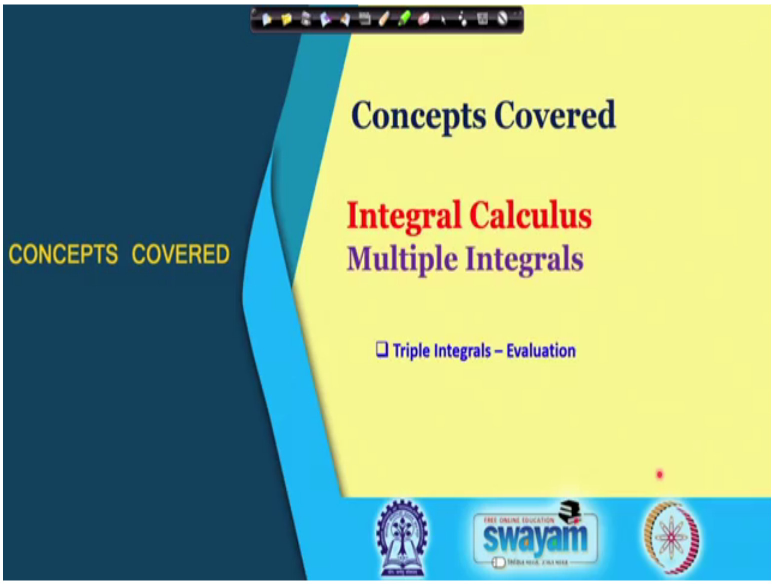
Engineering Mathematics – I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 34
Integral Calculus – Triple Integrals

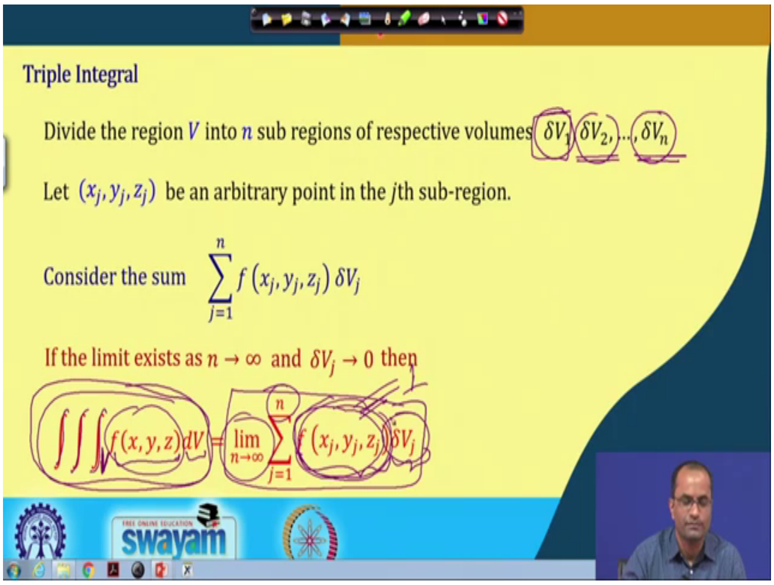
So, welcome back and we will continue now this Triple Integral. So, we have already finished evaluation of double integrals and many other applications of double integral.

(Refer Slide Time: 00:27)



Today we will learn about the triple integrals.

(Refer Slide Time: 00:31)



So, basically the evaluation part we will look into in this lecture. So, first how to define this triple integral? So, we need to similar to what we have done for the double integrals we divide the given region so now, our region will be a 3 dimensional in the 3 dimensional space. So, we will divide that region into n sub regions and we call the volume of these sub regions by this delta V 1 delta V 2 delta V n. So, we have a 3 dimensional a space now and we have divided that into small regions and for each region we are denoting its volume by this delta V 1 delta V 2 and delta V n.

And, now we let a point x j y j and z j an arbitrary point in j'th sub region. So, it is very similar to what we have done in case of 2 dimensional; we have divided the domain the 2 dimensional domain into a small sections, small cells. And, then in each cell we had taken a point and the corresponding value of the function at that point was multiplied by the area and that was summed up and taking the limit of that sum we introduce the double integral. So, here as well so, we will consider this sum now. So, sum over these all the volumes delta V j multiplied by the function value.

