Engineering Mathematics – I
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Lecture – 38
System of Linear Equations – Gauss Elimination (Contd.)

Welcome, back. So, this is lecture number 38, on System of Linear Equations and we will continue our discussion on this Gauss Elimination technique which was introduced in previous lecture. So, today we will go for more general case where we will see these echelon form, a very important reduced form where we can identify about the consistency of the solution or the system of equations.

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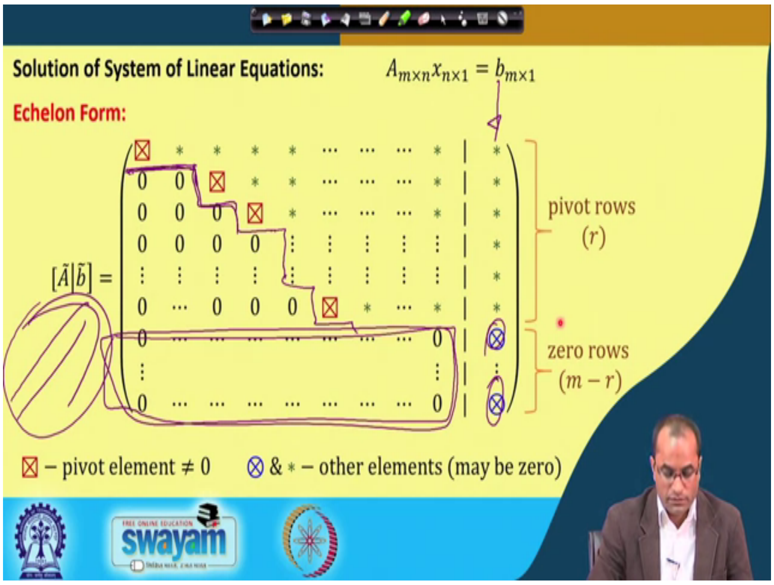

So, let us just go back to this what we have done in the previous lecture with the help of simple example now we will consider a rather general example where we have matrix A which has m rows and n columns and then we have this x n by 1 and the right hand side vector of order this m cross 1.

So, the echelon form what is this echelon form which we have introduced already in the previous lecture in general this will have this form. So, what exactly this let us discuss here. So, these elements denoted by this symbol here. These are the; these are the pivot the pivotal elements and these are nonzero elements. So, what is the property of this pivot element?

So, this number we will call pivot in this matrix when everything below this is 0 and also the left this is the first element. So, we will not be talking about the left to this one for instance this one here this is the pivot element because everything to the left all the elements are 0 and here also all the elements are 0 and in this sub matrix everything is 0; so this is we call the pivot element.

Again, in this reduced form in this echelon form which we call this matrix will be called or this element will be called a pivot because everything here is 0 everything here is 0 and in this sub matrix everything is 0 here. Similarly, this one is a pivot if it is a nonzero element and this everything to this one is 0 and everything to this one is 0. So, that is that the that is the property of the pivot it is a nonzero number and all the elements below these are 0 all the elements left to this are 0.

And, we have this special structure which of the matrix which we call the echelon form; we have these the first few rows are the nonzero rows this non nonzero rows and the last few rows here which we see are the 0s rows; so, this here 0s. So, all these rows on the bottom we have set these with 0s here the right hand side this symbol can. So, this can be 0 these elements or they may not be 0 that is the symbol we are using here just to represent this echelon form in general.

Here we have these pivot elements which are always nonzero elements and, now what exactly the same identification. So, here these are the pivot elements and they are nonzero and with this symbol or this star here these are the other elements they may be 0 and they may not be 0. But, what is important here to put into this echelon form we have bring into this structure which this matrix has.

So, here then this is step like structure and we need to this is the aim to put into this form so that we have these 0s on the bottom of this of this structure here this stair like structure. And if there are the zero rows they are they are taken to this bottom of this matrix and then from this augmented part which was correspond to this b here these elements may be 0 and may not be 0.

So, once we reduce and we will see in one of the examples a little more general example how to exactly get this echelon form. So, if you remember there this Gauss elimination was basically having three steps – the first one putting your system into this augmented matrix then reducing the augmented matrix exactly into this echelon form which we call and then we need to identify these pivot elements. The pivot element are the important elements of this matrix.

And, then we can characterize about the solution also whether we are going to have a unique solution or the solution does not exist or we have infinitely many solutions based on once we identify these pivot elements we can characterize the system in the form of this consistency. So, this putting into echelon form that is the most I mean the difficult part here. So, once we have the echelon form getting the solution from the echelon form was just through this back substitution which we have observed in simple examples in previous lecture.

So, we have these so called the pivot rows. So, here we have taken these pivot rows to the to the up and then if something is 0 here in our matrix A that we have brought down to this bottom of the matrix. So, these are the pivot rows because the each rows has a pivot here now, each rows has a pivot. These are the pivotal row pivot rows and the rest here these are called the zero rows.

So, if these are the r in number which usually the notation we use for pivot rows. So, if these are the r rows and we have total m rows, so, naturally these zero rows will be m minus r this is the structure which we will be using.

