Engineering Mathematics – I
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Lecture – 37
System of Linear Equations – Gauss Elimination

So, welcome back and this is lecture number 37. We will be continuing our discussion on System of Linear Equations and in particular, today we will look for the solution techniques that is the Gauss Elimination Method.

So, we will be talking about this Gauss elimination method and there we will see that how to get the solution using this very systematic elimination technique and with the help of the same we will also talk about the consistency of the solution; that means, about the existence and non-existence of the solution with the help of this technique we can identify whether the system has a solution and it is unique or it does not have a solution, all these identification we can also check with this Gauss elimination method.

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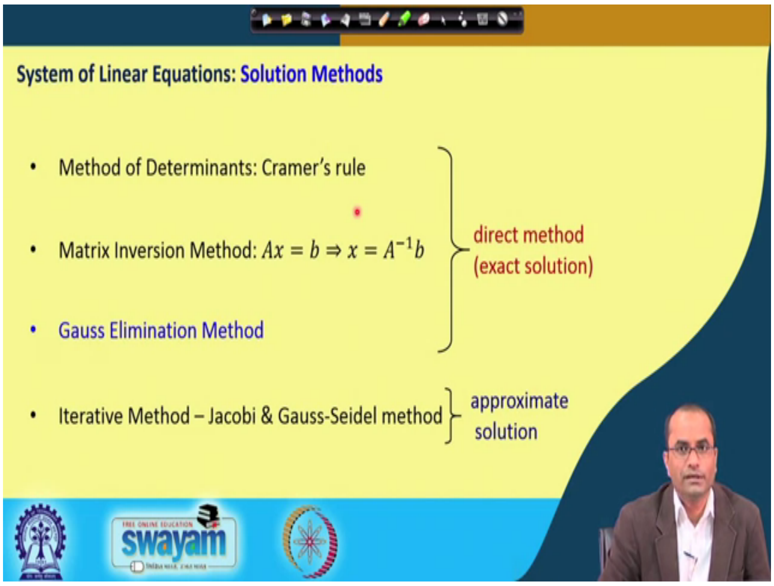

So, the system of linear equations, the solution methods we have basically the other methods to like the method of determinants you must be familiar with or we call this Cramer's rule. So, with the help of this determinant we get the solution and this is possible when we have a system where a unique solution exists, the same equation the number of equations the same as the number of unknowns and the system has a unique solution in that case this is possible to get the solution of a rather smaller system of equations using this method of determinants.

We have also the matrix inversion method where this Ax is equal to b this system we can solve once we have the inverse of the matrix. So, out of this Ax is equal to b we can write down like x is equal to A inverse b and once we have the A inverse. So, we multiplied by this b and then we will get the x. So, if we have the inverse of a matrix we can easily write down the solution of the system.

Now, this is a little more general case which we will be talking about this Gauss elimination method. So, here we will be also dealing the cases when we have non unique solutions meaning infinitely many solutions or the system does not have a solution. So, all these we can figure it out with this Gauss elimination technique. And, we have iterative methods that is the Jacobi and the Gauss-Seidel method for solving when we have a system of equations which is large in number so, really a very large system. So, these all these methods which we have this method of determinant Gauss elimination, inversion method they actually fails or rather they take longer time, so, we take these iterative methods.

So, the first three methods falls into the category of the direct methods. The direct methods meaning that we get actually the exact solution of the system using these techniques whereas, here the iterative methods the they give us the approximate solution of the given system of equation. So, we will be talking about in this lecture about this Gauss elimination method, which is a very general one in the category of this direct methods.