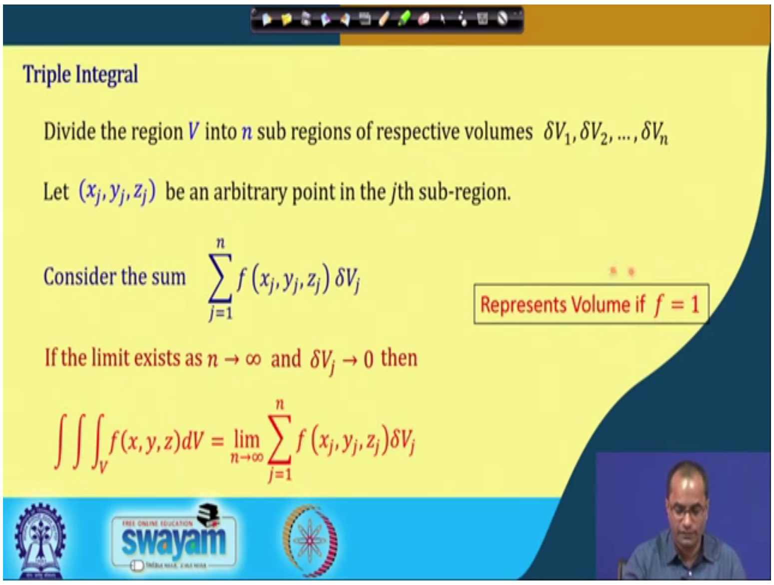
So, the function value at this point x j, y j, z j which we have considered in j'th sub region. And we are adding all these regions or this multiplication over these all regions; that means, this j is varying from 0 to n. And, now if we take the limit here as n goes to infinity or other way around that this delta V j we will go to 0 because the domain is fixed now. So, if we are letting this n goes to infinity then the sub regions will go to we infinity; that means, the volume of each sub region will go to 0.

So, taking if this limit exists as n approaches to infinity in that case we call this as integral and the notation for this integral in the 3 dimensional case or for the triple integral we use this notation. So, we have now the 3 symbols for this integral over that volume here which is a space in region in the 3 dimensional plane. Then we have the function here f x y z integrating over that volume in space and then the definition is coming again from the sum when we have added this f with this delta V j and at the end taking this limit as n approaches to infinity. So, again the physical interpretation if we want to take a quick look so, if for example, this f is said to V 1.

So; that means, there is no f here so we are basically adding this delta V j the volume of each sub region and then taking the limit. So, by considering this f as 1 we will get nothing, but the volume of that sub region again. So, using the double integral also we have computed the volume that was one of the applications of the double integral, but in that case we considered that f in the integrand. So, the double integral with that integrand that surface and was giving us the volume but, now in this case when we set this f to 1 then we are getting actually the volume of this domain there; the domain of the integration which is denoted by V in this case. And, we had also the application they are finding the area of the domain in case of the second in case of the double integral and there also precisely we have set this function to 1.

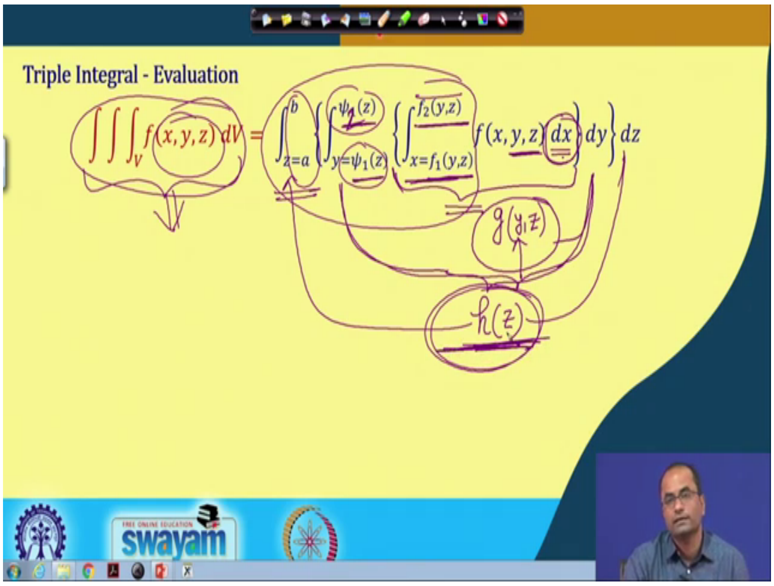
And, just integrating without this function we were getting the area of the domain of integrations. Or, in this case when we said this f to 1 you will get the volume of the domain because now, we have a 3 then we have a volume in this 3 dimensional space.

(Refer Slide Time: 05:29)



So, it represents the volume if we say at f is equal to 1. So, that also we can look into as one of the applications of the triple integral.

