Engineering Mathematics - I
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Lecture – 36
System of Linear Equations – Introduction

Welcome back. So, this is a lecture number 36 and today we will continue with a new topic that is a linear algebra a very important topic in these series in this series of lecture. And, we will start with the system of linear equations and again this is a very important to understand many concepts in linear algebra. So, we will start with a very basic Introduction to the System of Linear Equations.

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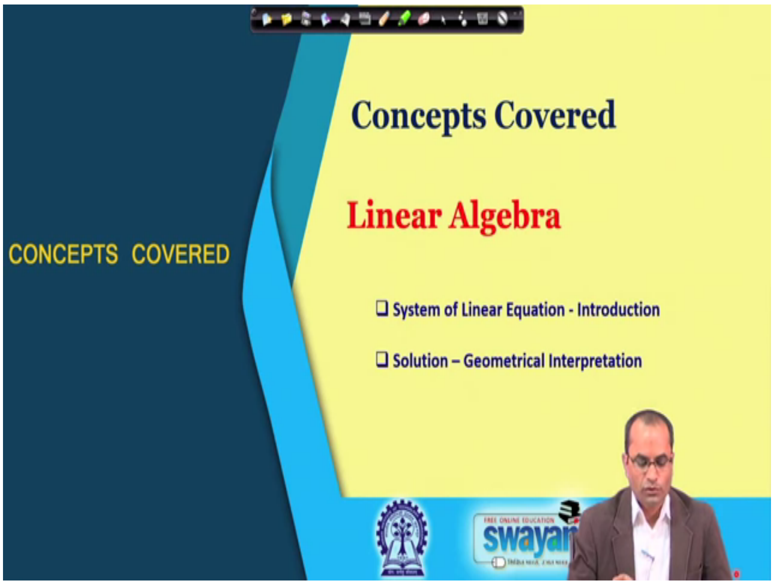

So, we will be covering the introduction to the system and also the solution of the system of linear equations and their geometrical interpretation.

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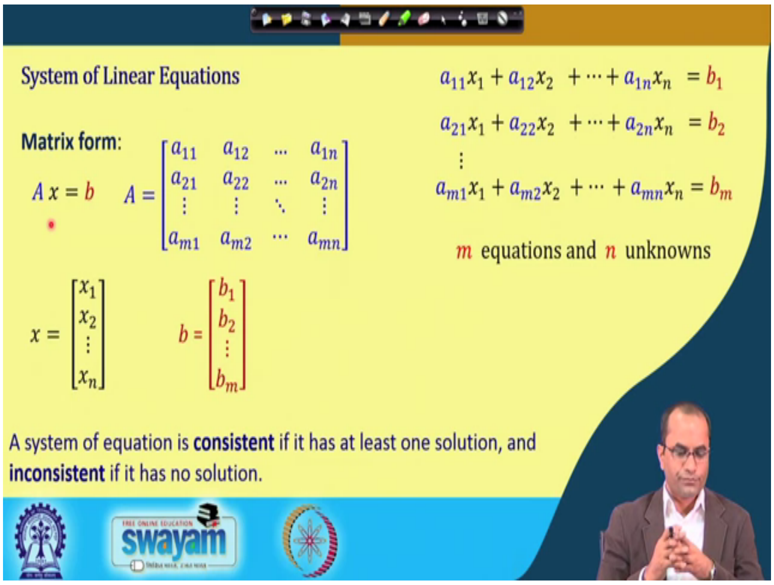

So, here the system of linear equations; so, this is a general system written in terms of the equations. So, this is the first equation where we have the coefficients here a 1 1 x 1 and this coefficient of x 2 is a 1 2 and so on; a 1 and r x n is equal to b 1. So, this is first equation where these are a 1 1 a 1 2 and a 1 n a 1 n these are the coefficients. So, they are the real numbers and the x 1 x 2 x 3 and x n they are the unknowns and the right hand side b 1 again some real number.

So, similarly we have the second equation which we have denoted by a 2 1 2 2 and 2 n these are the coefficients of x 1 x 2 x n respectively and the right hand side is denoted by b 1. So, here we have a m 1 the mth equations; so, we have considered m equations and n unknowns. So, here we have a m 1 a m 2 a m n and these are the coefficients of x 1 x 2 x 3 x n and the right hand side is b m. So, we have 1 2 and so, on m equations and x 1 x 2 x 3 x n these are the unknowns.

