Engineering Mathematics – I
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Lecture – 33
Integral Calculus – Double Integrals: Surface Area

Welcome back this is lecture number 33 and today again we will continue with this Double Integrals, but with a special application to surface computation of the surface area.

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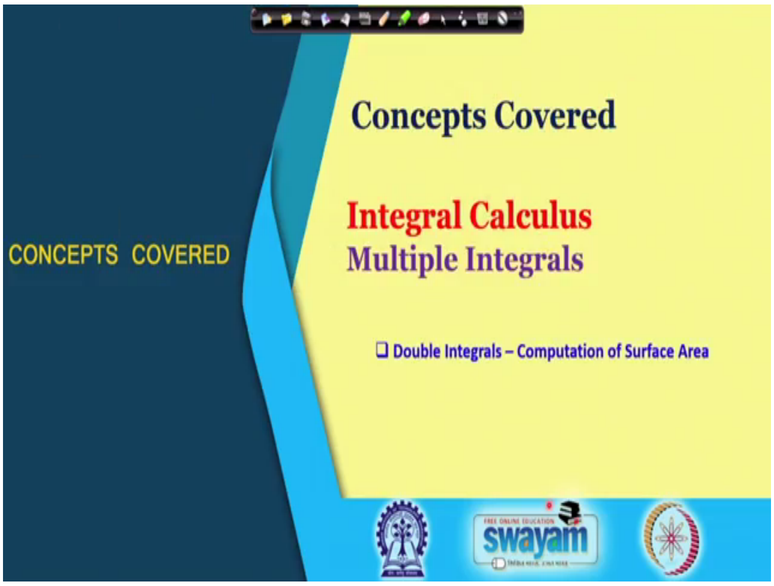

So, we have already seen the applications of double integrals for computing volume for example, and also the area of the domain of integration. And today we will continue our discussion for computation of surface area.

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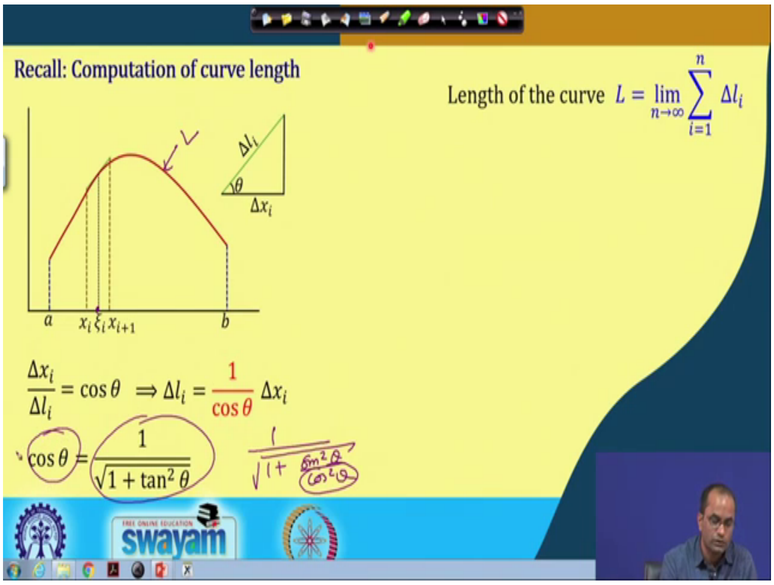

So, just to recall how do we compute the curve length in case of single integral. So, for instance we have this curve here or the function which is given by f x. So, this is x axis and then here we have the y axis and this is the function f x with respect to the values of x. And so, what we do we divide the whole domain from a to b into small pieces of intervals from x i to x i plus 1 that is the one interval shown here. But, we have discretized this the whole domain from a to b into n such intervals and let us consider this point for instance this x i i within the interval this x i to x i plus one. And, corresponding to this point x i we draw tangent at this point here which is given by x i i and f x i.

And this tangent the length of this tangent, if we add for all the intervals and later on we take the limit as the width of these intervals go to 0 in that case we will end up with getting the curve length. So, the idea is that here this is just shown this part of the tangent. So, this is the tangent line at this x i i point and this is the interval length here delta x i and this is suppose the angle which tangent makes from the x axis. So, that is we have denoted by theta.

So, we have this length here delta l I, this length here in the base here is delta x i and this is the angle theta, now the length of the curve. So, length of this curve l which is so, this is our curve l here and the length of this curve l will be given as when we sum all these parts of the tangents and then take the limit. Because, if we do not take the limit we have the error because this tangent is not representing the curve in this interval delta x i. But, when this delta x i will go to 0 eventually we will end up with calculating the length of this curve from a to b.