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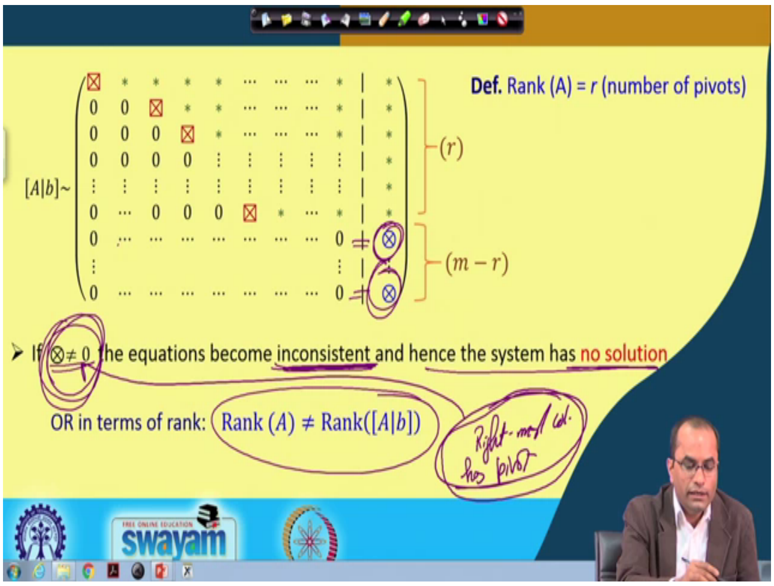

So, having this let me just introduce here one definition though I can you will be talking more about this rank because this is not the only definition for rank. There are many other definitions for this rank, but this is one of them that this rank of a matrix is defined as the number of the pivots because later on you will observe in an alternate definition that will again lead to this that rank A is equal to precisely this number of pinots and that is the reason this number of pivots or the identification of these pivots a pivot elements are very important.

So, here the rank of the matrix another very important concept of a matrix is defined as this number r which is the number of the pivots. So, with this definition, we go ahead and later on we will be talking more about this rank. So, if we say that the rank A is equal to this number r the number of pivots in our this augmented matrix then what we can observe from this augmented matrix we can identify so many things.

The one here is that if these elements here if these elements yeah if these elements here in this zero rows; if they are not equal to 0 and if they are not equal to 0 then the equations become inconsistent and why inconsistent because we have the situation that 0 is equal to something nonzero which we have already discussed in the previous lecture.

So, if in our echelon form we got these as nonzero number, then the system is inconsistent and meaning that the system has no solution. So, that is the first identification we are going to have from this echelon form or in terms of the rank we can define now because we have defined already the rank of the matrix by this number r.

So, what in terms of the r we will be talking always about now the this is corresponding to the matrix A this first part of this augmented matrix and the second one here this column was this b and the whole matrix we are calling this A b. So, this augmented matrix. So, the point here is now the rank in this situation what is the rank of A? Rank of A will be from this matrix how many pivots are there? There are r pivots; so, the rank of a will be r.

But, if these are the nonzero elements if these are the nonzero elements then there will be also a pivot here. So, we can just reduce it further to make all these 0 and they will be nonzero number here at least in one of the equations. The system is inconsistent, that is clear whenever we have one of them is a nonzero number here corresponding to this zero rows then the system is inconsistent, but in terms of the rank what we can call that the rank of A matrix which is this one here the rank of A matrix this one it is this r, the number of pivot elements.

And, it is not equal to the rank of the whole this augmented matrix because in this situation what will happen when these are the nonzero numbers then this rank of this whole matrix; that means, the number of pivots in the whole matrix will be equal to I will be more than the rank of A and in that case then we have the inconsistency.

So, if the rank of these two matrices rank of A and rank of this augmented matrix these are different then we have inconsistency of the system and this will exactly happen when we have these here corresponding to zero rows we have something nonzero, then there will be a pivot element here in this column as well in the last column or in other words we call that we have the pivot in the last column and this is exactly the case of this inconsistency.

So, in many ways we can discuss this the one was already here that because this is not equal to 0. So, inconsistent or we call this rightmost column has a pivot element and this will exactly happen when we have this here nonzero or sitting in one of one of these number is nonzero. So, we have either the rightmost pivot has rightmost column has a pivot or we call rank a is not equal to the rank of the augmented matrix or we call simply by looking at this that if this is nonzero then we have a case of no solution, clear.

