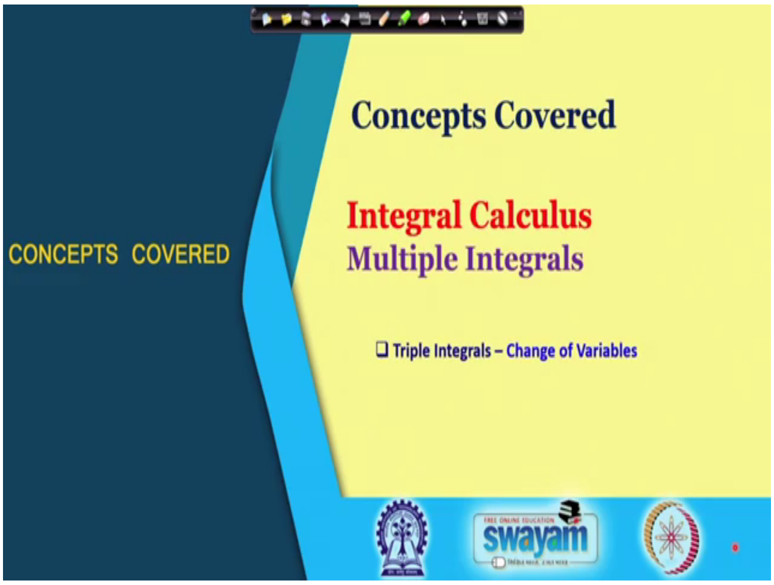
Engineering Mathematics – I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 35
Integral Calculus – Triple Integrals (Contd.)

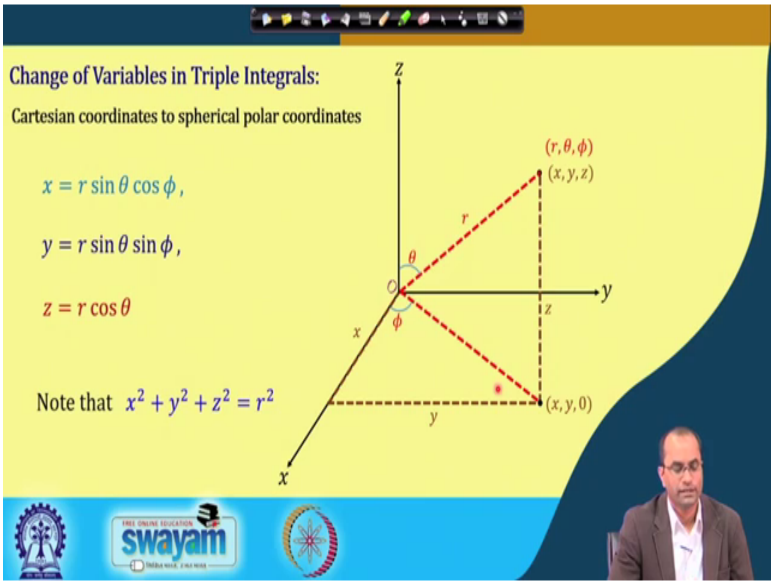
So, welcome back and this is lecture number 35 today we will continue again with Triple Integral.

(Refer Slide Time: 00:25)



And we will look for the change of variables in case of a triple integral.

(Refer Slide Time: 00:29)



So, first you will go through the example where we change from Cartesian to spherical polar coordinates; so a very important the change of coordinates. So, here what we have first let us take these Cartesian coordinates and suppose there is a point here x y z and when we are talking about the Cartesian coordinate; so, this distance now from here to the xy plane. So, if we draw a perpendicular here to this xy plane then this distance is the z co-ordinate from this xy 0 point to this x y z point. And, from the x axis along this y axis the distance of this point which was the perpendicular from this xyz point to the xy point is given by the y and then this x coordinate that is the distance from the origin.

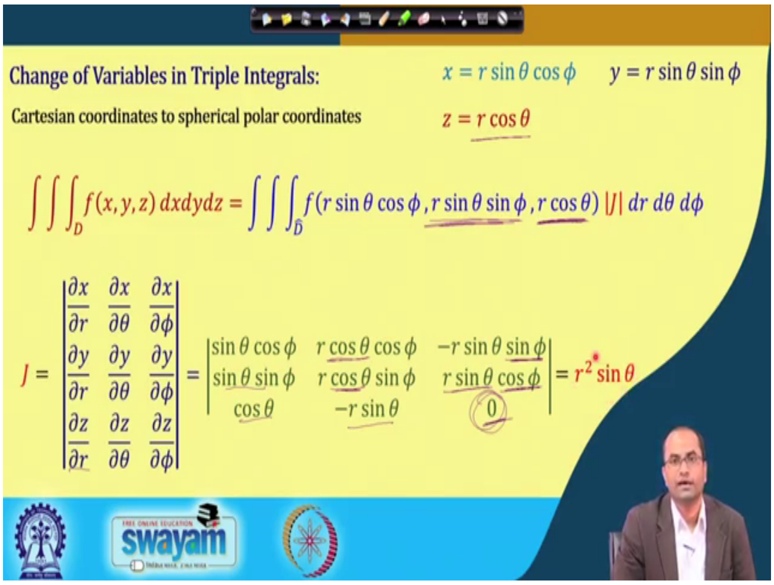
So, here we have in this point as the origin so, from the origin to this point that is the distance given by x. So, this is x here and then along this y direction we have this distance y and then this is the perpendicular distance along the z axis, this is the z. Now, our question is when we change into the; a spherical polar coordinates then what would be r theta and phi. So, the coordinates in a spherical polar coordinates we denote by r theta and phi. So, r is precisely the distance from the origin to this point. So now, in spherical polar coordinate this distance from the origin to this point to the given point will be denoted by r and then these two are the angles; these two are the angles theta and phi.