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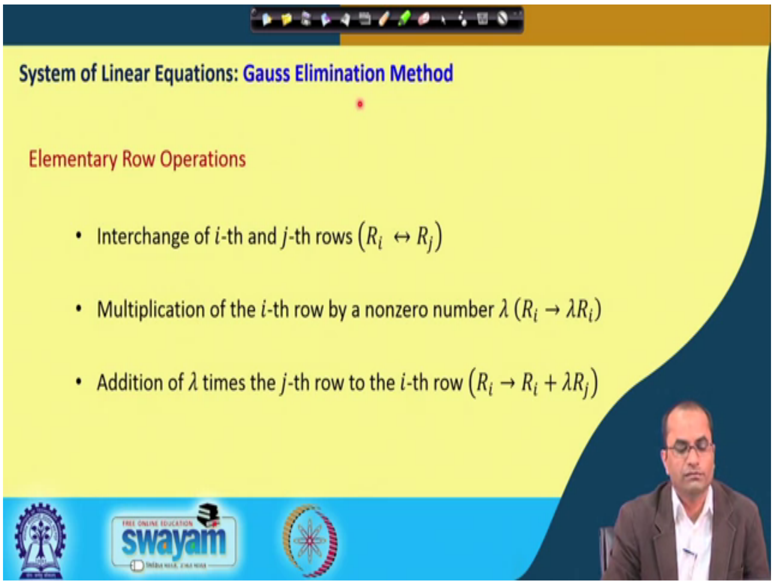

And, we will be also talking about in this technique some elementary row operations. So, what do you mean by elementary row operations? So, we can interchange the i-th and the j-th row of a matrix and if going back to the system of equations because for the system of equations we are writing the matrix A.

So, changing this row is nothing, but just the writing the second equation for example, at the first place and the first equation at the second place. So, it has nothing to do with the solution and that exactly in terms of the element row operations we will be doing with the matrix that we can change we can interchange any two rows of the matrix.

The second one we can multiply the i-th row means any row by a nonzero number lambda and this is precisely equivalent to saying that we have those equations and we are multiplying any equation by some number and again that system the solution of the system will remain we remain the same with that operation. So, in terms of the matrices we talked about this row operation that R i now the i-th row has become like lambda times R i. So, we have just multiplied lambda a non-zero number to the equation R i.

Here addition to the lambda times the j-th row to the i-th row. So, what we can do that we will multiply the j-th row by a number lambda again non-zero number lambda because if lambda is 0, then we are not adding anything to this R i. So, this lambda times the j-th row and we will add this to the i-th row and in terms of the operation we write like now the R i has become as R i plus lambda R j. So, we have added lambda times this

R j to R i. So, the row. So, R means the row. So, the i-th row we have added this lambda times this j-th row and the new R i will come up.

So, here these are the elementary row operations which we will be using for this Gauss elimination method.

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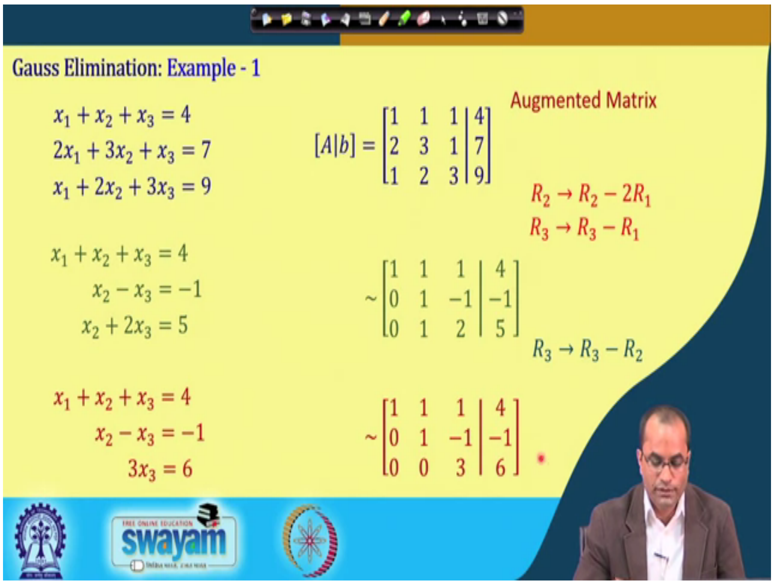

Let us explain this with the help of simple example and then perhaps in the next lecture we will go for more general problems. So, here we take a very simple example. So, we have three equations x 1 plus x 2 plus x 3 is equal to 4 and 2x 1 plus 3x 2 plus x 3 is equal to 7 and the third equation we have x 1 plus 2x 2 plus 3x 3 is equal to 9.