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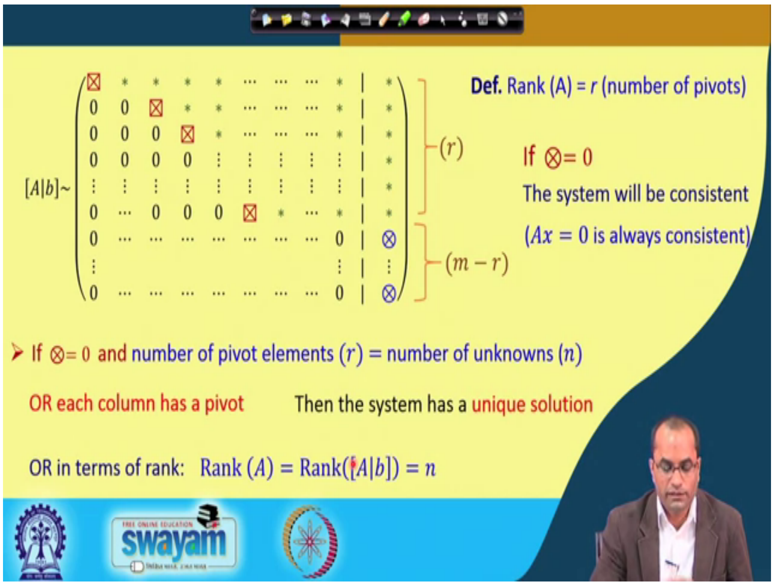

So, now moving further the next possibility will be when this is 0. So, if these elements are 0 here then we have that the system is consistent because there is nothing like 0 is equal to nonzero in that case. So, if these are 0 then we have two possibilities and you have one more point to be noted because Ax is equal to 0 is always consistent that is the inclusion coming from here because when we have this b is 0, so, this right this column everything is 0.

So, naturally these numbers will be 0 everything is 0 and we are doing only the row operations from the beginning. So, nothing will change. So, they will be definitely 0 when we have Ax is equal to 0. So, in that case the system is always consistent meaning this Ax is equal to 0 will have always a solution.

So, anyway so, we will be talking little more later and now here when we take this case that these are zeros meaning the system is consistent then there are two cases the first here the number of pivot elements is equal to the number of unknowns and this case also we have seen in the last lecture and this is the case when we have exactly or in other words that each column has a pivot because if the column columns are exactly the number of unknowns.

So, if each column has a pivot then the system has a unique solution because the column cannot have more than one pivot that because of this property that below this everything should be 0. So, the each column can have only one pivot and this number of pivot equal number unknowns meaning that each column has a pivot because a number of columns equal to the number of unknowns.

So, in that case the system will have unique solution we will discuss all these with the help of example today itself. So, in terms of the rank now if we talk about the rank of the A and rank of the A b, so, they will be the same now and that is equal to this r or equal to this n. So, that is the case of in terms of the rank which we will later on discuss more in details.

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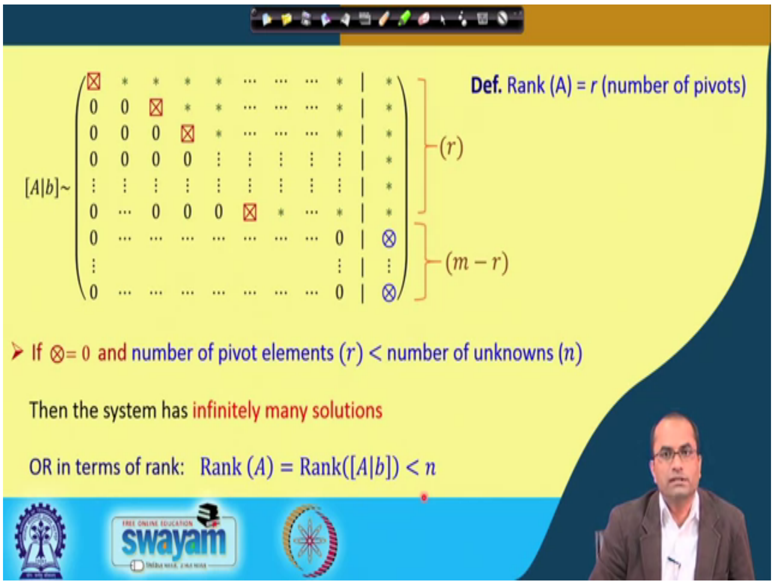

And, now coming to the another possibility that if this is 0 means we have the consistency and the number of pivot elements is less than the number of unknowns.

So, in this case what will happen, the system will have infinitely many solutions when this is the case and there we will introduce the concept of this free variables and dependent variables and then we can get the solution as well I mean some we can generate those infinitely many solutions or in terms of the rank what we will say again the rank of this A is equal to rank of b for the consistency, but this rank is less than equal to the number of unknowns. This is what we talked about this consistency in terms of the rank as well.

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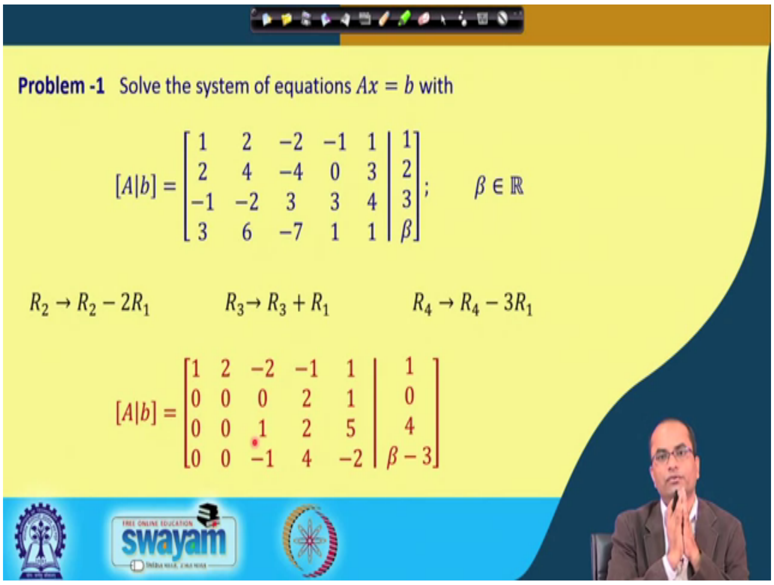