So, we want to get this limit from i 1 to n and then this summed up over this delta l i and then the limit n goes to infinity. And, this now this delta l i we will we will write in terms of the delta x i and the function f; so, here the relation from this triangle we have that this cos of this theta is this delta x i over delta l i. So, which is given here delta x i over delta l i is cos theta or we can get from here that this delta l i is nothing, but 1 over cos theta and delta x i. So, this cos theta this cos theta again from this trigonometric equality, we have the cos theta equal to 1 over 1 plus tan square theta. So, this 1 1 over 1 plus tan square theta is nothing, but the cos theta because we have 1 over and then the square root 1 plus this we can write down as sin square theta over this cos square theta.

So, the cos square theta plus sin square theta will be 1 and this cos square theta will go to the numerator; so, we will end up with this cos theta only. So, this equality here cos theta is equal to 1 over square root 1 plus tan square theta we can again get from here also this 1 over cos theta.

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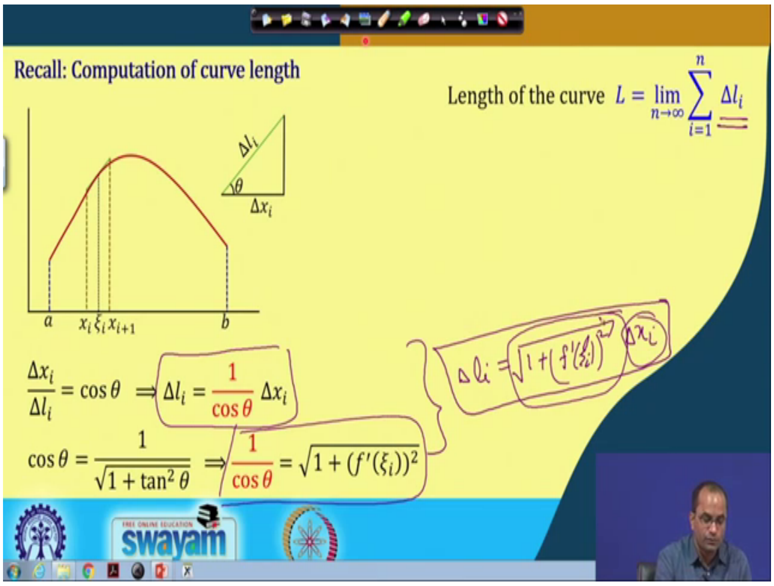

So, if we rewrite here if we invert it we will get this 1 over cos theta is equal to square root 1 plus tan square theta and this tangent theta. So, theta is the angle which this tangent makes with the x axis. So, that is nothing, but the tan theta is the first derivative that is the geometrical interpretation of the first derivative. So, this tan theta is nothing, but the first derivative at that point here x i i. So, we have replaced this tan theta here by the first derivative at x i i and this whole square.

Now, out of these two relations we can out of these two relations here we can remove this 1 over cos theta and we can get this delta l i is equal to. So, 1 over cos theta will be replaced by this 1 plus the f prime x i i and this whole square and then we have here delta x i. So, this delta l i in the summation here we will replace by the square root of 1 plus f prime x i and multiplied by this delta x i.

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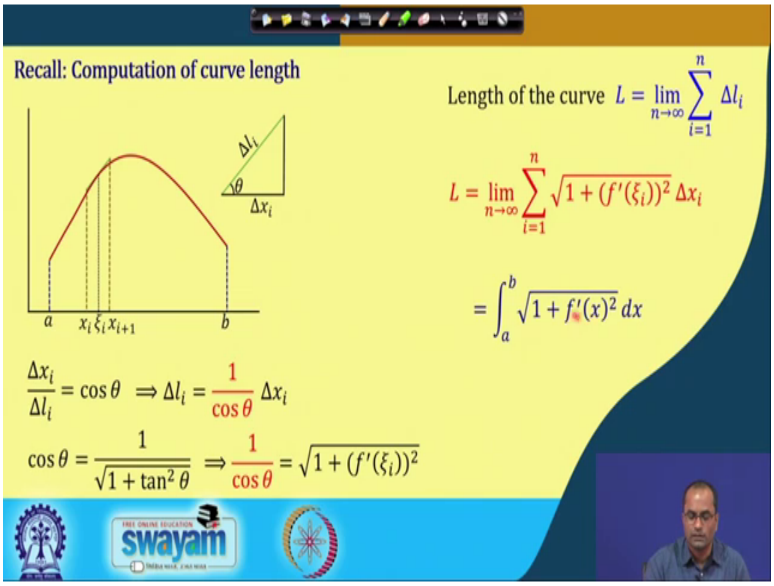

So, doing this now, we get this l is equal to this limit i from 1 to n and this delta l i is replaced in terms of the delta x and the given function f. So, 1 plus f prime x i i whole square delta x i and now we will write down this in terms of the integral as per the definition we have already learned. So, this if this limit exists we can actually represent this or denote this in terms of the integral from a to b the domain a to b and this is square root 1 plus the first derivative f whole square and dx. So, this integral here a to b and square root 1 plus f prime square dx will give us the curve length of this curve which is given by the function f in the range from a to b.