(Refer Slide Time: 05:43)

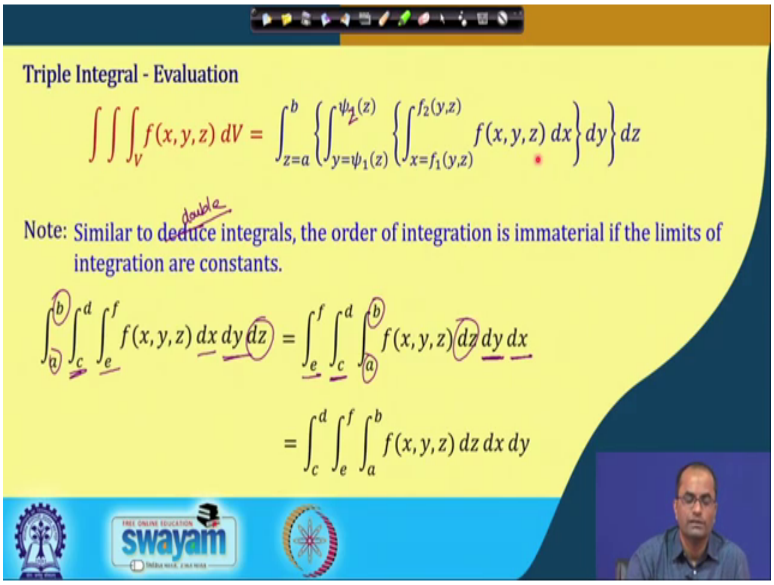


So, how to evaluate the triple integral? So, having the knowledge of the evaluation of the double integral, this will be also easier. So, this is the idea of this iterative computation of single integrals and then so, like for instance if we consider that is the most general form is given. So, this is the inner most inner one into integral this is taken with respect to x. So, the limits for this x in a very general case can be a function of this y and z to the function f 2 their y and z. So, as a result when we integrate the inner one we will get some function because, over x we have integrated. So, we will not see x anymore, but the limits can be here the function of y z here can be the function of y z and then we will have also the y z in the integrand.

So, in general the after evaluation of the inner integral we will get a function of y and z. So, then once having done the evaluation of the inner integral we will go to the outer one now with respect to y. So, we are integrating now with respect to y and the limits of the y can be the function of z can be the function of z. So, after the integration of this second inner integral, we will get a some function of z and now at the end we will integrate this x z with respect to z. And, now the limit has to be constant because always remember the value of this integral is a constant and you will not see anything of x y and z because, we are integrating over the volume this V or over all the points this xyz in our volume. So, at the end this will be free this integral value will not have anything of x y z.

So, once we have the evaluation of the inner 2 integral that we are getting like a function of z and now we will integrate with respect to z. So, for the outer integral now the limits must be constant. So, this is a general consideration which we should always keep in mind while putting the limits of the of the integral. So, the inner one when we are integrating with respect to x the limits can be the function of y and z both the limits are lower and the upper one. After this integration the x is removed we have the function of variable y and z and then we are integrating with respect to y the limits still can be the function of z like here psi 1 z and psi 2 z. So, these limits can be the function of z and the outer one then that has to be the constant limits for z and now we will get a value of this integral which is free from this x y and z. So, this structure we must keep in mind.

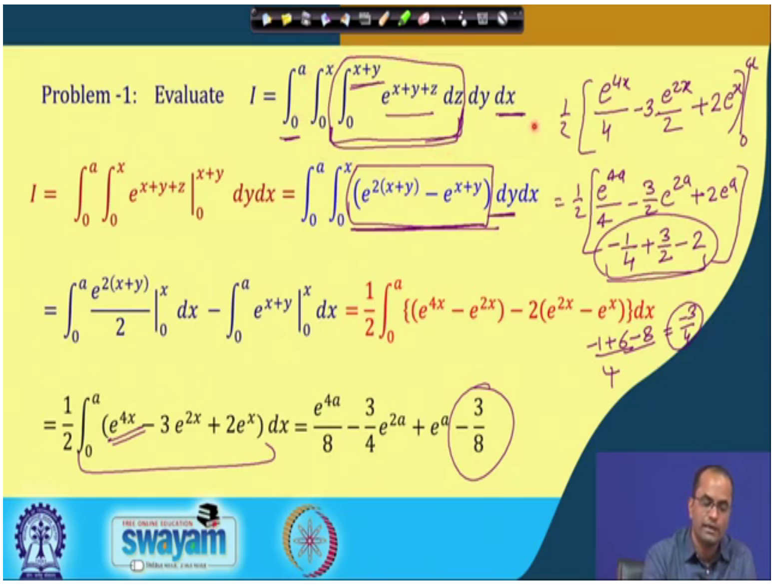
(Refer Slide Time: 09:09)



And then similar to the double integrals the order of integration is immaterial, if the limits of the integration are constant. So, similar to double integral; similar to double integrals the order of the integration is immaterial if the limits of the integration are constants. So, if we are working with the constant limits for instance here the limits of x are e to f these are the constants y is c to d and z is a to b. So, if these limits are constant then we do not have to worry about the order of the integration and simply we can change this order here. So, x y z is now z y and x and now the limits of z will be a to b because, in this integral here the limits of the of the z was a to b. So, here also the z will be from a to b; similarly the limit of y c to d here also the limits of y c to d and then the outer one is a limit of x.