So, coming to the problem here we will discuss now this solution of this system Ax is equal to b where the augmented matrix corresponding to the system is written as here. So, we have how many do we have 4 rows and we have 5 columns. So, this is a case where we have 5 unknowns actually. So, the number of columns means the number of unknowns because here this is A and we have our system this Ax is equal to is equal to b.

So, this x here must be to make it consistent with this a will have like x 1, x 2, x 3, x 4 and x 5. So, there are 5 unknowns and therefore, these 5 columns each column this will correspond to x 1, this is x 2, this is x 3, this is x 4, this is x 5. So, number of columns always represent the I mean of A here yeah not the whole augmented matrix this is corresponding to a this is corresponding to b, that is what we have here.

So, the number of the columns in this a will represent the number of elements in our system. So, these are their 5 unknowns and there is another beta here is sitting which belongs to this real number. So, this beta is a real number and we will discuss that for what values of beta we have the solution of the system or we do not have the solution all these consistency part will also depend on beta in this particular case, ok.

So, what is the idea of this Gauss elimination or the reduction technique to this echelon form we want to have echelon form out of this augmented matrix and the first step is to make these numbers here 2 minus 1 3, 0 out of this row 1.

So, if we subtract from this row 2, 2 times the row 1 then this will be 0, our focus is just to set this 0 rest everything will change accordingly, but we will set we will eliminate this x 1 from equation number 2, we will eliminate this corresponding to x 1, this coefficient from equation number 3 and also from equation number 4. So, for that we need to do this elementary operation.

So, the first elementary operation here we are doing that R 2 we are taking as R 2 minus 2 R 1. So, R 2 minus 2 times R 1. So, this will become 0, and this R 2 times this 4 minus 2 times this again this 0; here minus 4 and then this will be plus 4 so, again this will become 0. Here this will become 2 now and here 2 minus 2 this is 1 and 2 minus 2 that will be 0. So, this is the new row 2 now with this operation and the next yeah here as well we will continue this with this augmented column as well; so, here 2 minus 2, that is 0 here.

So, this first row operation is completed. We will take now the another one because for R 3 to make this 0 here to eliminate this one we need to just simply add into R 1. So, this will be now R 3 plus R 1 now the row 3. So, here when we add this is 0, when we add here also 0, when we add this you will get 1 and when we add this will get 2 here we add 4 and 1 will get 5 and this will be 4.

So, this is the second row operation we have done we have eliminated already this first variable and now we will do the third elementary operation that is on the row 4. So, this R 4 should be now R 4 minus 3 times R 1. So, R 4 minus 3 times R 1. So, with this will set to be 0 here and here 6 and minus 6 again this is 0 minus 7 and then we have plus 6 so, minus 1, here 1 and then plus 3 so, 4 and 1 minus 3 minus 2 and beta minus 3.

So, this is the first step of the elimination technique that we have eliminated x 1 already from the equation, x 2 is automatically removed which was usually we do not get such a nice matrix where this x 2 is automatically removed without any operation otherwise we have to repeat this now with the help of this second equation we have to repeat all these to get the to get eliminate the second variable from equation 3 and 4.

So, but this is already and now we will go to the third one that from now R 3 at the x 3 corresponding to this x I mean x 3 we have to make 0 all the coefficients; that means, this third column, but here what do we realize that there is already 0 there, but what happens I mean because remember this stepwise structure of this echelon form as well and this is a very systematic approach. So, with the help of equation 1, we remove first variable x 1 from equation number 2, 3 and 4; then equation number 2 will be used to remove x 2 from equation number 3 and 4 and equation then 3 will be used to remove x 4 and so on.

So, now what we realize here that if we change this if we interchange the two equations the R 2 and R 3 so, we will already get here 0 at this place and then we can set the other one also 0 to have this R 3 0.