First we to write down this equation in this matrix form where we also augment this right hand side of the equations 4, 7, 9. So, the right hand side here I will be basically doing precisely what we do in Gauss elimination method and the left hand side we will be doing the same thing with the equations directly. So, it is easy to understand that this Gauss elimination is nothing very very complicated, but it is like eliminating the variables in a systematic fashion.

So, here we have first that will be the first step that we have to write this combined matrix or so called the augmented matrix in this form. So, we have the coefficient matrix that this I will matrix A. So, if we write down the system of equations into Ax is equal to b form. So, this A is going to be in this case 1 1 1 and then we have 2 3 1 and 1 2 3. So, this matrix and then the right hand side vector is 4 7 9.

So, these two the matrix A and the matrix b basically have all the information of the given system of equations and what we do in the augmented matrix we basically collect this a here. So, this is precisely the A 1 1 1 2 3 1 and 1 2 3 and this b we have augmented here with the same matrix as extra column. So, the 4 7 and 9. So, this matrix we call as the augmented matrix.

So, here what we do now that the left hand side with the equations as I said also everything with the equation and then we will translate into the form of this matrix and so called the Gauss elimination technique. So, here what we are doing now with the help of this equation number 1, we are trying to eliminate this x 1 variable from the equation number 2. So, what we are supposed to do actually that if you multiply this equation 1 by 2 and then subtract this equation from this equation number 2.

So, we have now the equation number 2 will become the equation number 2 minus this 2 times and the equation number 1. So, this we have done this manipulation here. So, 2 times the equation number 1 and then we subtract here. So, this will cancel out here we will get 2x 2 and then here 3x 3. So, when we subtract we will get x 2 here and then we have 2x 2 and this x 3. So, we will get minus x 3 and then here 7 minus 8. So, that will be minus 1. So, we have just done this operation that we have multiplied equation number 1 by 2 and then this is subtracted from equation number 2 here.

Now, same thing we will be doing with this equation number 3, we want to again eliminate here as well this x 1 so, but this is easy again with the help of equation number 1. So, here we will just subtract the equation number 1 from equation number 3. So, we will get simply here x 2. So, 2x 2 minus x 2, 3x 3 minus x 3 that x 2x 3 and 9 minus 4 that is 5. So, we have a new system here where we have eliminated from equation number 2 and equation number 3 the variable x 1 and we will continue this process actually.

Now, from equation number 2 and equation number 3; so, with the help of equation number 2 now, so, equation number 1 we have taken to eliminate x 1 from 2 and 3 now we will consider equation number 2 to eliminate x 2 from equation number 3 that will be the next step. But, now before we go to the next step, let us look at this in terms of the matrix what we do? We will be doing now this elementary row operation that this row 2 will become now row 2 minus 2 R 1.

So, 2 times the equation 2 times the row 1 and this R 2, so, R 2 is 2 and then minus 2 times this one. So, 2 minus 2 that is 0. So, this will become 0 there and here again and we will have another operation which was corresponding to this, this elimination from the third equation, we will do simply the row 3 minus row 1. So, these two elementary operations we will be doing now which is parallel to what we have done with the equations.

So, what we will get? So, when we do the first operation here R 2 minus 2 R 1. So, R 2 minus 2 times the R 1 the equation number 1. So, here we have we have A 2 minus 2 that is 0, then here 3 minus 2 here it is 1 and 1 minus 2 that is 1 here and then we have 7 minus 8 that is minus 1 and at this place just we are subtracting row 1.

So, here 1 minus 1, 0; then we have this 2 minus 1, 1 and then 3 minus 1 this is 2 here and then we have 9 minus 4 that is 5 here. So, that is the elementary operation we have made here to reduce this augmented matrix into this form and now, the next one as I said before with the help of equation 2 we will eliminate x 2 from the equation number 3.

So, we have this first equation x 1 x 2 x 3 is equal to 4 we have the second equation we will not touch x 2 minus x 3 is equal to minus 1 and now what we will do eliminate this x 2 from this equation number three; that means, this equation number 3 minus equation number 2. So, this will become 0. So, no x 2, here when we subtract this will come 3x 3 and here it will come 6. So, we have this system now where from the third equation x 1 x 2 is removed from the second equation x 1 is removed and the first equation is intact as before.