So, in the matrix form we can write these equations as follows A x is equal to b. So, A will be the coefficient matrix x will be the vector of unknowns; that means, these x 1 x 2 x 3 x n. And, then we have this b the right hand side vector which is given there this b 1 b 2 b 3 and b n these are the components of this b. So, here the coefficient matrix we will collect all these coefficients a 1 1 a 1 2 a 1 n these are the coefficients from equation number 1.

The second row here contains the coefficients from equation number 2 and then the last row contains the coefficients of the mth equation. So, a m 1 a m 2 and a m n. So, this is the coefficient matrix and then we have the vector of unknowns ; that means, a x 1 x 2 x 3 x n. And we have the right hand side vector b which whose components are b 1 b 2 b 3 and b m. So, a system of equation is consistent that is the definition we will use the system of equation is consistent; if it has at least one solution and inconsistent if it has no solution.

So, because we will observe now in today's lecture that there are several possibilities for the solutions. One that the system does not have a solution at all, the second it has a solution and it is unique, the third one it has solutions and they are in finite in number; so, infinitely many solutions. So, these are the 3 possibilities for system of linear equations we will also and we will also look for the interpretation; so again getting back to this matrix form. So, this is easy to understand so, we have this matrix A and this x. So, if you multiply this A and this x.

So, we will get another vector whose first component will be this left hand side of the equation 1; the second component of that vector when we do multiplication of this Ax A and x. The second component of this vector A x will be the left hand side of the second equation and so on. And then we have the right hand side vector here b whose components are these b 1 b 2 b 3 and b m. So, comparing the each component we will get basically we will get back to these equations. So, this system writing in this A x is equal to b is equivalent to the given system of equations where, we have m equations and n unknowns in general we have considered here.

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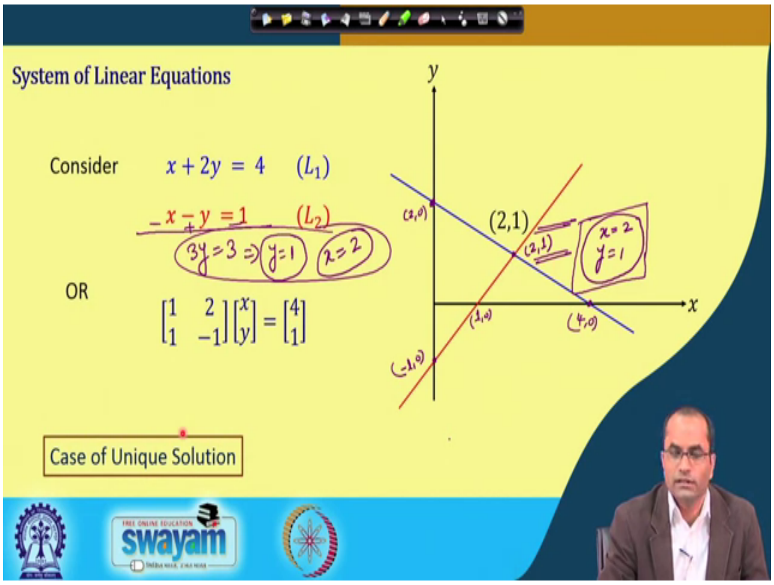

And then let us take a simple example. So, we have this x plus 2 y is equal to 4 one equation and then we have one more equation that is given by x minus y is equal to 1. So, considering this these two equations because, the geometrical interpretation will be very easy and then we can generalize the concept for 4 more variables. So, let us call this equation number 1 because, this is a line the equation of the line. So, we are calling this L 1 and here this is L 2 the equation of the second a line which is given by x minus y is equal to 1.

So, in this case when we are considering only 2 variables x and y these equations the equations of the system are nothing, but the equation of the lines. So, we can write down the system in the matrix form as well. So, this will be the coefficient matrix where we will collect the coefficient from the first equation that is 1 and 2 there. And in the second equation we have coefficient 1 and minus 1, the right hand side vector is 4 and 1 that is written here and then the vector of unknowns that is x y. So, we can write down this equation these equations in the form of the matrix vectors. And, now we look for the geometrical interpretation of these equations and the solution basically.