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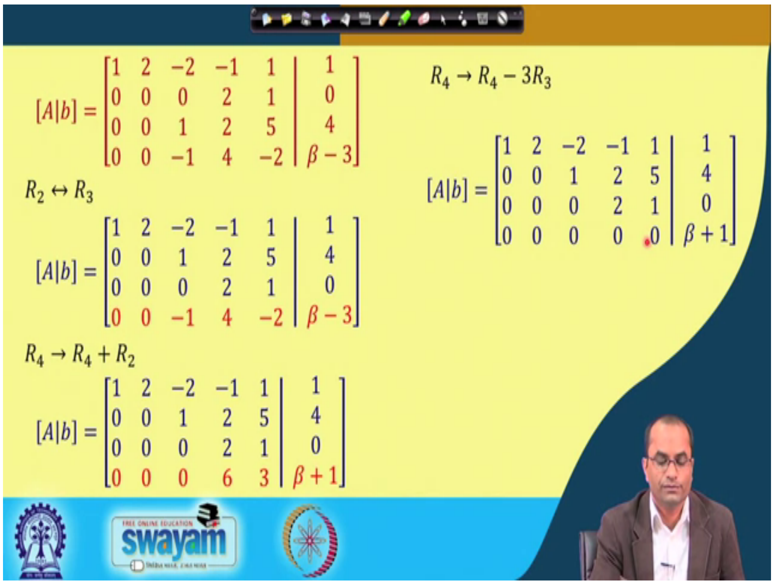

So, looking at this now, what we will do we will interchange these two row R 2 and R 3 now. We will have automatically without much efforts this is structure now. So, we have now one here and now the aim is to make everything 0 here. So, with the help of this equation 2, now this is a new equation 2 which was equation number 3 before, but now it has become equation number 2 by this interchange and now, we will set this element to 0. This is already 0 again, only we have to set this to 0 to eliminate this x 3 now; x 2 is eliminated. x 2 is automatically eliminated by the first operation itself and now, we are trying to eliminate x 3.

So, x 3 from this equation is eliminated that is also automatic and now this we will eliminate with the help of this equation number 2. So, if we add equation number 2 here, then this will be also eliminated. So, our next row operation is R 4 plus R 2. R 4 and then you will add this equation number 2 here. So, this will be a row number 2 this will be set to 0. So, now, by adding these two rows so, 2 and 4, so, what we will get minus 1 that is 0 here 2 and this 4, this will give 6 and 5 minus 2 this is 3 plus and 4 plus beta minus 3 this is beta plus 1.

So, we got this now this equation. Now, we will continue with the process. So, we have removed already this x 3, now we will we will remove the x 4. So, we will remove this x 4 here with the help of this equation number 3. With the help of this R 3 we need to remove R 4 the coefficient of x 4 in the row R 4. So, what we will do now if we take 3 times of this and subtract from this R 4 then we get the 0 here at this level.

So, doing this R 4 minus 3R 3, R 4 minus 3R 3 we will set this also 0. So, by doing so, so 3 times this is 0 and see this approach you if we go in this systematic way this will not be disturbed when we have all these 0 they will remain 0 because we are doing this operation with this row where left to this everything is 0. So, by doing so these numbers will not change these zeros will not change. So, here 6 minus 3 times 2, so, this 0 again and 3 minus 3 this is also 0 and this beta plus 1 minus 0 into 3 that is beta plus 1.

So, this is the reduced form we do not have anything now further to reduce in this case.

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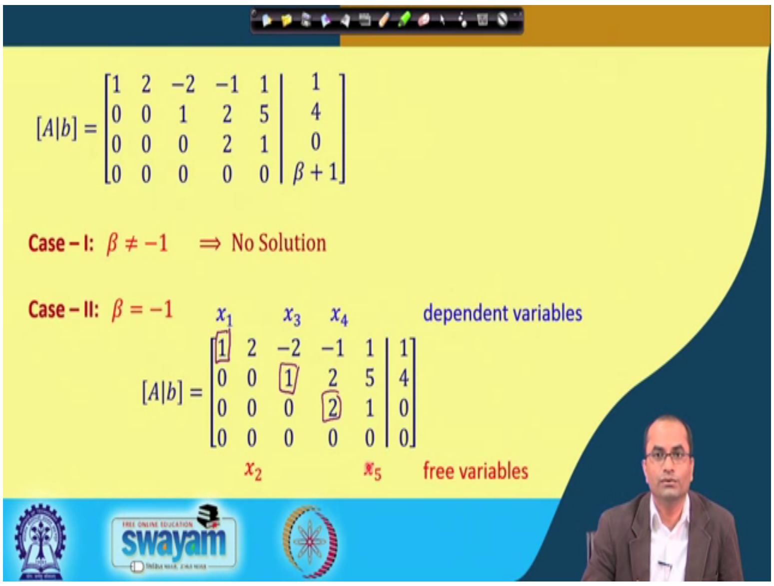

So, this is the reduced form and now we will identify we will take the first case when beta is not equal to 1. So, if we take this case when beta is not equal to 1; that means, something nonzero is sitting at this place here when beta is not equal to 1 something nonzero, nonzero will sit here. And, if something nonzero is sitting here; that means, the last column has pivot last column has a pivot meaning that we have an inconsistent system and or in other words we have 0 is equal to something nonzero. So, in either case when this beta is not equal to 1, the system is inconsistent and we do not have a solution for the system.

So, that means, this is the case of exactly no solution and the case 2 we will consider when beta is equal to minus 1. So, when beta is equal to minus 1, this is exactly the case where we will discuss about the solution because the system becomes consistent now; the last row has become 0. So, now, this system is consistent, we do not have pivot in the last column now. So, let us identify the pivots because that is another important step.