So, again the same operation here in terms of the row operation we will write like R 3 minus R 2. So, now, the equation number R 3 will become R 3 minus R 2. So, we write in this form. So, we have the first two rows as it is first two rows will be there and then the third one we will have here R 3 minus R 2. So, we are subtracting row 2. So, here naturally the 0 minus 0 here 1 minus 1 is 0 2 minus minus 1 that is 3 and 5 minus minus 1 that will be 6. So, we have this augmented matrix corresponding to this equation.

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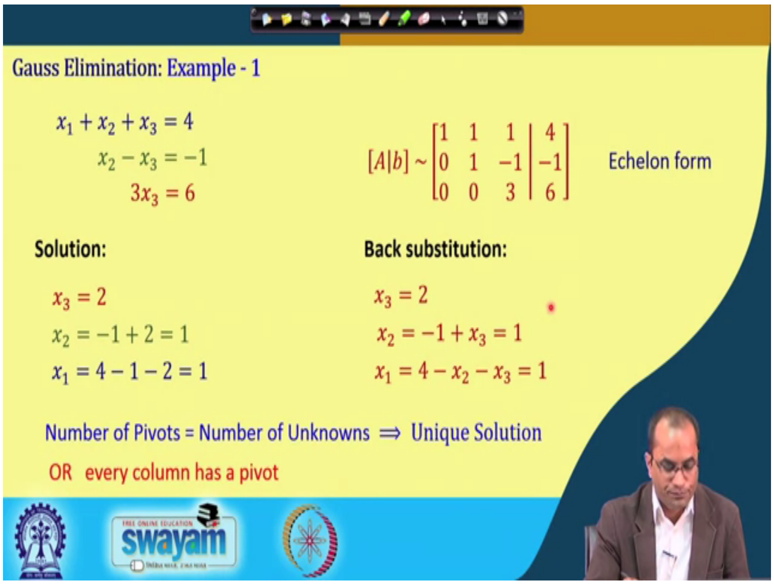

And, now the next one; so, we have this reduced system here or corresponding augmented matrix and now, this is the one step which we have to do in this Gauss elimination to start with this augmented matrix and to reduce into this form, which we call as echelon form. So, this is we will be continuing this discussion in the next lecture what is exactly echelon form in general.

So, in this case roughly speaking now, we have to see whether we got this echelon form. So, always we will have this structure for the echelon this step like a structure for this echelon form. So, here something non-zero, everything zero then here also then everything zero and then everything zero here and the left hand side naturally. So, we have this echelon form of the system and once we reach to this echelon form we can get the solution very easily. So, first let us see with the equations how to get the solution now.

So, the solution will be from the last equation we can directly write down our x 3. So, the x 3 will be 6 by 3 that is 2 and from the equation number 2, we can get now x 2 is equal to minus 1 and then this x 3 will be added that is 2. So, we will get 1 and from the equation number 1 since x 2 and x 3 is known we can get x 1 as 4 minus this x 2 1 and minus x 3 that is 2. So, 4 minus 3 that is 1. So, we got the solution here. Once we have this form we have to do only little manipulation to get the solution x 3 2, x 2 1 and x 1 is 1.

What we call in terms of the Gauss elimination we call it this back substitution that is the actually the third step. So, the first step was putting into this your matrix and the right hand side vector into this augmented matrix. The second one is to do reducing to this echelon form by doing these row operations, the last step of the Gauss elimination is going to be back substitution where we will get the solution of the system.

So, back substitution we will start from the bottom here where the 3 is equal to 6. So, the last equation yeah this is the last equation which says 3x 3 is equal to 6. We should always keep in mind that corresponding to this we have actually the system of equations you are given in the left. So, here we have actually 3x 3 is equal to 6; that means, x 3 is 2 here then this x 3 is known then from this we will go step by step to up now.