So, what we have learnt that in case of the curve length the formula is given by the square root of 1 plus and this derivative whole square. The extension of this we will have for the computation of the surface where instead of the tangent line we will have the concept of the tangent plane. And, and here we have taken the pieces of the tangent and then we have added them after taking the limit we got the curve length. In case of the 2 dimensional so, we have the surface there and we will take the tangent plane or the pieces of the tangent plane. And, then we will sum those pieces and after taking the limit basically we will get the surface area of the of the surface.

So, that is the trivial extension of this we will not go through the formal prove. But, if you understood the concept here in case of this one variable the integral which computes the curve length is given by the square root 1 plus this f prime square dx, we will just make an extension here now, for the computation of the surface area.

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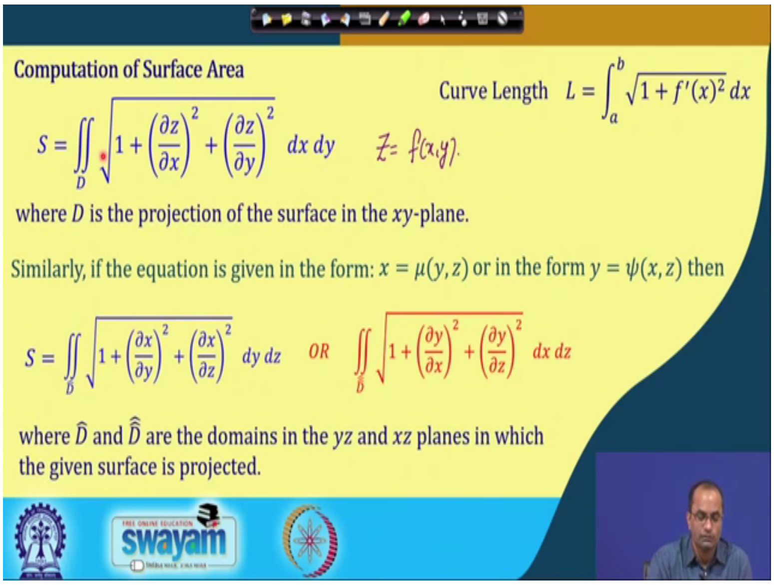

So, for the curve length we have just seen the formula was a to b. So, over the domain we are integrating this integrand which was 1 plus f prime square root of this. In case of the surface when we have a function z is equal to f x y so, our here the surface is given by z is equal to f x y.

So, this is the surface here and we can get the surface area again similar to what we have for the 1 dimensional case. We have the double integral, we will discuss this what is the domain here now and the square root we will take 1 plus like similar to here we have the derivative here also we have the first order partial derivative. So, the first partial derivative with respect to x whole square plus the partial derivative with respect to y of the given function whole square. And then we integrate this integrand over the domain which is in the xy plane and this is nothing, but the projection of the of the surface in the xy plane.

So, we have the some surface given or the z axis and if we project that surface into this xy plane then D will be exactly that domain in the xy plane. So, D is the projection of the surface in the xy plane and similarly because this equation we have formulated when the function was given as this z is equal to a function of xy. And, here we should note there the partial derivatives were taken with respect to x and y which was natural. And, for instance the function is given like x is equal to mu y z or it is given y is equal to psi x z in that case the formula will change obviously. So, in this case for x is equal to the function of y and z then the partial derivatives with respect to y and z will appear.

And, this domain will be the projection of the surface in the y z plane and in the case when the function is given by y is equal to psi x z; the integrand will have the derivatives with respect to x and z. And, again the domain of the integration will be in the xz plane where we are integrating the will be integrating this integrand. So, here in the case first when x is equal to mu x y and z we will have the domain here which is denoted by this D hat and now the integrand will become 1 plus del x over del y whole square and del x over del z because y and z are independent variables now. So, the partial derivatives will be with respect to y and z and dy dz.

And, now this d hat will be the projection in the projection of that surface in the y z plane or in case when the function is given by y is equal to psi z. So, we will have the partial derivatives of y with respect to x and z and then dx d and here this D double hat will be the projection of the surface in xz plane. So, this D hat and D double hat are the domains in the y z; so, in this first case and then in the second case in xz planes in which the given surface is projected.

So, with the help of this double integral which is given by in most of the cases S is equal to this domain 1 plus z x whole square plus this z y whole square and dx dy when the function is defined as z is equal to f x y we can compute the surface area. So, remember when we had the application for the computation of the of the volume under that surface over the xy plane, the formula was simply instead of this integrand we had the function here f x y.

And, if you want to compute the area of the domain of integration so, the area of D in this particular case then we will just set this integrand as 1; so we have 3 applications: the one was the area of this domain D, another one was the volume under this surface over the xy plane which was given by again double integral where, the integrand was simply f x y. And, now we have for the computation of the surface area where the integrand is square root 1 plus zx square plus z y square.