So, here the angle theta will be the angle to this point which was the perpendicular from this x y z point to the xy plane. So, this is the angle phi from the x axis. So, this here from the x axis to this line which is in the xy plane to this point xy 0 that is the angle phi. And, now this theta angle the theta is the angle with this z axis which this line which is joining this origin to this x y z point in the space that angle is denoted by theta. So, again to represent this point which was x y z in a spherical coordinate and now in Cartesian coordinate and, now in spherical coordinates this is denoted by r theta and phi and now r is the distance from the origin to this point theta is the angle from the z axis.

And, now this phi is the angle from the x axis to the point which was the perpendicular from this given point to the xy plane. So, with this notation of the spherical polar coordinates we will now introduce the change of variables in triple integral. So, the relation between these x y z and this r theta phi is given by x is equal to r sin theta cos phi and y is equal to r sin theta sin phi and z is equal to r cos theta. So, that is the standard relation between this spherical polar coordinates and the Cartesian co-ordinate. And, now we note that if we make this square here for x square and y square and z square because, this term will be r square sin square theta and this cos square phi. Here also we will have r square sin square theta the common and then here sin square phi.

So, again when we add all to this cos square phi plus sin square phi will make it 1 and then we have here also r square this cos square theta and from there r square sin square theta. So, again the sin square theta plus cos square theta will be 1 and you will get simply r square. So, that is the again a relation which we will keep in mind that will be useful in integration. So, whenever we see such a term x square plus y square and z plus z square square and we are going to change the coordinate system. So, we will simply put this x square plus y square plus z square is equal to r square. So now, with this introduction to the change from Cartesian coordinates to spherical polar coordinates we can talk about the triple integrals.

(Refer Slide Time: 05:51)



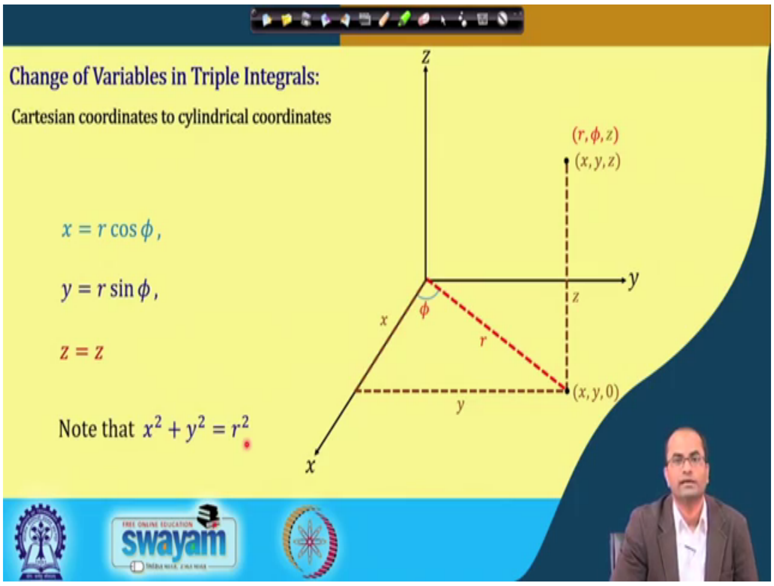
So, here if you have a triple integral given as f x y z and dx dy dz over some domain d and we want to change to the spherical co-ordinate; meaning that x will be r sin theta cos phi y will be r sin theta sin phi and z will be r cos theta. So, with this change of variables now, the idea is that this is the this simple idea which we have also explained in previous lecture; that we need to substitute we need to substitute x is equal to r sin theta cos phi. So, this is your r sin theta cos phi then for y we have substituted here r sin theta, this sin phi and for z this r cos theta which was the relation there z is equal to r cos theta.

So, with this we have now this Jacobian term that is the extra term which comes when we do this change of variables. So, that Jacobian we need to compute and then our dx dy dz will become dr d theta and d phi. So, and then correspondingly the limits will change or the domain in terms of now, r theta phi we have to represent. So, what is this Jacobian here? That is a given definition which we have already discussed that the first rho will be del x over del r del x over del theta and del x over del phi; so the partial derivative of x with respect to r theta and phi.

Similarly, the second rho of this determinant will be a partial derivative of y with respect to r theta phi and then here the partial derivative of z with respect to r theta and phi. So, del x over del r del x over del r when we compute this partial derivative you will have sin theta cos phi and del x over del theta; so with respect to theta so, r and phi will be treated as constant. So, the sin theta will become this cos theta here that is the only change. So, this with respect to theta and then we have del x over del phi so, x was r sin theta cos phi. So, this cos phi will become sin phi and with this minus sin because this derivative of cos phi is minus sin phi. Similarly, we can now compute other the derivative. So, del y over del r will be simply this sin theta sin phi which is the term here and then we have a del y over del theta.