So, here from this equation we will get x 2 variable. So, x 2 will be minus 1 and then plus this x 3 which is 2 and then we will get 1. Now, we will go further up here to x 1 and then we have a 4 minus x 2 minus x 3 4 minus x 2 and minus x 3 and that is equal to 1. So, we have x 1 is equal to 1, x 2 is equal to 1 and x 3 is equal to 2. So, we get the solution out of this back substitution once we have this echelon form of the equation.

There are a few more points to be noted here and we will explore them in the next lecture again. So, what we have observed that we have the we have got a unique solution we have got a solution of the system as 1, 1, 2 and indeed this is the unique solution of the system we do not see any other possibility directly getting the solution from the equations. But, what is now we are going to introduce this that these numbers here in this first row or in this first column this is called the pivot elements. This is called the pivot element and pivot element is always non-zero element and below this everything should be zero.

And, now, here when we come to the next row or next column this number is again pivot because everything here to zero and left to also zero and also in this sub matrix everything is zero. Going to the next this is also a pivot element because this is nonzero and everything left to zero and there is nothing here because the matrix was this 3 by 3. So, these such elements the such non-zero elements will be called pivot because they play a very important role now discussing about the about the consistency of the solution about the existence non existence of the solution.

So, if we have like written here the number of pivots, so, here how many pivots we have? We have one pivot, we have two pivots, we have three pivots. We have three pivots in our in our system here in our matrix in our this augmented matrix. So, the number of pivots here is equal to number of unknowns. How many unknowns do we have here? Three unknowns x 1, x 2, x 3.

So, this number of pivots is equal to the number of unknowns, then we have exactly the unique solution this is the case of the unique solution because our number of pivots is equal to number of unknowns or in other words we say because each column is parallel each column is associated with one variable. So, the first one with x 1, this is associated with the x 2, this is associated with the x 3 because if you remember we can write down this Ax is equal to b, I mean this product here in terms of this column here 1 0 0 multiplied by x 1 and then the second one will be multiplied by x 2, the third column will be multiplied by x 3.

So, this first column is associated with x 1 here, this is with x 2, this is with x 3. So, if the number of I mean each column has a pivot we can in other way we can put it as that if each column as a pivot in that case the solution will be a unique solution. So, every column has a pivot and this is exactly the case when we have the unique solution. So, we will be exploring a little more on the pivots and this is just a very very basic introduction to this concept here and we are considering only this 3 by 3 matrix. So, we will be talking about in the next lecture more about these pivots and echelon form.

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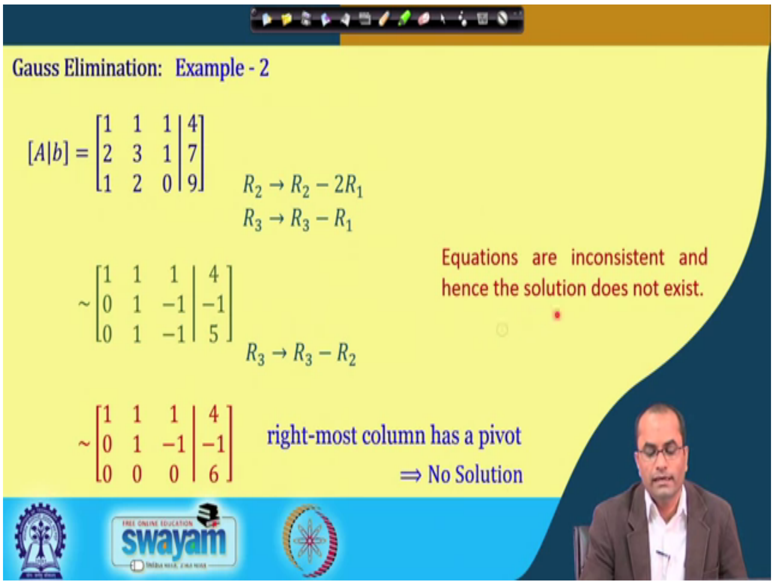

So, now going to the example number 2, we have changed the example now. So, this augmented matrix is taking this form and by doing those elementary transformation which we have done earlier the same transformation because there is only little change in the matrix will be again trying to make this element 0 out of the first row; that means, we have to do we have to subtract from this row 2 times the row 1. So, this is exactly written here R 2 minus 2R 1 and to make this 0 with the help of the row 1 we have to just subtract the row 1.