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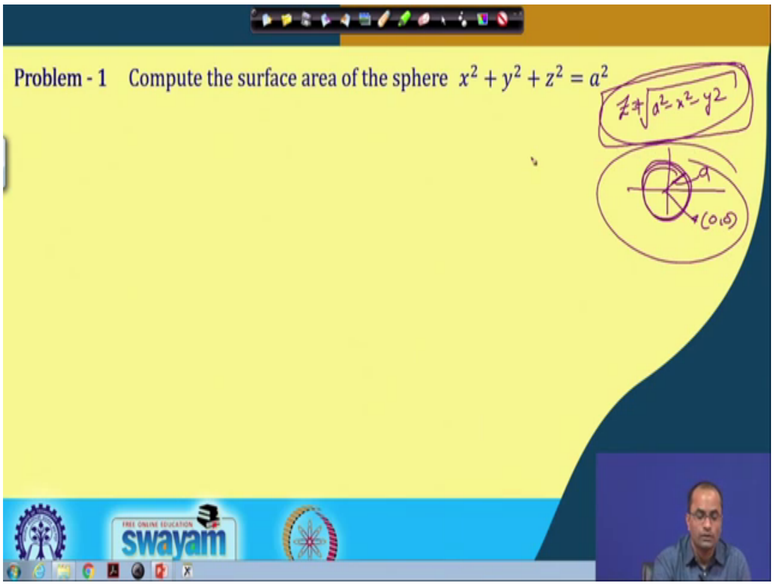

So, now we will take the problem here compute the surface area of the sphere x square plus y square plus z square is equal to a square. So, we have a sphere which is centered at 0 0 0 and given by this equation x square plus y square plus a square is equal to a square. So, the radius of the a sphere is a; so, we want to compute the surface area now. So, we will take for instance the upper half part of the sphere and if we project that upper half part of the sphere; this will give us the circle of radius a in the xy plane. So, though we can we can project this to any one of these planes we either y z or z x or x y. So, let us project into the xy plane. So, in the xy plane this will be a circle of radius a center at 0.

So, we have the domain now that is the circular disk in the xy plane and then the surface will be given by simply this z is equal to the square root a square minus x square minus y square. Because, we are considering the upper part of this sphere and we can make it the double to compute the surface area of the whole sphere. So, we are considering only the positive part; the upper half of the sphere and when we project into the xy plane we will get this circle of radius a here and the center at 0 0 ok.

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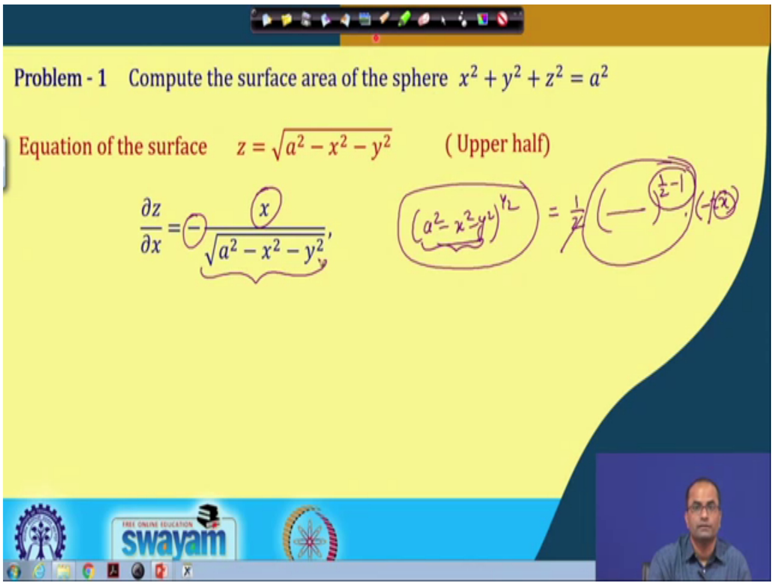

So, equation which will give us the surface here which is which I have just discussed for the positive for the upper half we will take z is equal to square root 1 minus sorry a square minus x square minus y square for the upper half. And, now we need to compute because in the formula we need the partial derivative of z with respect to x and also the partial derivative of z with respect to y. So, for the partial derivative of that with respect to x so, this is a power half here. So, we will get so, this is a square minus x square minus y square power half and when we differentiate this with respect to x.

So, we will get half in the same expression what is here and 1 by 2 minus 1. So, this is minus half here and then the derivative of this a square minus x square minus y square with respect to x that will be minus 2 x. So, this 2 will get cancelled and we will have with minus sign which is here and this x in the numerator and this power which is minus half here; so, that is coming in the denominator; so, that is the partial derivative with respect to x of this function.