So, this will be with cos theta the rest sin and r will remain and then with respect to phi the sin phi will become this cos phi and r sin theta will remain as it is. Similarly, now with respect to this z with respect to r we will have cos theta here. And, with respect to theta we will have minus sin theta and the r will remain as it is and then we have here with respect to phi of this z. So, z does not have phi it is z is equal to r cos theta and therefore, we have this 0 here because there is no phi term in z. So, del z over del phi will be 0. And, now we can simply a compute this determinant we can determine the value that will be coming here r square sin theta.

(Refer Slide Time: 09:49)



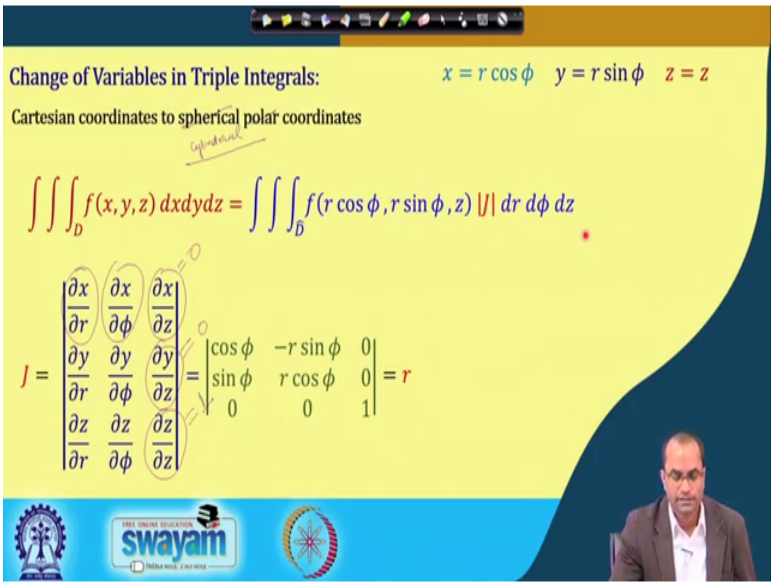
So, now we will remember that this Jacobian term which is coming because of this change of variables in this triple integral the value of this Jacobian term is nothing, but r square sin theta. So now, the Cartesian co-ordinate to cylindrical coordinates. So, again the representation for this point x y z in the Cartesian co-ordinate and it will be represented now in these cylindrical coordinate by r phi and z; so again this triplet with r phi and z. So, z is precisely the same which was in Cartesian coordinate. So, the distance from this point to this perpendicular drawn to the xy plane. So, that distance the third member of this point here is the same as this set in the Cartesian coordinate.

And, now this phi is similar to what we have in the spherical coordinate that is the angle precisely this one which we have already discussed. That is the phi which is from the x axis to this line which is joining the origin to this xy 0 point on the xy plane. And now this r, r is again the distance here from this point to this. Now, that is the difference here in the spherical coordinate we have this r from this point to the given point, but now this r is the distance in the xy plane.

So, basically in the xy plane we are represented by the polar coordinate r is the distance to this point and phi is the angle from this x axis and this z here; it is the same as the given in the Cartesian coordinate. So, there is no change in the z and this xy simply they are given by this r phi and that is nothing, but the polar coordinate in this xy plane.

So, x the relation again since it is the polar coordinate in this xy plane. So, x is r cos phi and y is r sin phi. So, in polar coordinate we used to call this angle theta, but now it is different representation we are calling it phi. So, the relation remains exactly the same x is equal to r cos phi and y is equal to r sin phi and this z is same as the z in the Cartesian co-ordinate. So, this is the simple relation when we talk about the cylindrical coordinate and here this relation again coming from the polar coordinate that this x square plus y square will be r square in this case.

(Refer Slide Time: 12:31)



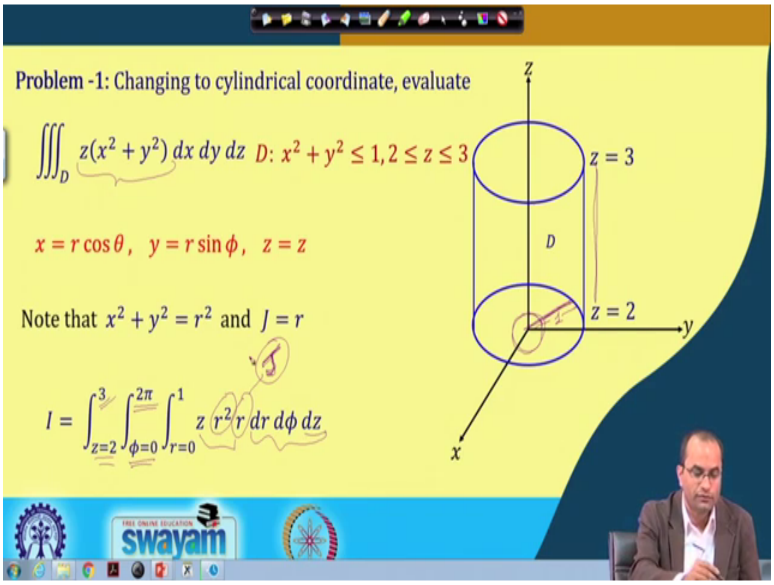
So, here now if we talk about the triple integral and the change of variables; so, we have this integral which is written in the Cartesian coordinates and if you want to make this change of variables here x is equal to r cos phi y is equal to r sin phi and z is equal to z. In that case this integral will be written as so, the direct substitution for x here r cos phi for y we have replaced by r sin phi and z because it was same there.