So, here the e to f so, here also we have e to f. So, simply we can change the order if the if the if the range here of integration or the limits of integration is or are constant. Also we can further change like the order dx dz dx dy and that corresponding limits will against it over the integrals. Once we have the functions here as above in a very general case then we have to be or it is more difficult to change the order of integration. And we have to be very careful now, once we change the order similar to what we have done in case of 2 variables. Once we change the order again we need to be very careful about the finding the limits of the integration, if these limits are not constant; if the limits are constant one can simply change the order.

(Refer Slide Time: 11:19)



So, here now we will go for the evaluation so, that is the first example we will consider the exponential function e power x plus y plus z the very simple function and we want to integrate with respect to z y and then x. So, the inner one is with respect to z, the limits 0 to x plus y and for y the limits are given as 0 to x and for x the limits are given as 0 to a. So, if you want to integrate this so, first we need to integrate with respect to z. So, with respect to z when we integrate this function e power x plus y plus z. So, naturally this x plus y will be treated as constant and we can integrate this. So, the integral of the e power x plus y plus z will be again the exponential function e power x plus y plus z and there is no factor or anything sitting with the z.

So, the integral of this will be the same function exponential function e power x plus y plus z and we have to put now the limits. So, the lower limit is 0 the upper limit is x plus y we can substitute that. So, once we put this limit of z x plus y so, we will get this minus times x plus y x plus y was already there then z is substituted as x plus y. So, we are getting exponential power 2 times of x plus y and minus the lower limit so, z is set to be 0. So, we will get e power 2 times this x plus y minus the z is said to be 0 and we get e power x plus y and now we have only the 2 integrals left with respect to y and then with respect to x.

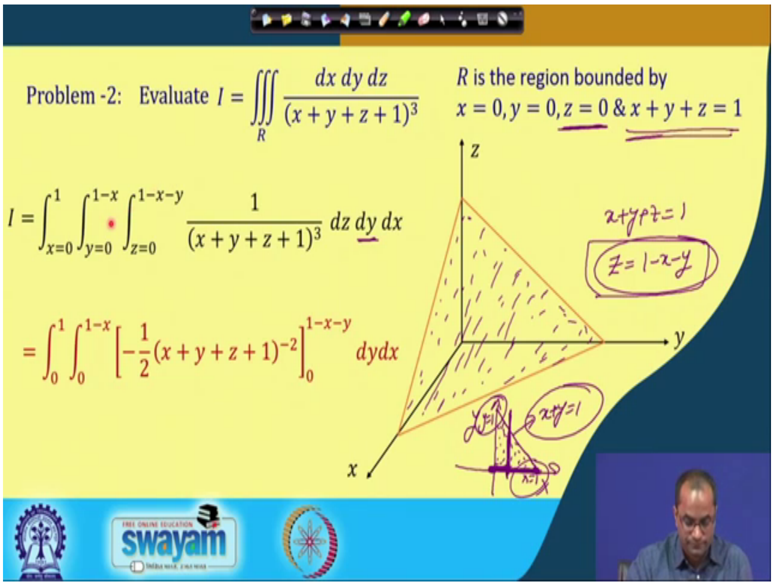
So, now we will solve for with respect to y. So, the order has to be the same we cannot integrate for example, first with respect to x because, we have the x in the integrand, we have also the x sitting in the limits. Same here we have to do it first with respect to z only because, the z the integrand contains x and y also the limit contains x and y. So, after this integration of the inner one we got the function of this x and y e power 2 times x plus y minus e power x plus y. So, the order has to be the same here first the inner one then the outer and so, on in the sequence. So, here when we get this function of x and y we can integrate now with respect to y; so, the integral of this with respect to y.

So, e power 2 times now we have the 2 here 2 times x plus y. So, the 2 will go to the denominator here the limits 0 to y, now the outer integral dx minus 0 to a this integral and the integral of this e power x plus y which is again e power x plus y and the limits of the inner integral this here 0 to x. So, we can substitute now the above limit here with the rest here y will be replaced for x. So, we will have 2 x and then we have 2 there. So, we will get this 4 x the exponential 4 x with the half factor already there and then we will put y 0s will get 2 x. So, this the integral will give us e power 4 x minus e power 2 x.