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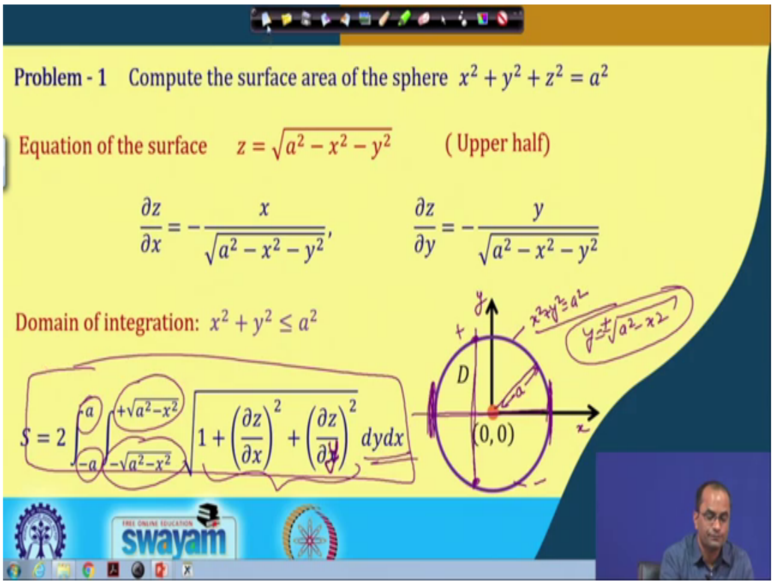

And then we have the partial derivative, similarly with respect to y; the only difference will be now instead of x we will get this y here and the domain of integration which we have just discussed when we project this upper the semiosphere into the xy plane. So, we will get a circle of radius of radius this; a here and the equation of this circle will be naturally the x square plus y square is equal to a square. So, let us say this is x axis and this is y axis here. So, the formula for computation of the surface we will make it double here 2 times because, we are considering only the upper part of this sphere. So, we have also the lower portion. So, we can make it this double here with 2 and then this minus a 2 a; so, the let us first fix the inner one with respect to y.

So, in the direction of y we are moving from this circle to the upper half of the circle so; that means, this is like minus a square minus x square. So, the y from here will be a square root a square minus x square with plus minus. So, this will be with the minus sign and the upper half will be the plus sign. So, we are moving from the lower with minus to the plus 1. So, minus a square minus x square to the plus here a square minus x square the upper part and then these lines here in the direction of x we are moving from this minus a to the plus a in the direction of x.

So, minus a to plus a and then this integrand which is 1 plus zx whole square plus z x z y. So, here this zy zy whole square and then dy dx. So, that is the surface we want to now compute this double integral which will give us the surface area of the sphere.

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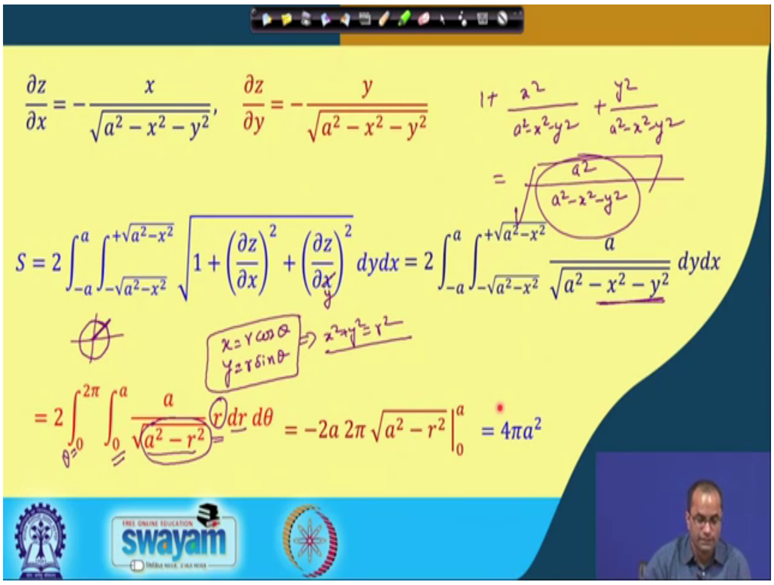

So, that is what we have now the partial derivative with respect to x is given by this partial derivative of z with respect to y is given by this and that was the formula; here this was with respect to y. So, if we if we substitute this now so, these are the whole square of these terms. What we will get? We will get simply 1 plus this x square over this a square minus x square minus y square will come and then here also will get y square over a square minus x square and minus y square. So, this taking this common x square and then minus this y square; we will get this x square plus y square will get cancelled and then we will get this one.

