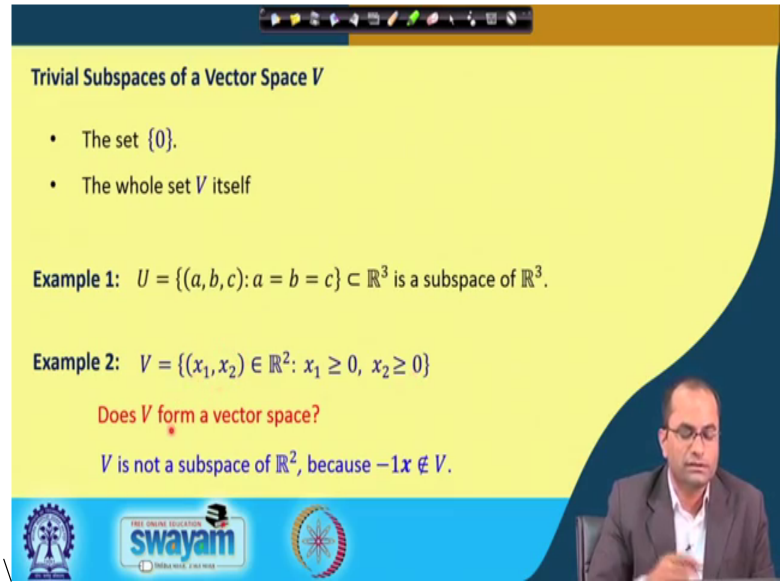

So, what are the trivial subspaces? So, here is a examples now of the subspaces here the set 0 itself. So, if we take just the 0 element of a vector space that will form also a vector space because of the closure properties. For example, you take the element the 0 and add to the 0 you will get 0 and we multiply by any scalar to the 0 the element will remain as a 0. So, this is a vector space because all these closure properties are satisfied and the whole set V itself we can call as a vector subspace of the vector space V.

Another example here when we take this set U a, b, c and all these a, b, c are equal. So, basically we have taken the elements when all these components are equal and this naturally the elements belong to this R 3; R 3 is a is a vector space and this is also a subspace of R 3 the reason is again clear if we take two elements from this set U. So, here this belongs to U because all three are equal and if we take b this is also an element from this U.

And, now this closure properties we need to basically focus on if we add the 2 here. So, what we will get a plus b and a plus b and a plus b. So, this new element here all the components are equal again this will also belong to the same set U or we multiply by the lambda to any element here, the new element will be also lambda a, lambda a. So, all three components are equal and that that is the reason it must belong to this U again. So, these closer properties are satisfied for this set U which is a subspace of now of R 3.

Example 2: So, if we take here the x 1, x 2 and x 1 is greater than equal to 0, x 2 is greater than equal to 0. So, here we are talking about this positive so, both the components are non-negative here; x 1 is greater than 0 here also this greater than 0. And the question is whether this V forms a vector space? And the answer is no, because if we take one element here and the closure property says that the lambda times this we must be there and if we take lambda minus 1, for example, so, the minus 1 into the this element from V which are having this non-negative components, but when we multiply with the minus sign it will become minus x 1 minus x 2 and this will not belongs to this set V.

So, here this closure property is not satisfied like here I have stated again the minus 1 when you multiply to any element here that will not belong to V because the V has the property that both are non negative. So, this is one example where we can see that this does not form a vector space though here the; this is a subset of this R square.

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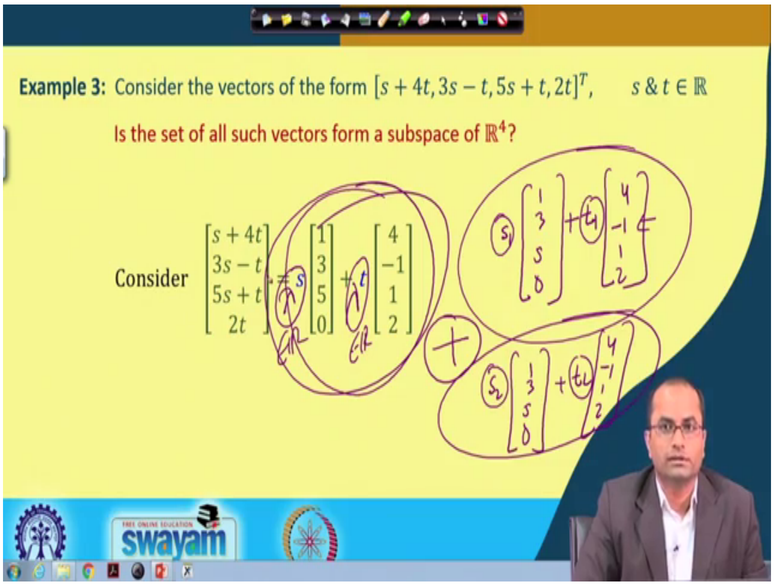

The last example here for the subspace we consider the vectors of this form s plus 4t, 3s minus t and 5s plus t 2t this type of vectors and this s and t they can take any real number. So, now, the question is whether this set the set of these vectors form a subspace in R 4. So, R 4 because there are four components here of this vector so, they are they belongs to R 4. Now, the question is whether this is a subspace of the R 4 or not?