So, it remains z and then this Jacobian term will come again and dr d phi dz instead of dx dy dz because our variables are r phi and z now in case of the cylindrical coordinate. So, this is the cylindrical not the spherical one, but the cylindrical coordinates; so, cylindrical coordinates. So, in this case we have this Jacobian now which we can compute by this determinant.

Again this first row will be partial derivative of x with respect to r then phi and then z, but there is no z. So, naturally this will become 0 here y also does not have z this will become 0 and this is 1 here. So, these are the trivial situations that this will become 0 because x does not have z, y also does not have z this will also become 0. And this will become 1 and then we can compute del x over del r del x over del phi. So, simply this is coming like cos phi minus r sin phi 0 0 0 and here 1. And, then of y with respect to r that will be sin phi and here sin phi will become cos phi with this derivative 0 and this z has only z. So, there is no r and phi so, naturally these members will be 0 here and once we compute this determinant; so, this will becoming simply r.

So, again because there was no change in z; so, basically this is similar to what we do in polar coordinate. So, we are changing from Cartesian to polar coordinates because the x and y are changed not the z; z remains as it is in the polar in this is the cylindrical coordinates. So, we have again the Jacobian which was also in polar coordinate as r. So, the only change when we do this change in this triple integral this x will be replace by r cos phi, y will be replace by r sin phi and z will remain as it is. And, this factor here this Jacobian will be r and dx d yd dz will become dr d phi and dz.

(Refer Slide Time: 15:23)



So, let us go through some problems where we can use the idea of this changing to cylindrical coordinates in this problem number 1. So, changing into cylindrical co-ordinates we evaluate this integral. So, the integrand is z and x square plus y square dx dy dz and this domain D is given by x square plus y square less than equal to 1. So, that is the disc circular disc and then we have in the direction of z as well; that means, it varies from 2 to 3. So, the z coordinates are directly given that z varies from 2 to 3 and then we have this circular disc here in the xy plane x square plus y square less than equal to 1.

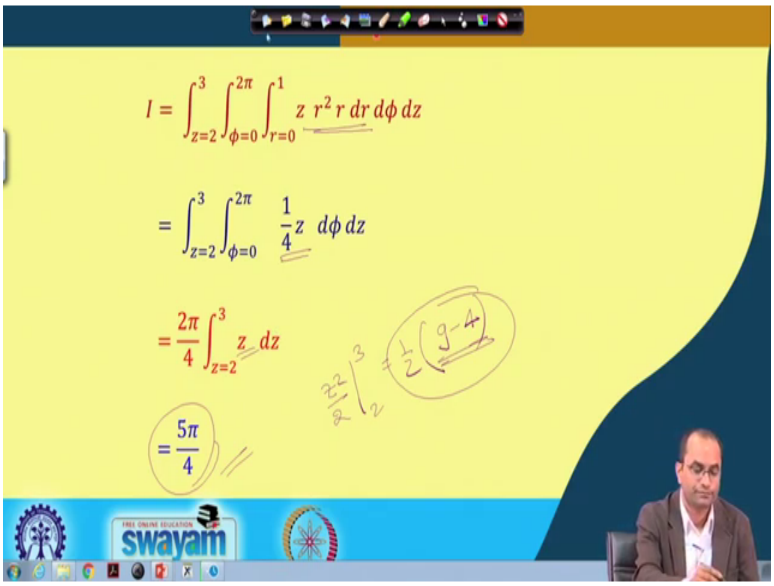
So, this is a cylinder here with radius 1 circular cylinder with radius 1 and it varies in the direction of this z from 2 to 3. So, this is the picture here the x coordinate y coordinate and this z coordinate. And, we have from z is equal to 2 to z is equal to 3 that is the height of the cylinder along the z axis and that is the axis of the cylinder as well. And, then we have the circular disc here which is x square plus y square less than 1. So, the radius of the cylinder is 1 and the height here is from 2 to 3. So, it is natural to use for instance the cylindrical co-ordinates because, this is exactly the cylinder is given and it will be much more convenient, if we use cylindrical coordinate in such problems.

So, we have the relation that x is equal to r cos theta and y is equal to r sin theta and z remains z as usual. We also know that this x square plus y square in this case will become this r square. And, also the Jacobian term which will be requiring for this change of variable of this triple integral and that is in cylindrical co-ordinate that is also r we have just evaluated in the previous slide. So now, this integral the given interval will be represented in this polar coordinate. So, this r here that is for the Jacobian. So, we have this term because of the Jacobian and then this dr d phi and d theta which will be replaced for this dx dy dz with this extra Jacobian term and then the first let us discuss these integrands. So, the z as it is and x square plus y square has become now r square; now we come to the limits here for r.

So, it is very clear or let us first put the limits of the z. So, the z limits trivially given there that z varies from 2 to 3; so, the limits of z 2 to 3. So, once we have covered the limits of z then what is left it is a projection to the xy plane and that is nothing, but the circle. So, here once we have fixed these limits of that z what is left now is a circle in the xy plane which we know already in polar coordinates what will be the limits for the circle.