So, here this is the first equation x plus y is equal to 4. So, if we look at here we can easily plot this. So, when x is 0 the y is 2. So, this is actually the point 2 0 and then we have when y is 0 the x is 4 so, this is the point 4 and 0. So, this is the equation number 1 and then we have the equation number 2 as well this x minus y is equal to 1 and that is given by this red line. So, here when x is 0 y is minus 1 so, this is the point minus 1 0 and then we can have another point for instance here 1 y is 0 x is 1. So, this is the point 1 0 and then we can draw this line given by x minus y is equal to 1.

Now, the question is that as we see in this picture that they intersect at some point here. So, let us compute the point of this intersection. So, if we add if we subtract equation number 2 from the equation number 1. So, we will get here the x will get cancelled and then we will get 3 y is equal to 3 which gives us the y is equal to 1. So, that is the coordinate of y and then we can put in any of these equations to get for example, x is equal to 1 plus y ; that means, 2. So, here these 2 lines intersect at this point here are 2 and 1; so, we can write down here 2 and 1.

Now, talking about the solution of the system of equations; so solution means a point x y which satisfy satisfies both the equations. So, here clearly this point here lies on the red line and this point also lies on the blue line. So, naturally this point where x is 2 and y is 1 it satisfies both the equation; it must satisfy both the equations because, it lies on both lines and what else we see here. So, the solution is x is equal to 2 and y is equal to 1 of this system. And, the other point here to be noticed that this is the only point, this is the only point where these 2 lines intersect or this is the only common point on both the lines.

So, that this has become the solution now. So, precisely in this case what we got we got the unique solution because these 2 lines intersect exactly at one point. So, there is no other possibility to have a solution for this given system of equations and this is what we call the case of a unique solution. So, in this particular situation when we have considered these two simple equations, we are getting unique solution. If we consider another case here for are not the another case.

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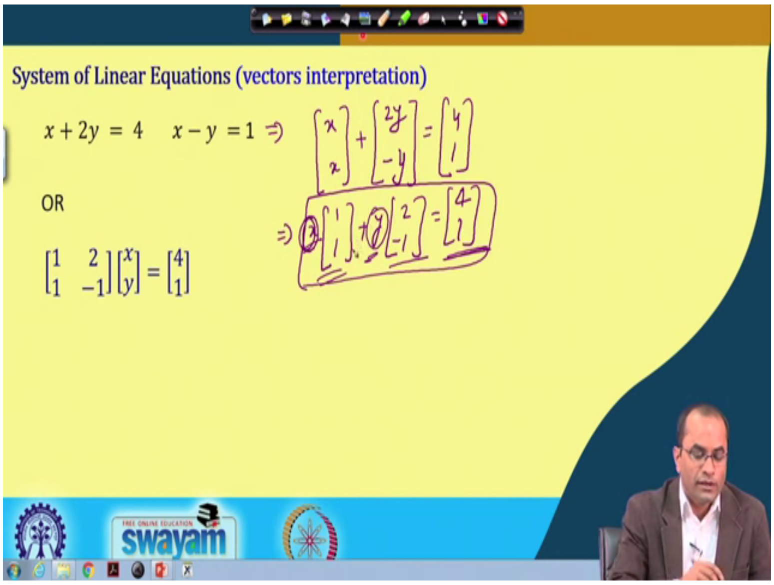

So, we will just look for one more interpretation of the same equation in the in terms of the vectors. So, we have seen that we had these two equations which we have also written in the form of this matrix a vector multiplication. But, there is another way of writing this equation because, if I consider here this x plus 2 y and the second equation is x minus y. So, we can also write down the system as like the first vector I will take x from this and also x from here and in the second equation the rest the y term.

So, 2 to y and then the minus y here and then the right hand side is 4 and 1 or we can write down this as x and the vector 1 1 plus this y and this vector 2 minus 1 is equal to this vector 4 and 1. So, we have written the given system of equations in this form where we see the vector scalar multiplication vector scalar multiplication and here the vector again the right hand side the scalar means this the number x here.