So, here now this it is easy to consider now in this way. So, our vector was s plus 4t 3s minus t 5s plus t and 2t which we have written in this in this format here 2s here 1, 3 and 5 the 0 in the last component and t is the 4 minus 1 1 and 2 here.

So, in this special form we have written and now, one can check easily the closure properties. So, if we take any two elements off of this of this set here that means in the two elements these numbers will change. So, in one we have taken for example one element is 1, 3, 5, 0 and the; I mean with some other other t here. So, this is one element of this set because the elements are of this type in this set.

The another one we can take with some other number 1, 3, 5, 0 with s 2 and this t 2 here this 4, minus 1, 1, 2. So, this is another element of the set and when we add these two elements when we add these two elements so, what will be added here s 1 plus s 2 though this will be again another real number. Here this will be added as t 1 plus t 2 into this. So, this new vector will be also a vector of this set which we are talking about and to any member of this set if you multiply by lambda so, it will be lambda s here will be lambda t and again these are the real number. So, this, the new vector, will also belong to the same set.

So, the both the closure properties are satisfied for the set and hence this will also form a vector space of R 4. So, this is a vector space of R 4 what is left within in the following lectures we will learn that this is a smaller set naturally of this R 4 and now how to how to quantify this in terms of what in terms of kind of dimension or some other criteria; so, that we will be also talking about in following lectures.

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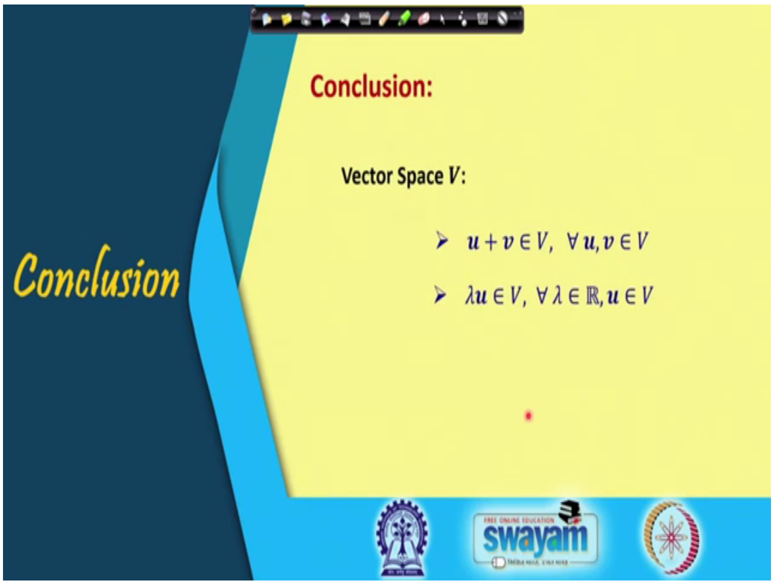

And, now coming to the conclusion, so, for the vector space the most important properties were these additive closer and this the scalar closer here. So, u plus v must belongs to V and lambda u must belong to V; these were the closure properties the most important properties of the vector spaces and we have seen several examples including the space vector spaces vector subspaces; so, these are the references used for preparing the lectures

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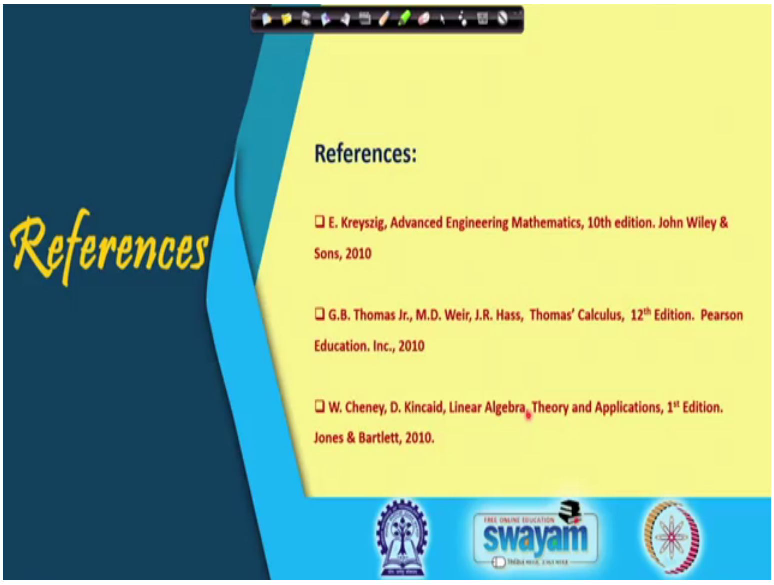

And thank you for your attention.