The r is going from 0 to 1 because, that is the radius here is 1 of the circle. So, r is moving from 0 to 1 and then the theta is moving from 0 to 2 pi the whole circle. So, or the phi in this case we take a in cylindrical polar coordinates here phi. So, phi varies from 0 to 2 pi and r goes from 0 to 1. And, then we have change the integral we have taken care for the Jacobian term and now we can evaluate this integral.

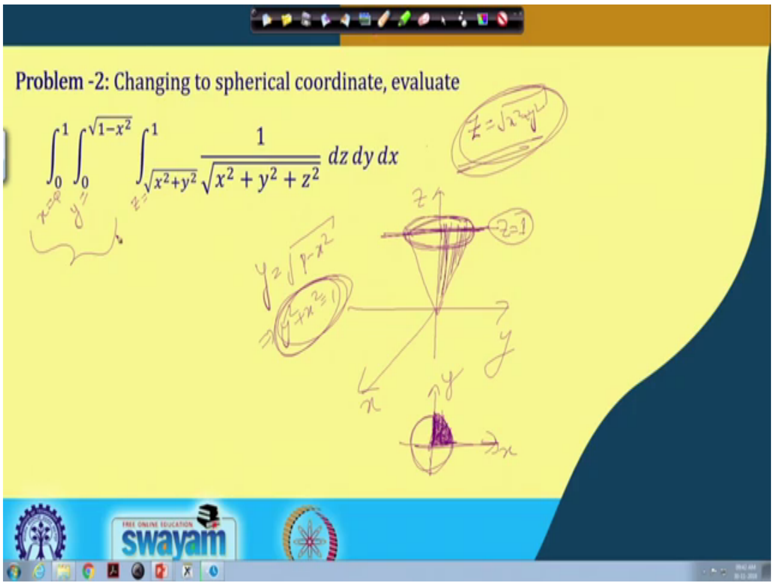
(Refer Slide Time: 19:45)



So, this is the integral we want to evaluate. So, first we will go with this evaluation of the integral with respect to r. So, here we have r cube dr that is the integral and r goes from 0 to 1. So, this r cube when we integrate will be r 4 divided by 4 and when we put the limits there 0 and 1. So, we will get nothing, but 1 by 4 and the z remain as it is and we have two more integrals now with respect to phi and with respect to z. So, with respect to phi there is no phi term in the integral. So, simply it will be phi and then we will put the limit 0 to 2 pi. So, the upper limit will give 2 pi; so, for this integral with respect to phi we will get 2 pi.

So, 2 pi and this by 4 and then we have z dz again this z dz is z square by 2 and then we need to put this upper limit and the lower limit. So, this will be z square by 2 z square by 2 and then 2 to 3. So, this is 1 by 2 and then we have here 9 and minus this 4 so, that is 5 here. So, 5 by 2 and then this 2 will get cancel so, we have 5 by 4. So, that is the value of this given interval and with the help of the cylindrical coordinate which was a natural because, of the domain there was cylinder. And, also the integrand was having this x square plus y square term and that becomes immediately this r square. And, then everything became very simple to integrate once we change to the polar co-ordinate.

(Refer Slide Time: 21:37)



So, we check another problem here changing to his spherical coordinates. So now, we will change to his spherical coordinate to evaluate this integral. So, here we need to be again very careful that what are the limits here? The limits are this x square plus y square to 1 and then we have 0 to square root 1 minus x square and we have for the x 0 to 1. So, we have to identify looking at the limits of the integral that what is the domain because, once we know the domain then it is easier to represent that in the spherical coordinates as well. So, in this case our the limits for this z that is a inner one. So, z goes from x square plus y square to 1 and here this is for y, y goes from 0 to a square root 1 minus x square and this x goes from 0 to 0 to 1.

So, these are the limits; so, here z is equal to x square plus y square that is the first the inner one the most inner one. So, x square plus y square or z square is equal to x square plus y square. So, this is nothing, but that is the equation of the cone we have. So; that means, in the direction of z we are going from this cone which is represented by z is equal to x square plus y square and that is the that is that is the cone here.

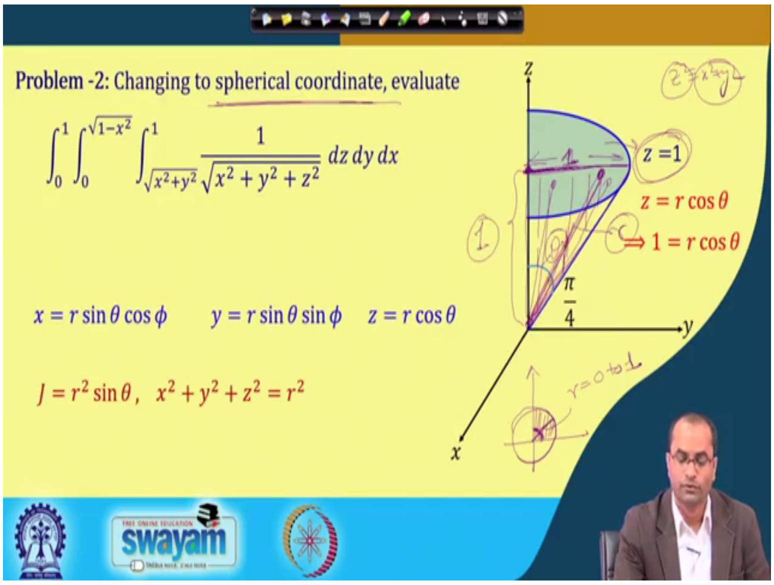
So, this is x and this is y then and this is z here. So, in the direction of z so, that is the that is the cone here this is represented by this equation z is equal to square root x square plus y square. So, what we realize from here? That z is going from this cone to 1 to fix number 1. So, z ends here in the direction of z at z is equal to 1. So, that is a plane a parallel to this xy plane at z is equal to 1.