So, with these two operations we will get this form 1 and then we have this 0 0 that was the objective in this algorithm to eliminate this x 1 variable from the equation number 2 and equation number 3. And, now, the next objective is that using the equation number 2 we will eliminate x 2 from the equation number 3. So, here; that means, R 3 will be now R 3 minus R 2 and so, we have when we subtract this will be 0, this will be also 0 and this will be 6 here.

So, what do we see now, it is something different than the previous case if we look at this closely to the last equation, what are we getting from this last equation? We are getting here so, 0 times x 1 0 times x 2 0 times x 3. This is exactly what we have here x 2 and 0 time x 3 and the right hand side is 6; meaning what we are getting that 0 is equal to 6 this is what we are getting from the last equation. And, that is the point we are this Gauss elimination technique or this reduction to this echelon form helps because here we can now observe that there is an inconsistency in the system because we should not get we should not get 0 is equal to 6.

So, by reducing to this echelon form we have also realized that the given system is inconsistent in consistence mean again that it has no solution it cannot have a solution here. So, by just reducing it to this echelon form we observe that the solution has knows that the system has no solution. Because from the last equation we are getting 0 is equal to minus 6.

So, by observing this we conclude that this system the given system has no solution and in other words in terms of the pivots we call that the right the rightmost column has a pivot because now this is the pivot here and this is the pivot in the second row or in the second column; the third column has no pivot because this cannot be pivot because this is a 0 element. The pivot has to be a non-zero element, but looking at this number here this is a pivot. So, it satisfied that everything left to the 0 and there is nothing to go to the bottom.

So, we have this right here right most column has a pivot. So, which is which is the case exactly of the inconsistency or the system has no solution because this rightmost column has a pivot, this should not happen that. The pivot should be there in this main matrix here not in this rightmost column which corresponds to this right hand side vector b ok.

So, this is the case of no solution and the equations are inconsistent and hence the solution does not exist.

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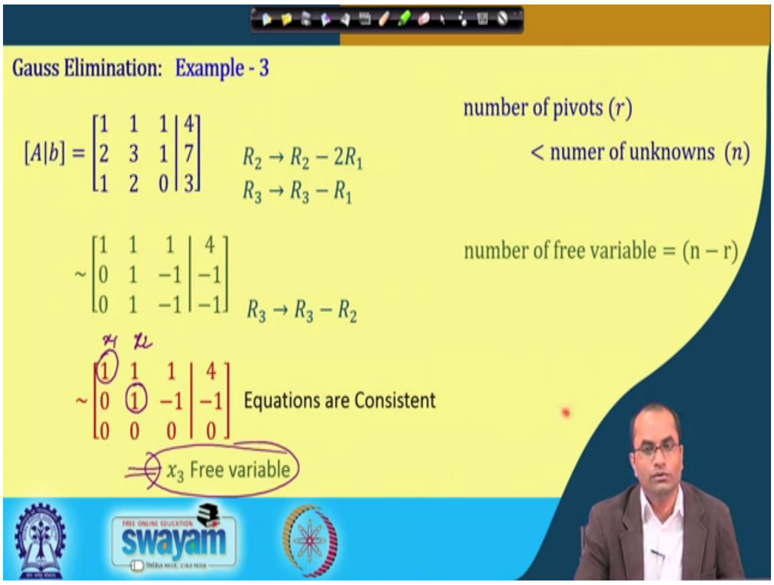

Now, we will go to the third example which is again slightly changed. So, here, but we will be doing the same operations this R 2 minus this 2R 1 and R 3 minus this R 1, the same operation because the aim is to eliminate x 1 from equation number 2 and equation number 3. So, by doing so we got this and then the further again the same operation because this equation I mean this row 2 or we want to get rid from equation from row 3 this element 1 here. So, we will just subtract. So, R 3 will be R 3 minus R 2 and in that way we will get this 0 here also 0 and 0.