So, this will be a over the square root a square minus x square minus y square. So, again the same limits what we have there and the integrand become this a over square root a square minus x square minus y square and dy dx; so now, if we change to the polar coordinate because, that will be easier looking at the limits of the integrals and oh as well as the integrand. It suggests that we should convert to the polar coordinate which was discussed in the last lecture. So that means, this x is equal to r cos theta we will substitute x is equal to r cos theta and we will take y y is equal to r sin theta. So, with this substitution now and also we have to find the limits. So, for in case of the circle it is trivial to gap. So, for the theta we have the 0 to 2 pi because this is our circle here.

So, the theta moves from 0 to 2 pi and for the r we will have 0 to a. So, r will move from 0 to the circle; that means, 0 to a we have the integrand a over square root a square and this x square minus y square. So, with minus sign x square plus y square which will give us from there x square plus y square is equal to r square. So, we have a square minus r square and this factor r which is Jacobian or directly we have also seen from the polar coordinate that we get additional factor here with r. So, we get r dr d theta with this integrand and now it is easy to see because this r is sitting there and we have this r square term here. So, we can easily integrate now this integrand with respect to r.

So, what we will get because this power was a minus half; so, when we integrate this minus half and plus 1. So, this square root will be in the numerator now. So, we have a square minus this r square and this 2 factor is there and here what we will get. So, we will get this we need 2 r here. So, the half we will multiply and multiply by 2 there so, but here when we take the integral so, minus half plus 1 half. So, that will cancel out. So, we will not get we will get this integral as square root a square minus r square, this 2 is already there with this a so, 2 a. And when we integrate the outer one because, the integral the inner integral will not have anything of theta.

So, we can integrate the outer integral as well. So, in that case we will get simply 2 pi because the integral will be theta and the upper limit is 2 pi lower limit is 0. So, you will get 2 pi from the outer integral and the inner one is giving us this a square minus r square square root with the limit 0 to a and, now if we substitute this limit. So, the upper limit here we will give us we will give us 0, the lower limit of r when set to 0. This we will get here a simply with minus sign. So, this minus will become plus and the answer would be 4 pi a square and that is exactly the surface area of this sphere which we already know. So, but with the help of this double integral and with the help of the polar coordinates we could very easily compute this surface area.

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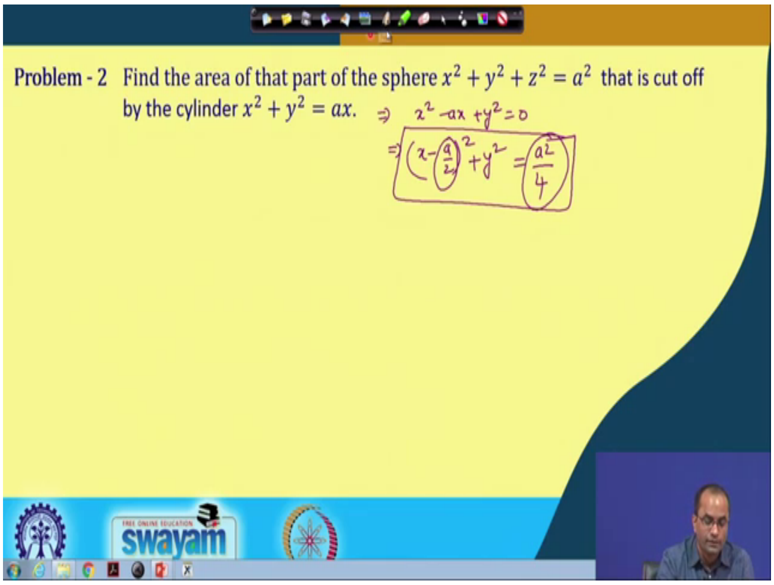

So, moving to the next problem we will find now the area of that part of the sphere. So, we have again a sphere x square plus y square plus z square is equal to a square that is cut off by this cylinder x square plus y square is equal to ax. So, this is a equation of the circle.

So, here x square plus y square is equal to ax is x square and then minus ax and plus this y square is equal to 0. So, this we can make it whole square so, a by 2 whole square. So, you will get x square there and there will be a term minus ax which is already there. There will be additional term here a square by 4 which we can now put it right hand side to compensate it and then we have here dy square. So, this is the equation of the circle which center is at a by 2 and 0. So, a by 2 and 0 is the center and a by 2 is again the radius so, a by 2 whole square; so with that so, let us clean it.