So, here always the z moves from this cone to this z is equal to 1. So, that is the direction of the z and then if you look closely here y is equal to square root y is equal to square root x 1 minus x square; that means, y square plus x square is equal to 1. So, that is the circle and here x goes also from 0 to 1. So, the y that is a projection of this cone here in the xy plane that will be the projection in the xy plane.

So, here we have x and y if we project this cone we will get this xy I am circle in this xy plane whose radius will be 1. So, this the two outermost limits suggests now that y is moving from 0 to the circle y is from 0 to circle and x from 0 to 1. So, that is the portion of this circle. So, one-fourth of the circle here then the positive quadrant and then we have in the direction of this z up to z is equal to 1 and from the cone.

So, only this portion of the circle and then the corresponding cone here will is represented by these limits.

(Refer Slide Time: 25:17)



So, this is the cone here up to z is equal to 1 and this start from the origin, but as we have just discussed that if this x is equal to 0 to 1 and y is equal to this 0 to square root 1 minus 6 square such that we are in the first quadrant of this xy plane. So, naturally this is the cone which is represented here and now we need to convert into the a spherical coordinates.

So, for a spherical coordinates we have the relation x is equal to r sin theta cos phi y is equal to r sin theta sin phi and z is equal to r cos theta. So, with this relation we can easily compute this Jacobian which we have already discussed that is r square sin theta. And, we have also this relation here x square plus y square plus z square is equal to r square. And, now we need to compute because we need this angle here which is which is required in this is spherical coordinate which is represented by this theta. So, the theta limits so, just to note this that z is equal to 1.

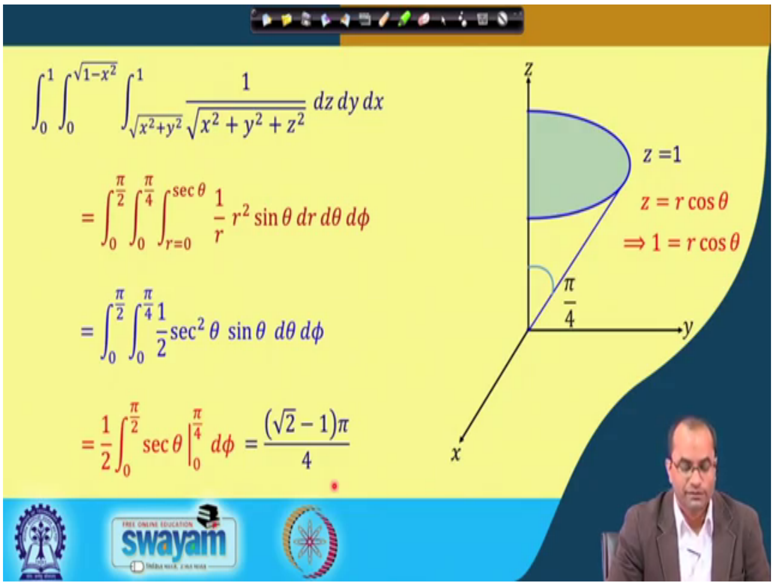
So, this is the z is equal to 1 here. So, from here to here we have this distance 1 and then this circular part also has radius 1, that has also radius 1. Because, that is what we have observed here that this z square is equal to x square plus y square. So, when z is 1 this is basically ending with the circular disc as x square plus y square is equal to 1. So, this distance is 1 this distance 1 so, this is the maximum angle we have for this theta. So, theta angle to cover the whole this given cone will be from 0 to pi by 4 that will be the limit for the theta. And, for the xy axis for this phi and r we have already observed that this is the portion we are going to cover it now. So, the r will go from 0 to 1 that will be r from 0 to 1 and for phi now.

So, phi will be 0 to pi by 2 to cover this portion there. So, once we have covered in this one direction as discussed before; that means, in the direction of z we are; yes we have not discussed for the z here. So, the limits of this that will be from the cone and to this z is equal to 1. So, from this 0 to we are going to this yeah we are talking about the spherical coordinates. So, we need r so, r will be this distance from here to the points the maximum distance there on the circular disc.

So, z is equal to 1 that circular disc is nothing, but z is equal to 1 there. So, we are moving from 0 for r we are moving from 0 to this disc here 0 to that disc here 0 to this disc here. So, that z is equal to 1 will suggest us now for the limits of the r because, z is equal to already given r cos theta. So, this r cos theta is equal to 1 r cos theta is equal to 1 will give the r because, it depends on this angle theta.

The r depends on this angle theta here; for example when theta is pi by 2 this r is nothing, but 1; but here it is different and it is varying with r theta. So, the r will be given by 1 over cos theta and this theta we have discussed already 0 to pi by 4 and for phi we have discussed that it will go from 0 to 0 to pi by 2. So, this was for r phi and theta we have discussed for all.