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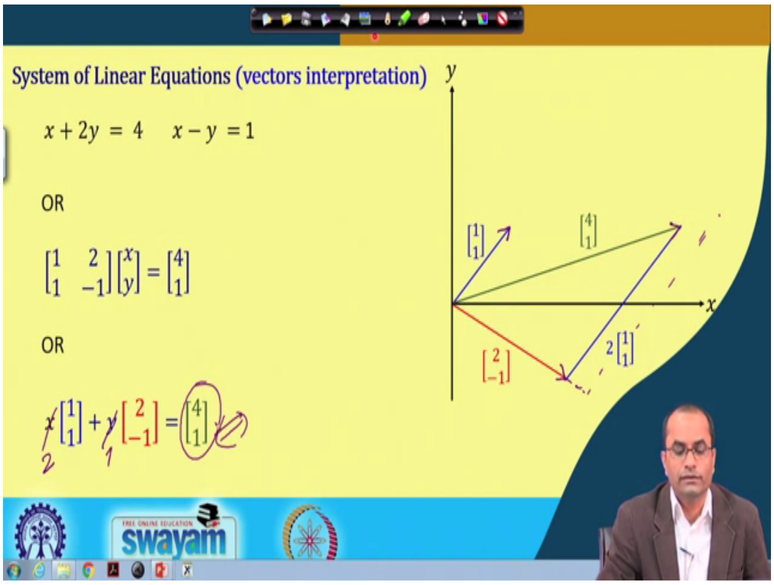

So, now so, this is exactly what is written here that the system of equations whether it is given in the matrix form or we can also write down in this vector form where, x is multiplied to this vector 1 1 y is multiplied to this vector 2 minus 1 is equal to 4 1. So, this is another way of looking at this product here because, in this case we had the matrix vector product. So, we can also look into in this form that this x is multiplied to this first column, y is multiplied to this second column that is a way we also do the product of this matrix and the vector. The right hand side we have this vector 4 and 1.

So, now we will look into this aspect of this vectors that we are adding this vector here 1 1 with some multiplication plus another vector with some multiplier here and that should give us this 4 and 1. So, we will see whether it is possible to have the combination of these two vectors with the addition of these x and y we can get this 4 and 1. So, here we will plot this vector here 1 and 1. So, in the direction of x we have 1 here in the direction of y we have again 1. So, this is one vector which is given by this 1 1 and now we have another vector here this 2 1; so, on the x axis 2 and then 1. So, we have another vector here this 2 1 and the third vector which is seen here in this equation that is the 4 1.

So, let us plot that 2 here; so, that will be 4 1. So, we have these 4 4 vectors and 3 vectors. So, one is 1 1 then 2 minus 1 and then we this 4 1. Now, we will look for the possibility that whether it is possible to have a combination of these two vectors to get this vector third vector this 4 and 1. And, in this particular case this is very easy to see that if we add in this red vector here 2 1; the 2 times of this blue vector. So, 2 times of this blue vector so, you will get exactly that point here or we will get that vector 4 and 1.

So, if we do so; that means, we are adding in this 2 1 vector 2 times of 1 1 vector. So, this length is doubled now 2 times of this vector means the direction will be the same and then the length is just double. So, here we have this 2 times 1 1 and then we exactly getting this vector which was 4 and 1. So, what do we see here in terms of the vectors that 1 time 2 times of this 1 1. So, the 2 times of this 1 1 and the 1 time of this 2 minus 1 will give precisely the right hand side vector.

And what else we see that this is the only possibility to have this vector 4 and 1 because if we add for example, if we disturb this red vector by some multiplication. So, we will be somewhere else and then in this direction whatever way we go we cannot reach to this point. So, this is the unique representation of these two vectors to get this 4 1 vector and that is what this is the case of the unique solution, which we have already observed in the another interpretation of those equations.