From here again we will get e power 2 x and minus e power x. So, this minus sign and this half we have taken common from both the term. So, we are mixing now these 2 integrals here to 1. So, this 2 factor will come because we have taken this half outside the integral. So, this 2 will come there and e power 2 x minus e power x. So, that is the integrant now which we can simplify. So, this is e power 4 x and these two will give minus 3 e power 2 x and then we have here minus plus. So, this is e power 2 x so, 2 e power 2 e power x. So, this again very simple to integrate which we will get e exponential here this 4 x. So, again let me just evaluate little bit. So, here we have e power 4 x. So, this integral will give e power 4 x over 4, the second one we have the 3 e power 2 x by 2 and here we have 2 and exponential x.

Then we have the limit 0 to a the limit 0 to a and this half factor is also sitting there so, half. Now, this half and when we put a there so, e power 4 a by this 4 minus 3 by 2 e power 2 a and then we have 2 e power a as it is, then with the minus sign the 0 will put this 1 over 4 and here again the 0 will give 1. So, 3 by 2 and here this will be minus 2. So, and this is precisely what we have here e power 4 a by 8 and minus 3 by 4. So, 2 2 4 here we have e power 2 a and this 2 with this half so, we get e power a here and once we make this calculation. So, this is 4 then we have minus 1 then 6 and minus 8. So, we are getting minus 3 by 4 and this 1 by 2 is there so, minus 3 by 8. So, that is the evaluation of this simple integral. So, we have to again iteratively solve the integrals like and then these becomes single integrals. So, first with respect to z, then with respect to y and at the end with respect to x.

(Refer Slide Time: 18:09)



So, next we have this problem 2 where we will evaluate this integral and this R the region R is given bounded by this x is equal to 0 ah. So, y is equal to 0 line and z is equal to 0 line or basically these are the planes now. So, we are talking about the 3 dimensional space. So, this x is equal to 0 is the y z plane here this y is equal to 0 is the xz plane and this is xy plane and we have another plane here x plus y plus z is equal to 1.

So, this is cutting at this x axis by 1 and also at z again at 1 and x again at 1. So, we have this slanted plane here and we have these 3 coordinate planes x is equal to 0 y is equal to 0 and z is equal to 0. So, this is the situation the x axis y axis and z axis and then we have this plane which is given by this x plus y plus z is equal to 1, that is this plane given by this. So, we have the 4 planes the 3 a standard coordinate planes and then we have the one this plane x plus y plus z is equal to 1.

So, now we have to fix the limit for this region R and the R is the bounded by these 3 3 planes. So, putting the limits now; so, we have the integrand 1 over x plus y plus z plus 1 power 3 with let us fix first with respect to z then with respect to y and then with respect to x we can change the order also and accordingly the limits will change. So, with respect to z now so, in the direction of z; so, we have the x axis and we have the y axis and then we have the z axis and there is a plane here x is equal to y x plus y plus z is equal to 1. So, now in the direction of y we are fixing first so, we are going always from z is equal to 0; we are entering this 3 dimensional space and we are exiting through this plane here x plus y plus z is equal to 1.

So, this x plus y plus z is equal to 1 so, from here we can get this z so, 1 minus x minus y. So, we are entering into this 3 dimensional space this volume through this z is equal to 0 because that is the plane we have as xy plane. So, we are entering through this xy plane and then leaving the domain always from this plane which is given by x plus y plus z is equal to 1 so; that means, there z is 1 minus x minus y. So, for the limits of this z for the limit of z we have z is equal to 0 to this z is equal to 1 minus x and minus y so, the limits of z are fixed now. So, now what we have to do, we have to project this now in the xy plane.

The our volume we need to project now in the xy plane because, in the z axis along the z axis we have computed the limits. So, z goes from 0 to 1 minus x minus y and now if we project this in the xy plane so, what we will get. So, if we have the x axis here we have the y axis then this will be the projection; this is cutting at x is equal to 1 here also at y is equal to 1. So, this is now the projection in this xy plane. So, we need to now fix only the limits here for x and y; this equation of this line will be just setting this z to 0. So, we have x plus y is equal to 1. So, this is the projection now and we can easily because now we are in the 2 dimensional case and we know how to fix the limit.