So, now this is different than the previous case we are not getting anything like 0 is equal to 6 we were getting the last equation is completely 0. So, 0 is equal to 0, there is no problem. Or in other words we do not have pivot in the last column here because what are our pivot this is a pivot elements and as I said they play away play very important role in discussing several properties of the vector and mattresses. So, here this is pivot everything 0 here and there is nothing left.

Now, coming to the right one this is pivot see this is not the pivot here because the left you have a non-zero element. So, for the pivot it has to be 0 to the left and also to the bottom and everything to the left has to be 0. So, what we have we have this pivot here, we have this element as pivot and now going to the column number 3 or the row number 3 we do not have any pivot. So, no problem. So, but, but at least the last the rightmost column also does not have a pivot. So, the system is consistent.

So, the system is consistent means we will have solutions and in the indeed in this case when we see that the equations are consistent and this number of pivots are less than actually the number of unknowns. So, in this case we got in only two pivots and there were three unknowns. So, this is the case where we have infinitely many solutions because in the case of unique solutions the number of pivots will be equal to the number of unknowns.

So, in this case we will have infinitely many solutions and in terms again a very systematic approach to solve this, when we have this case of infinitely many solutions. So, this column has the pivot this column has a pivot the third column does not have a pivot and as I said this we say this corresponding to x 1, this is corresponding to x 2 and this is corresponding to x 3.

So, here since this x 3 corresponding to the x 3 the third column does not have a pivot, we call that this x 3 is a free variable because corresponding to this x 3 the column does not have a pivot and we call this like a free variable; free means we can choose this as we want. So, this is exactly the point where we are entering into the discussion of the infinitely many solutions and since this is free variable so, we can choose anything we want and that is what we will do now.

So, here the number of pivots in a less than is equal to number of unknowns because there are two pivots and there are three unknowns and the number of free variables will be precisely the left one because this is occupied here we have pivot x 2 has a pivot, but x 3 does not have a pivot. So, this is free variable. So, number of free variables will be n minus r. So, n is the number of the columns there in this main matrix here a or the number of unknowns and minus this r, that is the number of pivots we give this name to the number of pivots.

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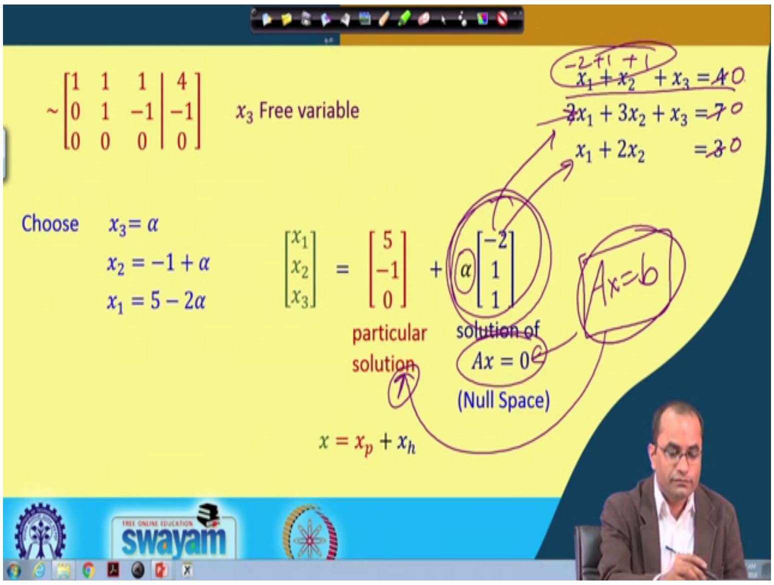

So, now what we have? We have our system which is reduced in this form and where x 3 we have denoted as marked as free variable and then we can choose this x 3 some alpha alpha as a real number and then once we have x 3 then from equation number 2, we will get x 2 because these are the dependent variables now. So, the x 2 will be minus 1 and plus 1. So, plus plus alpha minus 1 and plus this x 3 that is alpha. So, minus 1 plus alpha that is our x 2 and we have x 1 again from this equation number 1, 5 minus 2 alpha.