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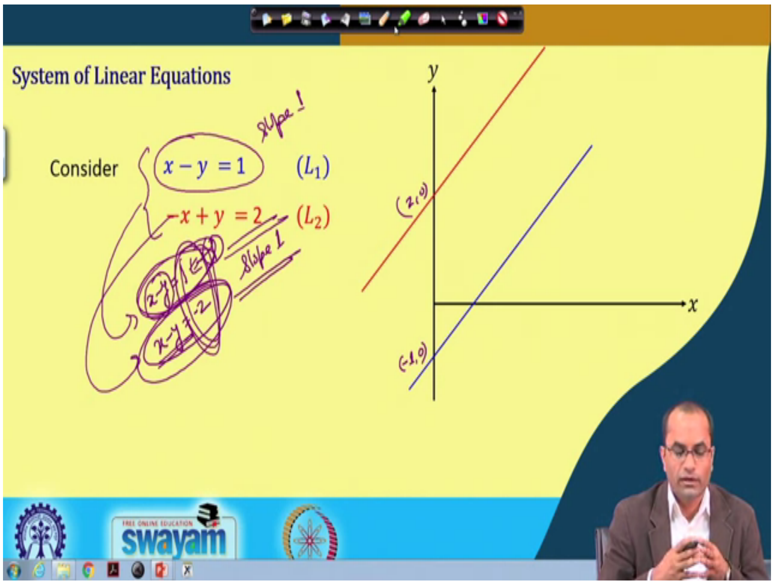

So, now coming to another example. So, we have now x minus y is equal to 1 and the second equation we are considering minus x plus y is equal to 2. So, this is equation number 1 and then we have an equation number 2 here. So, if we plot now these two equations as earlier; so, we have this first equation here where x is 0 and then y is minus

1. So, this is the point here minus 1 and x at 0 and the second line minus x plus 2 is equal to 2.

So, in this case when x is 0 here we are getting the point y x 2. So, this is the point 2 0 here and then we can draw this line which has slope 1 and this line also has slope 1. So, basically what we realize now because both the equations this equation has slope 1 and this equation also have has slope 1. So, both the equations are basically parallel to each other.

So, if we talk about the solution of this system; so, it does not exist in this case because there is no common point on the lines where, we can we can say that this point will satisfy both the equations equation number 1 and equation number 2 because they do not intersect. So, they are the parallel lines and in this case we do not have a solution of this given system of equations. So, with the help of this very simple example the geometrical interpretation itself says that we do not have a solution of this system of equations.

In terms of the equations if we want to see directly because, the first equation is x minus y is equal to 1 and if I multiply here the equation number 2 by minus 1 we will get x minus y is equal to minus 2. So, we have these two equations x minus y is equal to 1 and x minus y is equal to 2. So, this is not possible because in equation number 1 we are telling that x minus 1 the difference of x and minus x and y will be 1 whereas, in the second equation we are telling that the difference of x and y will be minus 2.

So, this is not possible to have a point which satisfy both the equations. And, the geometrical interpretation also says the same that these two lines they do not intersect and hence, we cannot have a solution for this given system of equations.

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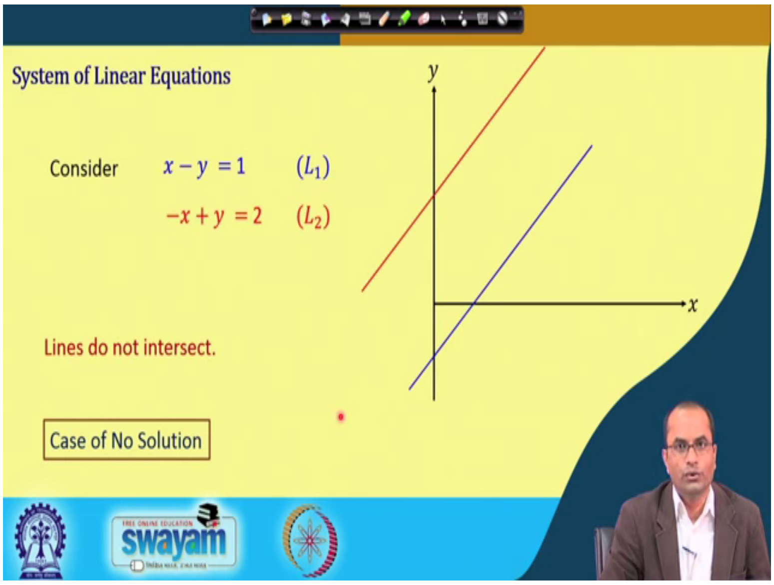

Well so, these lines do not intersect and this is the case of no solution.