This is the pivot here in the first column, this is not pivot. So, each row and each column will have at most one pivot that is also true this is the fact here because if this is the pivot then nothing else can be the pivot because 2 cannot be pivot it is left to this something nonzero is sitting.

So, this cannot be the pivot now because left to this so, this is a 0 element or cannot be cannot be pivot because this is a 0 element. So, this cannot be pivot. So, this is not pivot. Here we have pivot because now it satisfy all the properties everything left to this 0 here 0 and here also 0. Now, coming to this one this is also pivot here 0 here 0 and then this last column also does not have a pivot because now this cannot be a pivot and this cannot be a pivot.

So, what do we have finally, we have we have one pivot here, one pivot here and one pivot here. So, 1 2 3, we have three pivots. We have three pivots and we have five unknowns. So, this is the case where we get infinitely many solutions and coming now to this one we will introduce these free variables and again just to recall from the previous lecture and again this is more generous who will understand here. The first column has a pivot. So, this is not this we will not take as a free variable the second column does not have a pivot.

So, this x 2, we have marked here as free variables and this column has a pivot so, this is corresponding to x 3 you are not taking as free variable. So, this is; so, called the dependent variables we can call. So, this x 3 is dependent, this is not free we have the pivot there. So, wherever we have a pivot we marked them as dependent variables and wherever we do not have a pivot we will mark them free variable that is a that is a simple algorithm we will follow here though it is not necessary that we have to do in this way, but this is a very systematic approach.

So, go to the column number 1, if it has a pivot element mark this as a dependent variable; go to the next column 2, it does not have a pivot element. There is no pivot element in column 2 we will mark as a free variable; column number 3 has a pivot then this is marked as a dependent variable; again column number 4 has a pivot. So, this we have marked as a dependent variable, column number 5 again we have marked as a free variable because it does not have a pivot.

By doing this now, we have so many things in hand because free variables means we can choose, we can give the values whatever we like and whenever we have free variables in the system ah; that means, we have infinitely many solutions. So, this is the case of infinitely many solutions since we have the free variables; free variables again means they are free, you can choose whatever you like.

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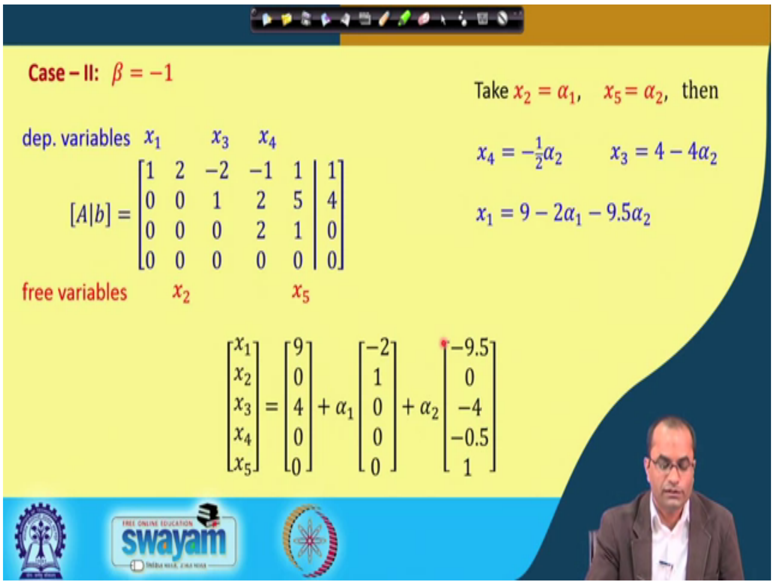

So, in this case we have two in fact, free variables. So, we have more possibilities to actually choose the choose the variables now here. So, this x 2 and x 5, these are the two free variables. So, we will take like x 2 alpha 1 and x 5 as alpha 2. In that case now from this equation number 3 we can write down x 4. So, from here 2 times x 4 is equal to minus this x 5 and x 5 we have taken x 2. So, this x 4 we can write down in terms of alpha 2.

Similarly, the x 3 from this equation number 2 we can write down in terms of this the dependent where these free variables and also the x 1 from equation number 1, we can write down in terms of these free variables alpha 1 and alpha 2 in the vector form we can place them. So, x 1, x 2, x 3, x 4 the constant from each; so, from x 1 we have this 9 as constant from x 2 was alpha 1 only. So, there is nothing constant from this x 3 we have 4 as constant and from x 4 there is nothing exactly and from x 5 also we do not have anything to take as a constant.

And, then this alpha 1 and then the rest from x 1, for example, we have 9 minus 2. So, minus 2 was with alpha 1. So, minus 2 here and with alpha 2 we have with alpha 2 here in x 1 we have minus 9.5 this is taken here in this vector. So, we can write down now these x 1, x 2, x 3, x 4 and x 5 in this particular form where we have first vector free from these free variables and then this is with alpha 1, the one free variable the vector again coming here and x 2 again this free vector with this three variable again this vector is coming up.