(Refer Slide Time: 29:29)

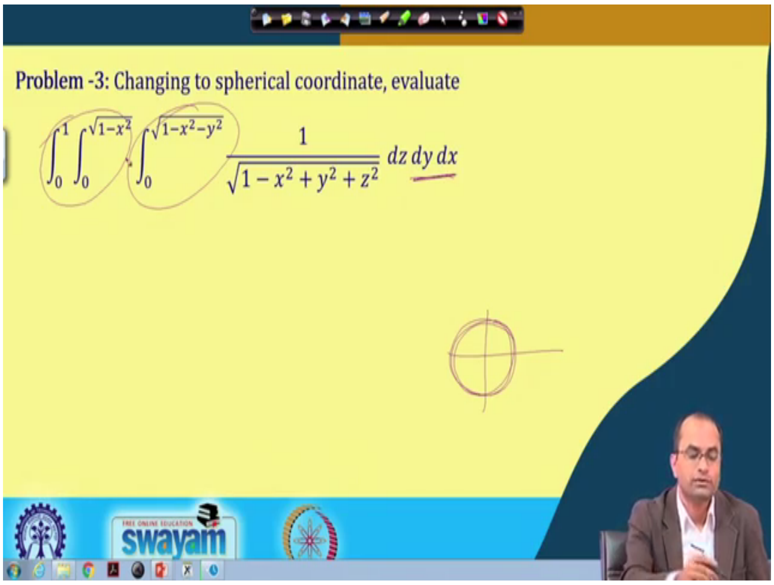


So, we can now change the variables. So, this was the integral there and that will be converted. So, for r that is the distance from here to that point in a spherical polar coordinates not in the cylindrical. Cylindrical we have only in the xy plane we have to look for the theta and phi and r. So now, we are in the spherical coordinates. So, that r is the distance from this point to this where we exit this domain. So, that is exactly the z is equal to r where r is given by 1 over cos theta. So, this is r from 0 to 1 over cos theta that is a sec theta here and for the limits of the theta we have already discussed that is the 0 to pi by 4. And, for phi we will be moving from 0 to pi by 2.

And, that is 1 over this r here because x square plus y square plus z square is r square. So, that is 1 by r and this r square sin theta that is the Jacobian term r square sin theta that is the Jacobian term and then we have this dr d theta d phi. So, now we can easily integrate this one. So, first with respect to r because this r dr so, we will have r square by 2; that means, sec square theta by 2 and then we have sin theta here. So, this is nothing, but the sin theta and this 1 sec theta we will go to 1 over cos theta; so the tan theta sec theta.

So, sec theta tan theta with this half again we want to integrate with respect to theta. So, the integral of sec theta tan theta will be again sec theta and the limits here 0 to pi by 4 and this factor half is sitting there already we can substitute these limits. So, this sec pi by 4 will be square root 2 and then minus 1 there and 4 phi you will get this pi by 2. So, this 1 by 2 will make this 4 so, that is the value of the given integral square root 2 minus 1 pi by 4.

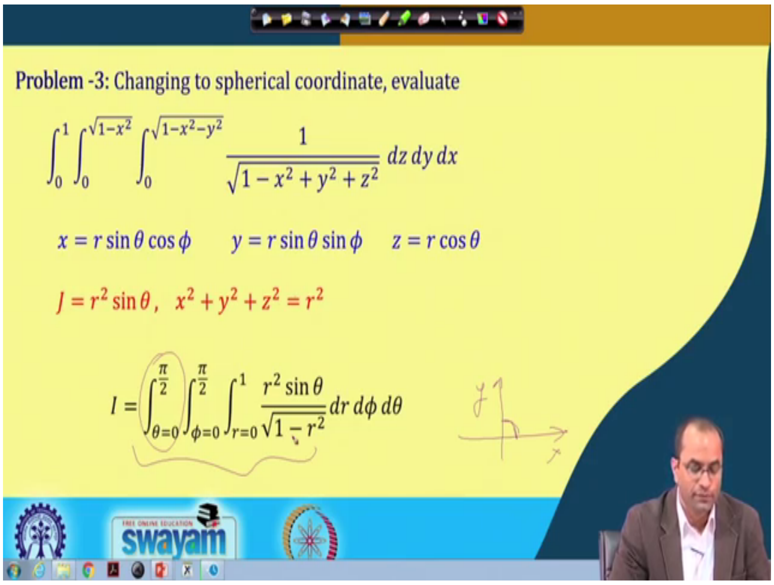
(Refer Slide Time: 31:35)



The last example where we will be changing again to its spherical coordinate to evaluate this integral. And, again the change into a spherical coordinate we have to look for the geometry what we are having in this integral limits. So, for the inner one z goes from 0 to square root 1 minus x square minus y square; that means, z square plus x square plus y square is equal to 1. So, we are moving from in the direction of z from 0 to that is sphere of radius 1, that is the equation z is equal to square root 1 minus x square minus y square that is the sphere.