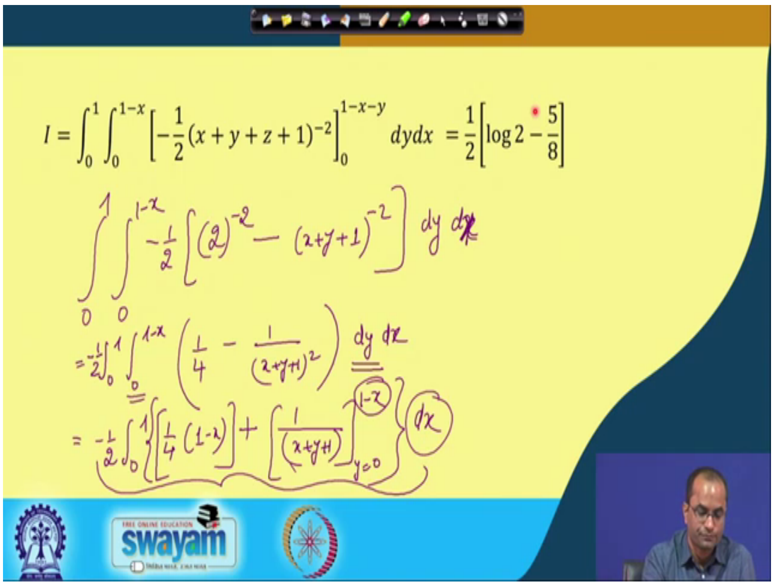
So, the limits of the y now will be; so, y for y we are now entering into this domain here the projected domain from y is equal to 0 and we are leaving to y is equal to 1 minus x. So, the limits of y are also clear from y is equal to 0 and leaving the domain to y is equal to 1 minus x there and now what is left its 4 x. So, naturally again we can project everything to the x axis that is another way of looking at. So, if you project now this again to the x axis we will get simply this 0 to 1 this line segment over the x axis from 0 to 1.

So, for the x part for the limits of x will be now from 0 to 1. So, the most difficult part for in this 2 dimensional computation will be always this finding the limits. The trick is for the inner one for the z we have to look into the hole this 3 dimensional volume and see that with respect to z, where we are entering the domain and at what point we are at which surface we are exiting this a volume. So, in this case it was rather simple, the surface where it enters that was the xy plane; that means, the z was 0 there. So, it is entering to this volume from the z is equal to 0 and the exit point from this volume along this z axis was this plane z x plus y plus z is equal to 1 or z is equal to 1 minus x minus y.

So, the fixing the limits of the z was trivial in this particular case that z goes from 0 and it leaves at z is equal to 1 minus x minus y. Once we have fixed the limits for y were for z now we can project our this domain of integration to the x y plane because for z we have fixed already. So now, we will project to the xy plane while, projecting this volume here to xy plane we will get this triangle nothing else, but this triangle which cut here at x is equal to 1 and at y is equal to 1 in along this x and y axis. And, now in this 2 dimensional we know how to fix the limit. So, in this case for the limits of this y we have from 0 to 2 this line which is 1 minus x and outer integral the x was then left from 0 to 1 which is given here.

So, having these limits now we can easily compute because this is 1 over x plus y plus z plus 1 power this 3 here and which we can integrate. So, we will get minus power 2 and this 0 to 1 minus x and minus y.

(Refer Slide Time: 25:41)

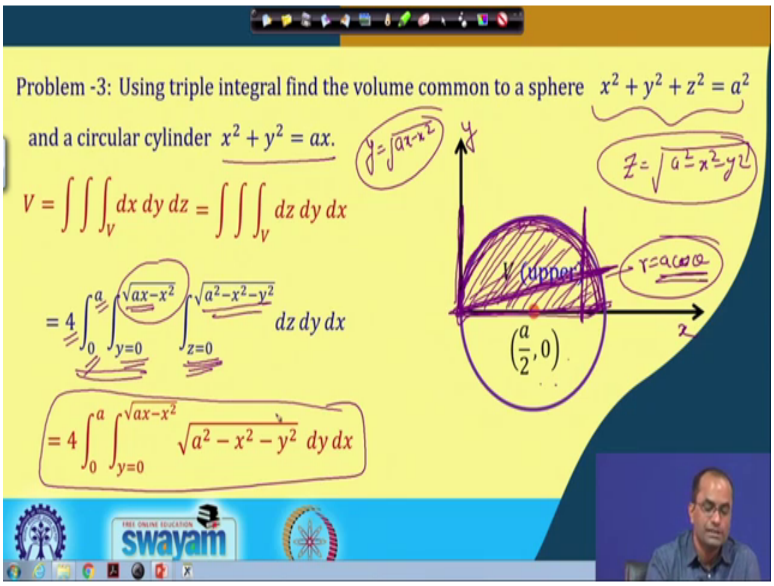


So, this is the integral we want to evaluate now so, this when once we put the upper limit here with respect to z. So, what we will get this minus 1 by 2 will be outside here. So, when we put z is equal to 1 minus x minus y so, y and x will cancel and this 1 will add. So, we will have 2 power minus 2 and then minus this z will be set to 0. So, we will have x plus y and plus 1 power minus 2 and then we will have the dy and this dx dy and dx there and the outer this integral 0 to 1 and 0 to 1 minus x.