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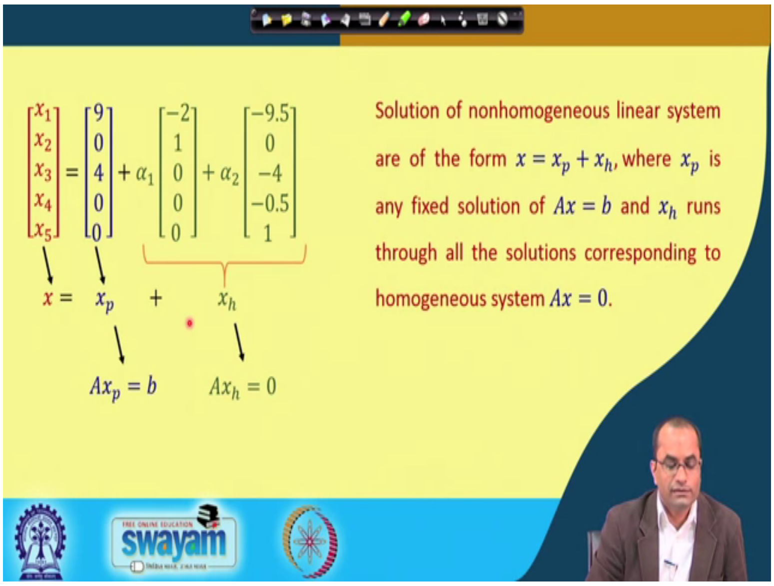

And, what we have also seen that we have a very special structure now when we write down in this form that this is a particular solution a particular solution of the given system Ax is equal to b, whereas this part with this alpha as alpha 1 and alpha 2 these the these vectors or the combination of these vectors they satisfy the homogeneous the corresponding homogeneous equation; that means, Ax is equal to 0.

So, here this x p this particular solution so, if we substitute here in instead of x this x p this Ax will give exactly be the given system and here these this part actually satisfy Ax is equal to 0 and that is for as a total also we have the solution. So, this x is a solution because A and this x has two parts a particular plus this x h and then we can use this product rule here. So, Ax p plus this Ax h and this is 0 anyway and this Ax p will give b.

So, as a whole also this x will satisfy Ax is equal to b, but if we go little more into the detail what we observe that actually this part is satisfying Ax is equal to b and this part is satisfying Ax is equal to 0. So, Ax is equal to 0 and then as some also they will satisfy naturally because of this reason here Ax h is 0 and Ax this particular is b. So, we have at the end this b plus 0 which is b.

So, but this is interesting now to note that we have this structure and this is always possible to write down for whatever system. In fact, we have this nice result that the solution of a non-homogeneous linear system; non-homogeneous we call this Ax is equal to b when the b is not 0 we call non-homogeneous, when b is 0 we call it is homogeneous system.

So, the solution of non-homogeneous linear system is always of the form x p plus x h where x p is any fixed solution of Ax is equal to b and this x h this for homogeneous part it runs through all the solutions corresponding to homogeneous system Ax is equal to 0, because you can keep on changing the values of alpha 1 and alpha 2 and we can produce the infinitely many solutions of this ax is equal to 0 equation or Ax is equal to b equation because the whole here x p plus this x h is also satisfying the Ax is equal to b system of equation.

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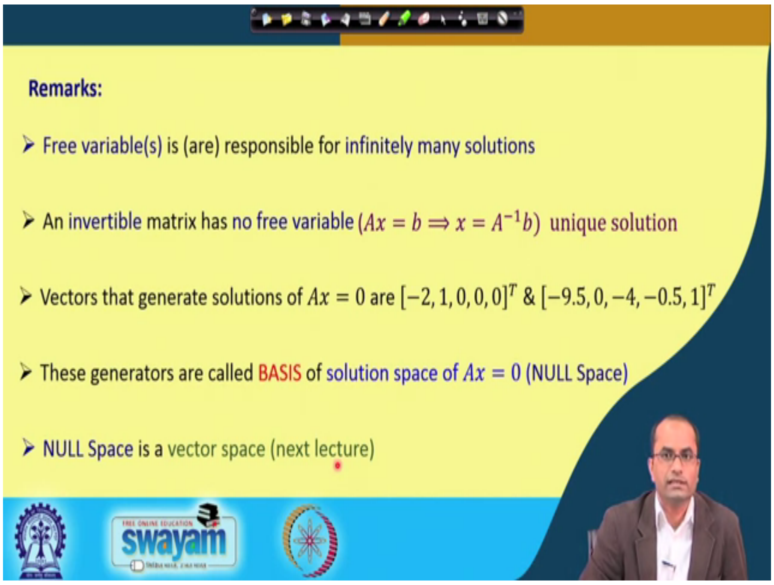

So, just to remark, that these free variables are the responsible for infinitely many solutions, so, in conclusion whenever we have free variables in our system; that means, some columns do not have a pivot and in that case there will be a case of this infinitely many solutions. An invertible matrix will have no free variable the reason is clear because once the matrix invertible we get actually unique solution and if we have a unique solution definitely we will not have a free variables or other way around if we observe that there is no free variable; that means, this A is also invertible.