So, which we can write down in this form x 1, x 2, x 3 x 1, x 2, x 3 all the constants from everyone every component. So, x 1 has 5 as constant and this minus 2 with this alpha. So, here we have minus 1 and in with alpha we have 1, and there is the from x 3 equation we have 0 as constant and this alpha with this 1.

So, this is the solution of Ax is equal to 0 and this is null space which we will call as null space and more discussion on this we will have a little later, but the solution of this Ax is equal to 0 and that is a very particular structure. Here we do see once we have written this solution here. It has two components; one without this alpha and the other one with alpha and this part here 2 minus, 1, 1 this exactly satisfy Ax is equal to 0 and if it satisfy x is equal to 0 it the alpha times this also satisfy Ax is equal to 0 because it is alpha is just a constant and if x satisfy this Ax is equal to 0, A into alpha x will also satisfy Ax is equal to 0.

And, this the solution the solutions of this Ax is equal to 0 because there are so many solutions here for each alpha, alpha you take 1, you take 2, you take 1.5, anything. So, that is exactly the solution of Ax is equal to 0 and this later on we will come to this, this is called the null space of a. So, null space will be nothing, but the solutions all solutions of this Ax is equal to 0 together will be called as null space.

Here this constant part without alpha this is a one particular solution of given system Ax is equal to 0 or sorry Ax is equal to b. So, this 5, minus 1, 0 will satisfy our equations and this minus 2 1 1 or multiplied by any number this will satisfy Ax is equal to 0 that the so called the homogeneous system of equations, a corresponding to our given system. So, homogeneous means the right hand side we have set to 0.

So, this a very very special structure we will be talking about more later. So, always you will have that this our x we can write down x p. So, one particular solution of the of the given system plus this of the solution of the homogeneous system when we set the right hand side to 0, and one can easily check that this to minus 2 1 1 actually satisfy when we have this homogeneous one ; that means, the right hand side 0. So, this minus 2 1 1 so, here minus 2 1 and 1, so, this is equal to 0, it satisfies.

So, it will satisfy all these three equations this part and any number also we can multiply it to this, it will not affect because the right hand side is0. So, this part is exactly the solution of the Ax is equal to 0 corresponding to this, Ax is equal to b equation. So, this part here is corresponding to Ax is equal to 0 and this is one particular solution which exactly satisfies this Ax is equal to b equation.

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So, more on this we will continue in our next lecture. So, what we have seen a very basic introduction to this elimination technique and in particular very important is this echelon form which we have seen at least for this 3 by 3 system and based on this echelon form we can talk about the solution we can talk about the consistency of the solution or inconsistency of the solution. So, all these cases of unique solution, infinitely many solution can be discussed with the help of this echelon form.

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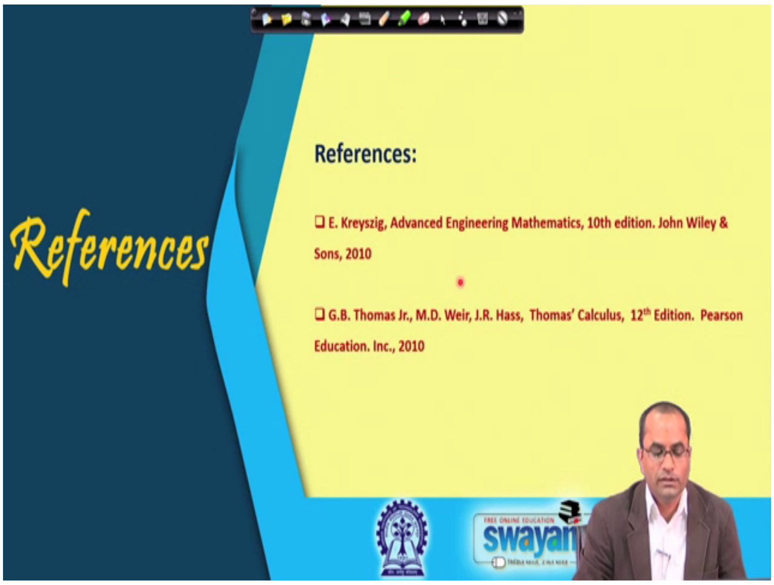

So, these are the references I used for preparing the lectures and.

Thank you very much.