So, we can simplify further so, we have 0 to 1 and 0 to 1 minus x with this minus half there and this will be 1 by 4 this will be 1 over this x plus y plus 1 minus 2. Then plus 2 in the denominator and then dy dx, now with respect to y we need to integrate. So, here in this place we will get 0 to 1; 1 by 4 and with respect to y. So, this will be just y and when we put the limits so, we will get 1 by 4 and this 1 minus x here and with minus sign. So, we need to integrate first this one so, we will get minus 2 and this plus 1. So, we will get 1 over this x plus y plus 1 minus 2 and plus 1. So, this will be also with the plus sign and then these here we have already substituted the limit.

So, in this case we have to put these limits for y 0 to 1 minus x and then everything will be integrated over x. So, by putting these limits here 1 minus x for y. So, we will get this again this 1 by 2 here and then when putting this limit 0 we will get x plus 1 and that will give us the integration with respect to x in terms of the logarithmic. So, this evaluation is again simple with respect to x we can do that. And, the answer to this calculation will be 1 by 2 the log with base e naturally here 2 which will be coming out of this integral here and the rest here the 5 by 8.

(Refer Slide Time: 28:27)



The last problem so, using this triple integral we want to find the volume which is common to this is sphere. We have this sphere x square plus y square plus z square is equal to a square and this circular cylinder which is given by x square plus y square is equal to a square. So, this is again the circle which we have discussed a couple of times before. This is a sifted circle where radius is a by 2 and the center is a by 2 and 0 and I shown here in the in this figure. So, the cylinder here whose objection on the xy plane is given by the circle and the center here is a by 2 and 0. So, the volume if you want to find using this triple integral so; obviously, there will be no integrand now. And, we have this integral over this v which is given by the cylinder and then we have the sphere.

So, the sphere is the cylinder is cutting the sphere and that portion we want to find the volume. So, the most important part again here is finding this limit for this v and what we do now first with respect to z again we are putting. So, in the direction of z let us take the 4 the multiplication of 4 and we will just consider only the upper half of this cylinder. So, we will make the double to get the lower portion as well and then since we have the sphere so, again there are 2 portions: one the upper one and the lower one. So, overall if we just consider this upper half here then we have to multiply by 4 to get the whole volume because, this will give us the half volume because, we are integrating in the half and that too upper half.

So, it will be one-fourth of the total volume. So, we have to multiply by 4 here the limits now we have to fix the limits so, for the z. So, this is the cylinder which is in the direction of z and it is cutting exactly the sphere there. So, the for the limits of z for this upper half portion will be we are entering into the z from z is equal to 0 and we are leaving this to the sphere. So, on the sphere here because the cylinder cuts that is sphere and our interest is to get this common volume. So, we are exiting the domain through the sphere where the equation is a square minus x square minus y square for this z in the upper half portion.

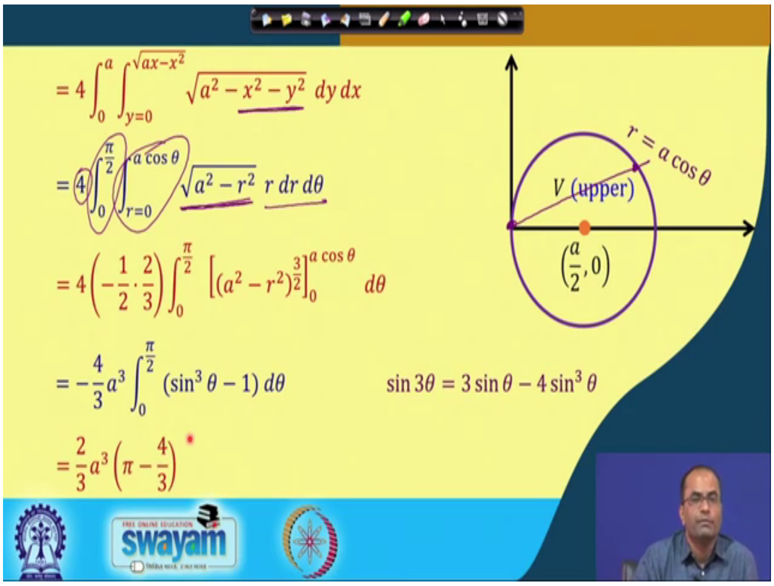
So, the limits for the z are clear now from 0 to that is sphere because, we are leaving that domain to that sphere. And now this z direction is done and we can now project this to this xy plane and this is precisely the projection here. So, we have to now do the limits over this semicircle circular disk. So, in this case now the y enters through this 0 here and leave this to the circle. So, y goes from 0 to that circle because, the circle equation was this x square plus y square is equal to ax. So, from here we can get a x and minus x square and then the square root.