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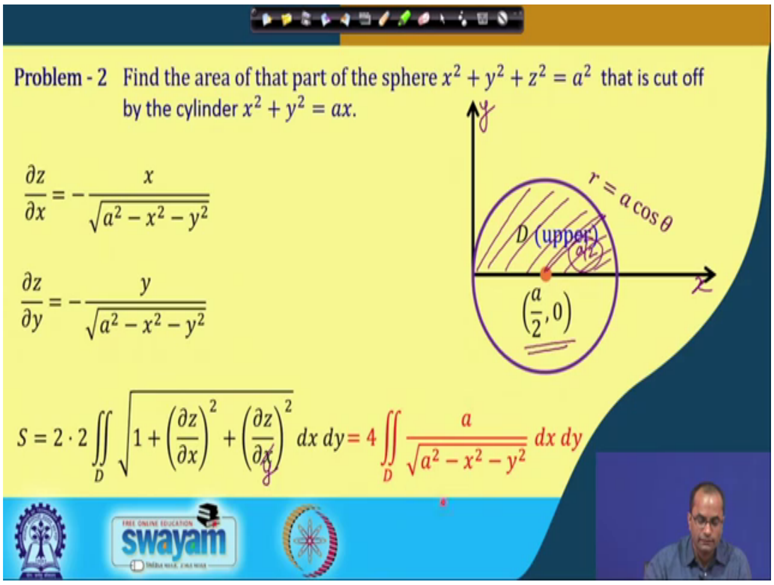

So, here we have this circular cylinder. So, this is just given in the xy plane, but we have the cylinder which is also in the in the along the z axis. So, here this is the projection of that cylinder in the xy plane. So, this is your the x axis and then we have the y axis; the center is at a by 2 0 and the radius is again this a by 2 of this circle.

So now, you want to so, again this is from the last lecture also we have seen that this r is equal to a cos theta is also the polar equation of this circle which may be required in our computation. So, here we need to compute again zx which we have evaluated in the previous example also the zy and we want to get the surface here for this is sphere which is cut off by the cylinder.

So, again here we have 2 times because this sphere will have 2 part the upper one and also the lower one and this is cut by this cylinder here. So, and we are considering in our computation in our integration only the upper part of this here. So, we have to also make double because of this lower part. So, basically the 4 times because we are considering only this portion which is cut by this half cylinder and because of that we have to double it. And since we have the sphere so, it has the above portion and also the lower part. So, for that we have to again multiply by 2. So, this is 4 times the domain D is the upper part of this upper part of this circle here, this is our domain now and so, this is again this del z over del y.

So, we want to we will put it there and as in the previous example we will get a over a square minus x square minus y square dx dy with 4 times which we can again evaluate.

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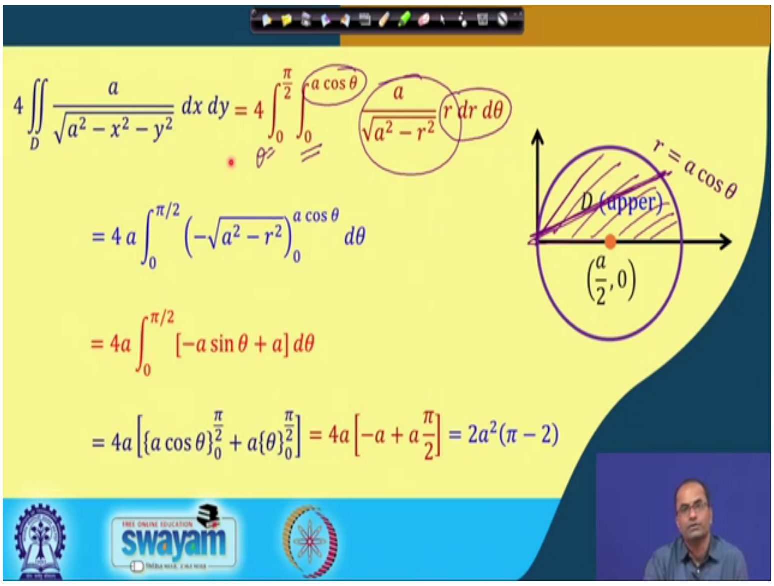

So, with the help of with the help of the polar coordinate we can now prescribe the limits for this upper half of the upper half of the circle here. So, the theta goes from 0 to pi by 2. So, this is theta is equal to 0 and theta is equal to pi by 2 and then the r goes from 0 to the circle. So, r goes from 0 to the circle a cos theta and we have a over a square minus r square with r dr d theta.

So, again this is very simple to integrate that is like we have done already in the previous example. So, once we integrate this we will put the limit 0 and the upper limit this a cos theta which will give us here the minus a sin theta; when we put the upper limit a square minus a square cos theta which will give us a sin theta. And this minus minus plus and when r is set to 0 we will get a. So, this is minus a sin theta plus a and then we will integrate here with respect to theta. So, this sin theta we will give cos theta and a theta these limits 0 to pi by 2 and once we compute this we will get this 2 a square and pi minus 2.

So, the only the difficult part here is locating the limits for the integral and that we have to be careful. So, in this case the sphere was cut by the cylinder. So, this projection on the xy plane is nothing, but the circle because it was a circular cylinder and the projection is given as here by this circle. And, then once we have figured out this projection on the xy plane then we can easily put the limits of the integration.