So, we are moving from 0 in the direction of z to that sphere. So, we have to now look a little bit more careful. And then in the direction of this y and this x this is nothing, but that circle only when this is sphere is projected to this xy plane. And, then we have x from 0 to 1 and y from 0 to 1 by square root 1 minus x square. So, that is precisely in the xy plane what we have that is the values for this y and x that is a circle circular part of these limits and then here in the direction of this z we are moving from 0 to the sphere 0 to the sphere. So now, again this is natural to consider a spherical coordinate which will convert this.

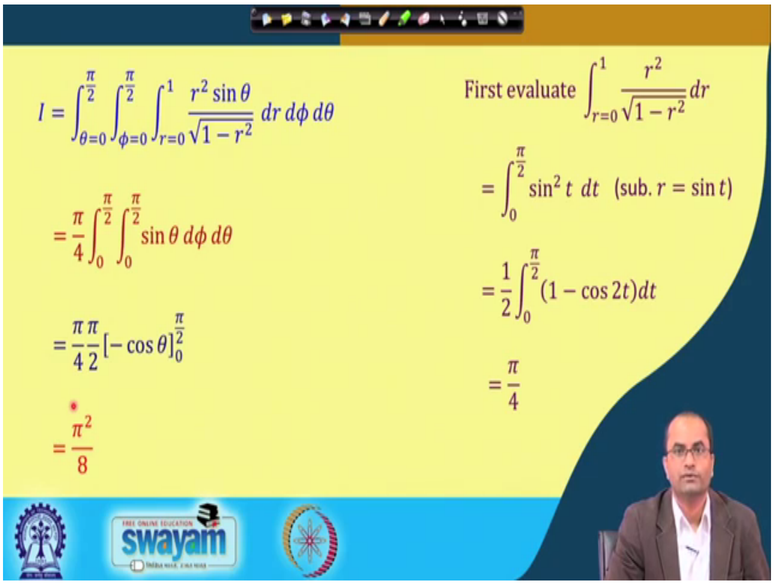
(Refer Slide Time: 33:13)



So, x is equal to r cos theta y is equal to r sin theta and z is equal to r cos theta, the J will be the r square sin theta and again this relation. So now, this integral will be converted to this integral. So, we have this Jacobian term we have 1 over square root 1 minus r square and then dr d phi d theta. So, now for this r the limits of r always from this 0 to that sphere 0 to that sphere; so the distance from origin to that sphere is constant that is the radius of this sphere because, the sphere has the centre 0 and radius 1.

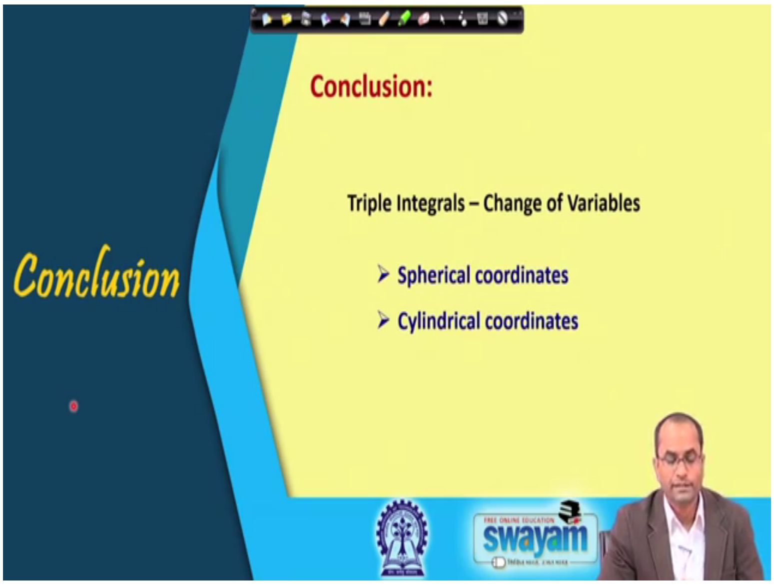
So, that radius is 1 so, r goes from 0 to 1 to that is sphere and then we have to see about this angle of phi which is yeah the angle phi first. So, which is from this x axis so, that is 0 to pi by 2 because, that was 0 to this square root 1 minus x square. So, that is in the xy plane and then we have the limits of the theta. So, again to cover this sphere in this positive this octant so, we have to go from 0 to pi by 2. So, with these limits we have covered all these the whole given spherical part.

(Refer Slide Time: 34:43)



And now we can easily integrate because with respect to r we have here r square over 1 minus r square. So, first we are evaluating the inner integral r square divided by square root 1 minus r square. So, here by substitution this r is equal to sin t we can easily get this sin square t which will be converted to 1 minus cos 2 t which we can integrate now and the value will be pi by 4. So, the value of this inner integral is pi by 4 and then we have sin theta a d phi d theta. So, this again this d phi will give us pi by 2 only and then the outer one sin theta will be cos theta with minus sin and 0 to pi by 2. So, we will get pi square by 8 the value of this integral.

(Refer Slide Time: 35:29)



So, what we have seen now at least two special cases where the we have considered this spherical coordinate and also the cylindrical coordinate; in this triple integral the concept was this change of variables where the Jacobian we need to convert. And, the most difficult part again in this triple integral is finding the limits corresponding to the given space here. In spherical we have to represent the coordinates in spherical coordinate when we are changing or in cylindrical coordinates, but the evaluation become much easier if we have suitable change there in variables.

(Refer Slide Time: 36:11)

So, these are the reference we have used.

Thank you very much for your attention.