So, this vector that generates solution of Ax is equal to 0, what we have just seen that for Ax is equal to 0 all the solutions we can generate with the help of those two vectors. This was appearing with alpha 1 and this was appearing with alpha 2 we have just used here this transpose because they were given in this column form. So, these vectors they are so called the generators of the solution of this Ax is equal to 0, that is what we have observed.

And, these generators are the BASIS are the BASIS of; so, this term we will introduce little later. So, some technical terms I am just mentioning here, but they will be discussed in the next lecture. So, these generators which we have just seen before these are called the BASIS because these are the main component that generator of the solution. So, these are called BASIS of the solution space again one more term has come; the solution space of this Ax is equal to 0 because this Ax is equal to 0 has infinitely many solutions in that case for instance. And, these vectors are called basis of the space where we have so many solutions there and that space is also called the null space which again we will be discussed in the next lecture yeah exactly here the null space is a vector space.

So, again one more term here the vector space has come. So, this null space which is the solution space here is a vector space and there are some properties of a set here, the solution set which we are talking about which will be later called as a vector space like for instance here we see that if we add 2 solutions the new solution will be also solution of this Ax is equal to 0 equation. Because A and if x 1 satisfy this and A and the x 2 also will satisfy this and then the A and this x 1 plus x 2 will also satisfy that equation because Ax 1 plus Ax 2 both are the 0, 0 so, that will also satisfy.

So, this solution set here of Ax is equal to 0, has some nice properties like you add any two solution that will be also solution or you multiply by n number to this solution that will be also a solution. So, there are so many other properties which are vector space has; so, that we will discuss exactly in the next lecture.

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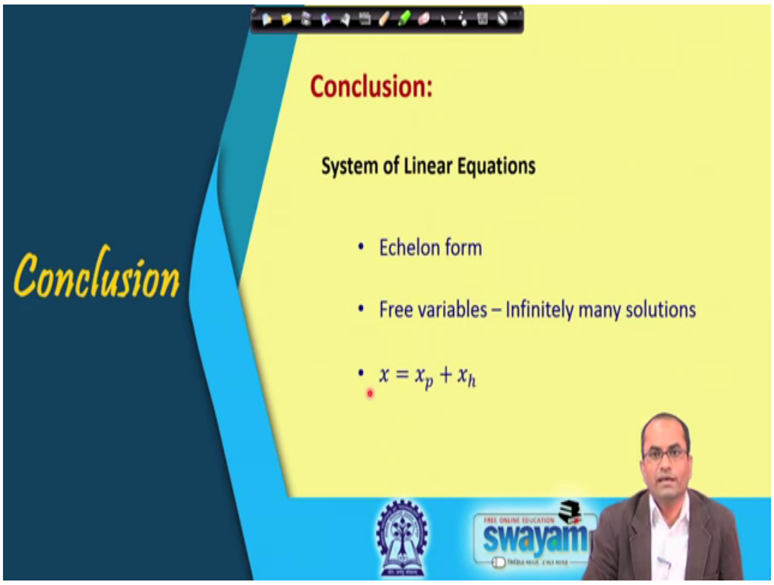

And, the conclusion here; so for the system of equations this echelon form was very important and we have introduced this concept of the free variable and that lead to basically infinitely many solutions and the solution of non homogeneous system of equation has this special structure that we get this x p which satisfy the given Ax is equal to b and we have a part here x h we satisfy Ax is equal to 0 and as a whole also this x means this satisfy Ax is equal to b.

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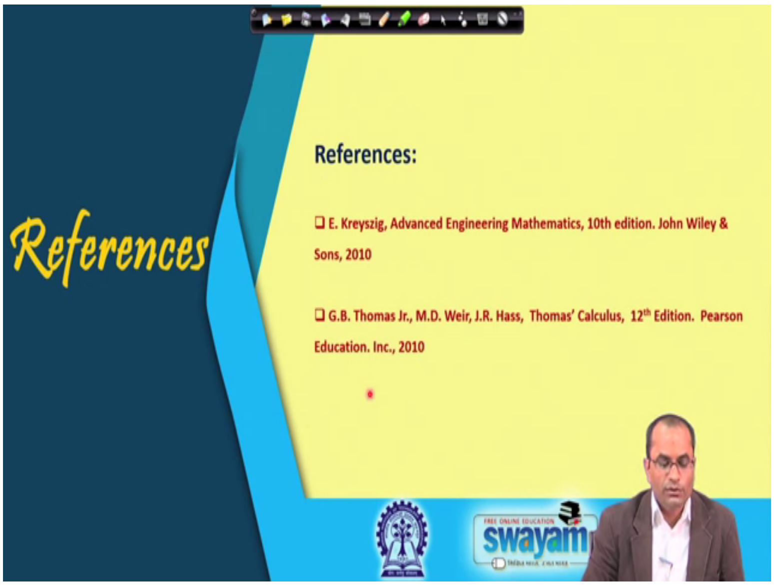

So, these are the references used.

Thank you.