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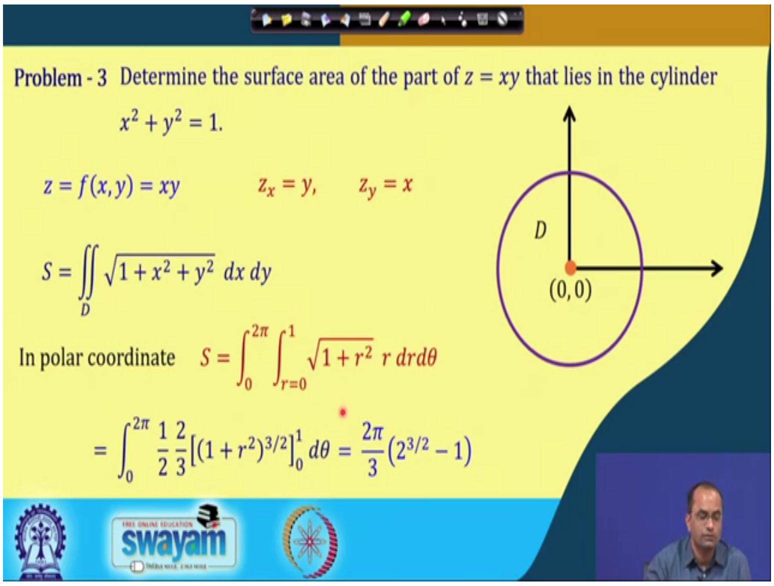

So, the last example where we will determine the surface area of the part z is equal to xy that lies in the cylinder x square plus y square is equal to 1. So, again we have the cylinder, but it is it is a cylinder with center 0 0 and radius 1 and we have this z is equal to x y we want to get the surface area. So, this is much simpler than the earlier examples.

So, again the projection of this cylinder because the cylinder cut that surface; so the projection of that surface over the xy plane will be nothing, but the circle because the cylinder is this circular cylinder and the cylinder is cutting the surface there. So, that will be the projection here and we can get out of this z is equal to x y this partial derivative with respect to x, partial derivative with respect to y which are required in the formula for the computation here.

So, over the domain D which is given here and then we have the square root 1 plus zx square at y square. So, 1 plus x square y square and dx dy and in polar coordinate if we change this one we will get for the circle here 0 to 2 pi r moves from 0 to 1 from this point to the exit point here which is circle r is equal to 0 to 1 and we have a square root 1 plus r square and then r dr d theta; so which we can again integrate this easily. So, we have 1 plus this r square and this was power half.

So, when we integrate 1 will be added there. So, we will have 3 by 2 and this will be 2 by 3 there to compensate this and then we have the limit 0 to 1; putting these limits and then there is no theta here. So, because of theta we will get also 2 pi so, 2 pi by 3 this 2 gets cancelled. So, 2 pi by 3 and we have this when we put r so, 2 power 3 by 2 and when we put 0 we have 1 there. So, that is the surface area of this surface xy which is which is lies within the cylinder here of radius 1 and centre 0 0.

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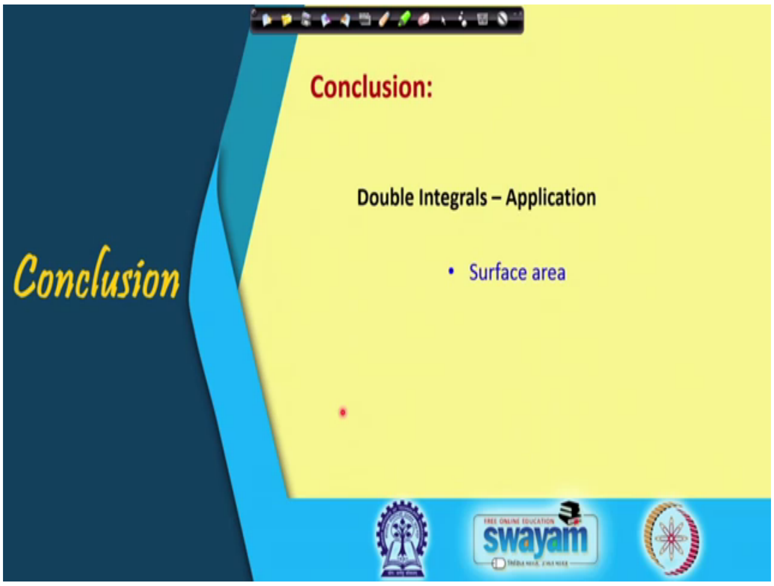

So, what so, we have seen this so, another application of the double integral for the computation of the surface area and, with the help of this simple formula over a square root 1 plus zx square plus zy square and dx dy only we have to be careful that this domain of this integral is the projection of that surface in the xy plane or we have or it could be in the in the y z or zx plane as well. Once we have that projection, if we identify that projection putting the limits will be will be much easier and we can compute the integral.

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So, these are the references we have used for the computation for this preparation of the lecture and.

Thank you very much.