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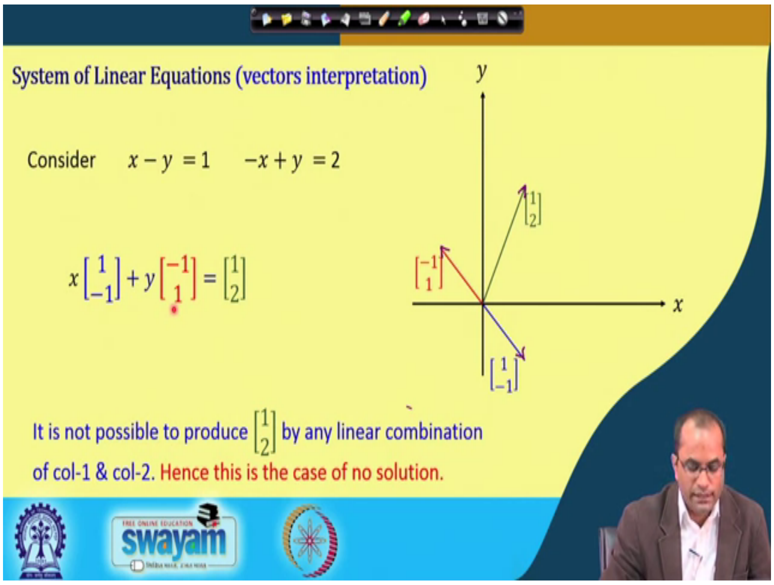

So, now coming to the next interpretation which we call the vectors interpretation. So, we can again rewrite these equation or the system of equation in this form in the vectors form. So, x and multiplied by 1 and minus 1 and y will be multiplied by minus 1 and this 1 the right hand side vector is 1 and 2. So, similar to the previous case let us draw this vector here 1 minus 1. So, this is the vector 1 minus 1 and then the other vector here we have minus 1 and n 1. So, these two vectors so, these two vectors and the third vector is 1 and 2. So, the third vector is this 1 and 2; this is the third vector 1 and 2.

So, now the question is can we add by multiplying some number to this vector minus 1 and 1 And, then the multi we can multiply some number to this vector 1 and minus 1. The question is can we add these two by multiplying some number to get this vector 1 and 2 and which is clearly visible here that this is not possible because, these two vectors are parallel. So, we can get anything in this in this parallel to this these two vectors, but we cannot get for instance here 1 and 2 which is not parallel to these two vectors the given vectors. So, certainly by any combination of these two vectors or rather we usually call it linear combination. And, later on we will come up with more or a better definition of this.

So, here any linear combination of these two vectors will not give us the vector 1 in 2 and hence this is the case of no solution. So, it is not possible to produce this vector 1 and 2 by any linear combination of this column 1 and this column 2 and hence, this is the case of no solution.

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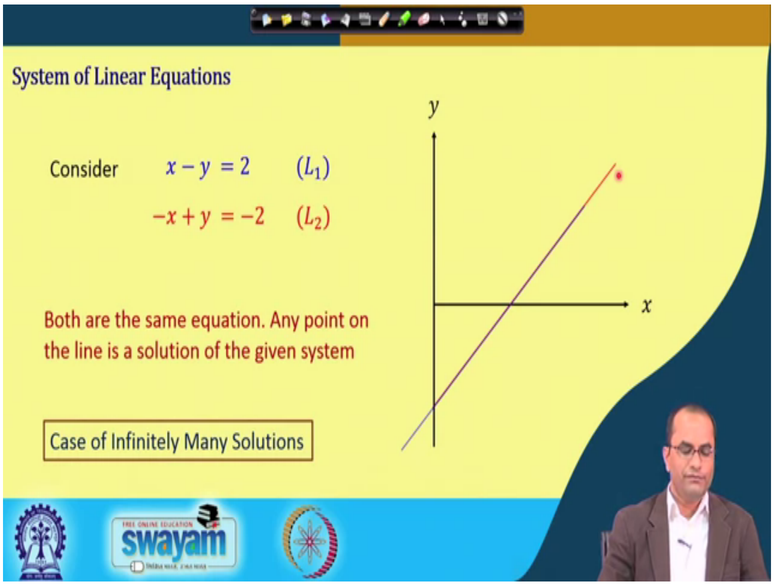