So, this is the upper limit and now for the a we are moving from 0 to this a so, for this x 0 to a. So, again the limits were not difficult in this particular case only thing we have to be careful for the z limit because in case of the z we are entering through this xy plane, but we are leaving through this sphere there in the direction of z. Having this once we set the limits for the z we project that this common volume to the xy plane and that will be precisely the semicircular disk for our consideration and the limits for this we have we can easily here set here.

So, now we need to just do this evaluation here for this integral and we can do for the in inner one first with respect to z. So, that will be z and then the upper limit. So, a square root a square minus x square y square dy dx and now this is the double integral which we are well familiar with. So, we can easily evaluate this integral; indeed changing to the polar coordinate would be much easier we do see here x square y square and also this circle which is given already. And, we know the polar equation of that circle also that will be r is equal to a cos theta.

So, that will be the circle here r is equal to a cos theta. So, if we put into this polar coordinate then this will be easier because, r is from 0 to this a cos theta to the circle and theta will be from 0 to this pi by 2. So, having this in polar coordinate the evaluation will become easier and we have done already similar evaluation before.

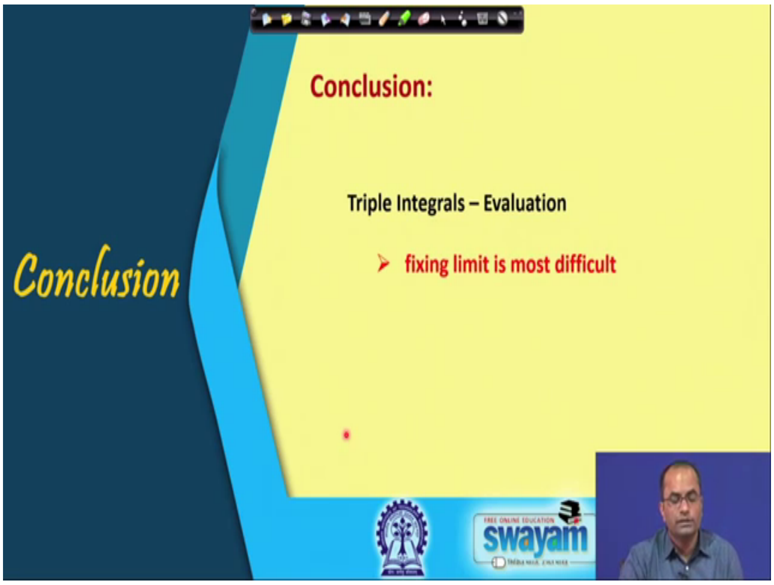
(Refer Slide Time: 34:09)



So, to evaluate this we will change to the polar coordinate the r will go from 0 to this circle so, from 0 and to this circle here. So, r is equal to 0 and r is equal to this a cos theta that is a limit for this r and then for theta we have 0 to pi by 2 and this 4 to get that complete volume and we have a square minus this r square. So, for x square y square we have r square r dr d theta.

So, this we can again this is a simple well known function which we can integrate because, r is already given there and then we can put the lower and the upper limits. So, we will get in this case the sin cube theta minus 1 and d theta with the factor of minus 4 by 3 a cube. But, we know this identity here that sin 3 theta is 3 sin theta minus 4 sin cube theta. So, this sin cube theta we can write down in terms of sin 3 theta and sin theta which are easy to integrate; we can integrate sin theta we can also integrate sin 3 theta. So, this sin cube theta we can convert from this identity in terms of this sin 3 theta and sin theta and after this evaluation of the single integral we will get 2 by 3 a cube and pi minus this 4 by 3.

(Refer Slide Time: 35:41)



So, the conclusion here we have gone through the simple the basic evaluation of the triple integrals and what was the most difficult part was fixing the limit. And, the trick here is that first we will fix the limit for instance of this z, we can do with x or y as well. But, fixing the limit for the z we have to see that where we are entering the whole volume and at what point, at what surface we are exiting the volume.

So, that will be the limit for this direction the z for instance and once we fix this with respect to 1 we can project the whole volume to the other plane. So, if we have fixed already with respect to z then we can project the geometry, the volume to the xy plane. And, once we project to the xy plane we are in the 2 dimensions and we know how to fix the limit for 2 dimensional case.

(Refer Slide Time: 36:45)

So, these are the references we have used for preparing the lectures and.

Thank you very much.