So, coming to the to the next and the last case here where we consider the system as x minus y is equal to 2 and minus x plus y is equal to minus 2. So, this is our system now and again if we look for this geometrical interpretation we need to plot these two lines. So, the first x minus y is equal to 2 which is given by this blue line here x minus y is equal to 2. And, now if we draw this minus x plus y is equal to minus 2 again we get the same line because, this is nothing, but the same equation. Here if you multiply by minus 1 to the second equation you will get the same equation.

So, basically these are not two different equations this is the same equation we have only 1 equations. And therefore, we get this line the red line is sitting over the blue line here. So, they say these two are the same lines and now naturally any point we take on this line I mean they are the same line. So, we can take a point here on the line and that will satisfy both the equations. So, here we have infinitely many possibilities for the solution of the given system of equations.

Because, when these two equations are given it was easy to see that these two equations are the same equation. But, when we have more equations and this is not always the case that we can identify that which equation is a combination of the others for example, example. So, that we will deal little later all those cases, but here for this very simple case we can see this geometrically also that now, in this case any point on the line is the solution of the given system of equations.

So, now this is the case of this infinitely many solutions which we call. So, we have see in these 3 possibilities of the solution. In one case we have the unique solution, in another one we have no solution here we have infinitely many solutions.

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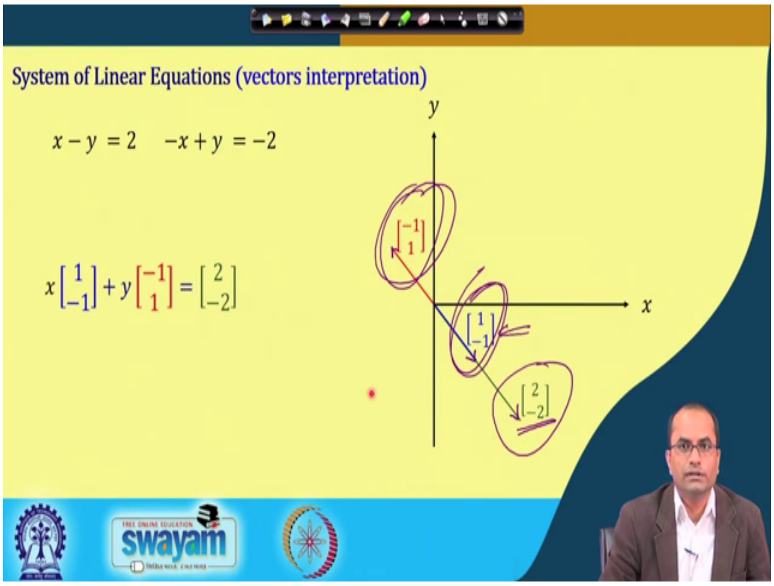

So, let us look for the vector interpretation for this case as well. So, we had these two equations and now we can write down in terms of the in terms of the vectors. So, we have x multiplied by 1 and minus 1 again these coefficients when y multiplied by minus 1 and 1 the another vector and then the right hand side vector we have 2 and minus 2. So, now if we plot these vectors the one here 1 minus 1 then we have minus 1 1 and the other one is minus 2 and minus 2. So, we have these 3 vectors the 1 minus 1 and then we have this 1 minus 1 and the other vector here we have this 2 and minus 2.

So, it is very clearly visible now that this vector 2 minus 1 we can get by one of these vectors or by the linear combinations we can multiplies. There are so, many infinitely many possibilities to have this linear combination of these two vectors to get this vector 2 and minus 1 because, this is parallel to the given vectors. And, and there are infinitely many possibilities now to combine these two vectors to get this vector 2 and minus 2. So, from the vector interpretation as well we can see that there are infinitely many solutions in such cases.

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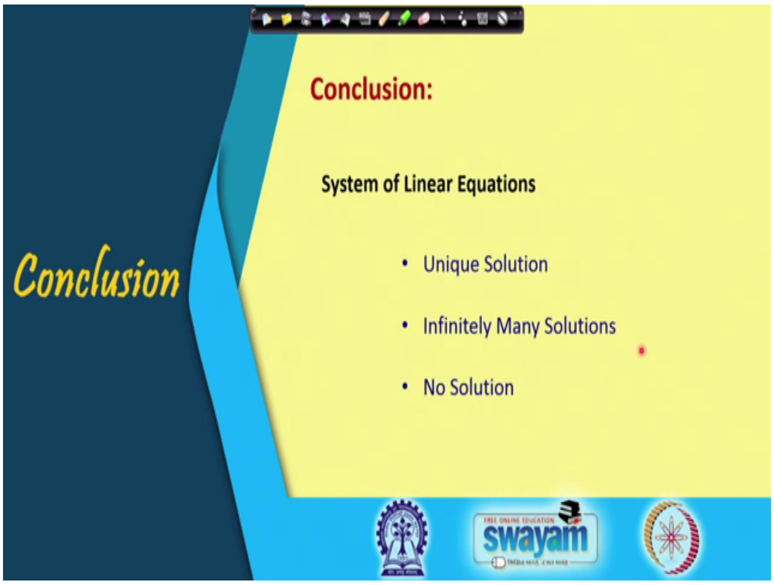

So, and now coming to the conclusion. So, what we have seen? We have seen the 3 cases for the system of linear equations that can happen to a system that the system will have a unique solution and that was the case in terms of the these two equations which we have consider or with two unknowns that there are 2 lines and they intersect exactly at one point. And, this is what we call the case of unique solution that we can also interpret when we have the 3 variables. So, in this case we will have instead of a line we will have a planes. So, there are 3 planes now and they intersect exactly at one point; these 3 planes and that is the solution that unique solution of that system.

And then we have also seen that if the lines are parallel for example, that they never intersect and the same thing we can interpret in case of the 3 variables also that we have the parallel planes. So, here we have seen the in the second case that the lines are parallel. And hence we do not have any solution to the given system whereas, the third case we have seen that that the both lines were basically the same and then any point on the line was the solution of the system of equation. So, we have seen the case of the unique solution, we have seen the case of the infinitely many solutions and we have seen the case of no solution.

And, these all are the possibilities which can happen for a system of linear equations when we consider a large system of m equations with n unknowns. But, we have seen here with the help of very simple examples where we have considered only two unknowns. So, in the next lecture we will be talking about this now the solution techniques to find the solution when we have a large system; not just 2 by 2 which we have considered here for geometrical and the vector interpretation.

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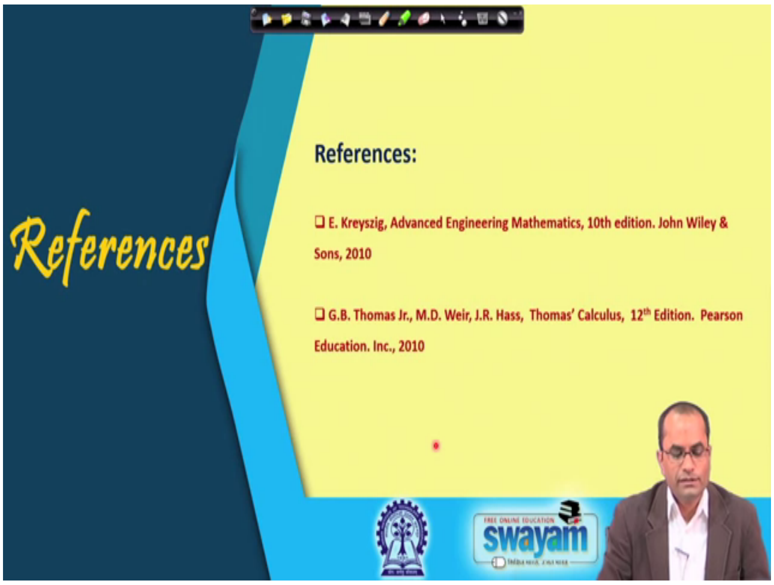

So, these are the references we have used to prepare these lectures and.

Thank you for your attention.