Engineering Mathematics - I
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Lecture - 32
Integral Calculus – Double Integrals: Change of Variables

Welcome back. And this is lecture number 32 and we will continue discussion on the double integrals and in particular today we will discuss an very very important topic that is Change of Variables in Double Integrals.

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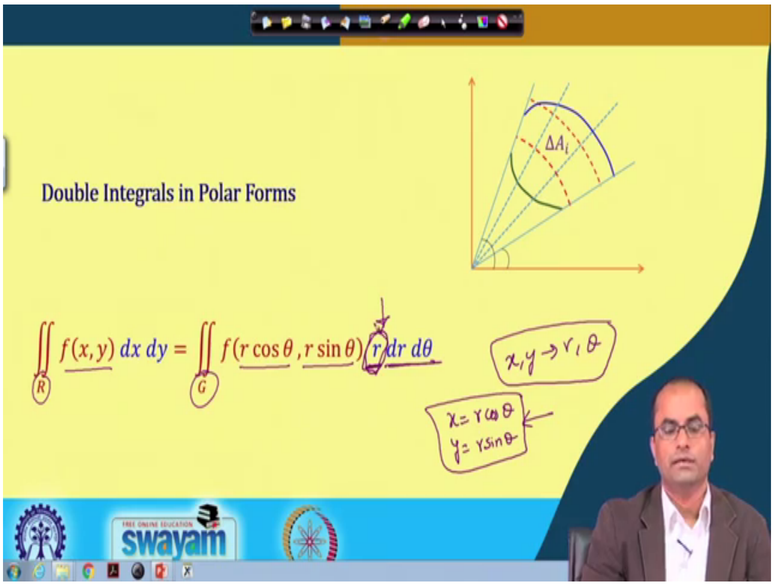

So, in the last lecture we have seen the double integrals in the polar forms and the idea was to again break this integral the domain of integral into smaller parts of region, this delta i which we have denoted here for this small portion of the area and which was coming to be the r times the delta r and delta theta. Delta theta was the angle here and the r was the radius from this to this circular arc. And in that case the whole idea was that if we have the integral here in the Cartesian coordinate, we can convert into the polar coordinate by just substituting this x and y to r cos theta and r sin theta and this dx dy will become the r dr d theta.

So, here also we have done eventually the substitution from xy to the r theta coordinates by this relation that x was r cos theta and y was sin theta. So, this was awesome substitution, but directly with the help of the basic the fundamental definition we have seen that how this r is coming to the picture. And so, it is not just the substitution of the variables here x and y 2 r cos theta and r sin theta, but also this vector r had appeared there and then we have r dr d theta and corresponding to this r we have to also substitute now the limits of the r and theta.

So, now, we will go for a more general case and this will be a particular situation when we go from Cartesian to polar coordinate. But in general, if we have any other type of change of variables then what how to deal with this factor and we will see now in today's lecture.

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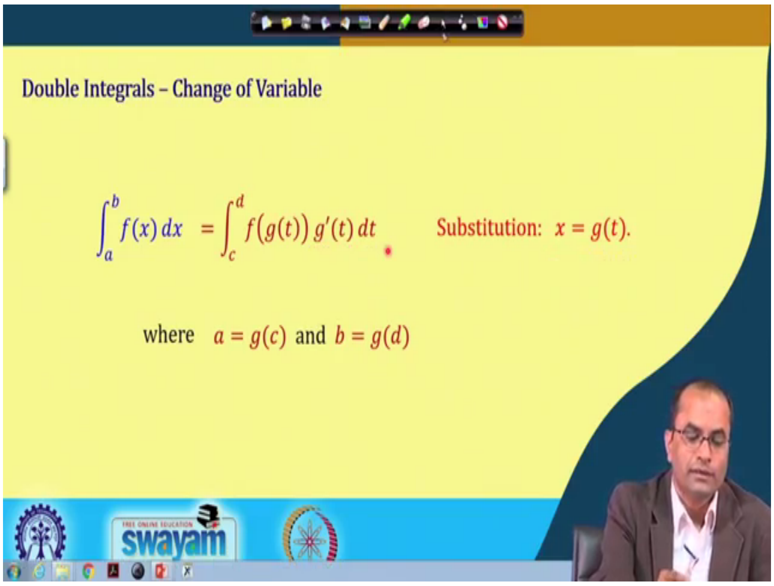

Just to recall if we have a single integral f x dx and the limits of x varies from a to b and we want to substitute now to a new variable. So, x to let us say the t here by, this relation x is equal to g t. So, what we do? We immediately get that what would be our dx from here, so dx would be g prime t and dt in this form and then we substitute this x as g t and dx will be the g prime t dt and then the corresponding the limits will appear there.

So, the idea used to be that we have the corresponding limits now of this t and this x was substituted by the new function this g t and for dx we have got here g prime t dt. So, that is again here to recall that we got now there is something else here which is in case of a single integral it is g prime the derivative of this g and then dt. And these corresponding limits will be computed again from this substitution only. So, this a is going to be this g of c and this b is nothing but g of d. So, in that way we have also computed these new limits and we have substituted the variable and instead of this dx dt has come, but within another factor which was in terms of the derivatives of this g. So, a similar kind of derivative we will also get when we have a integral in two variables or the double integral.

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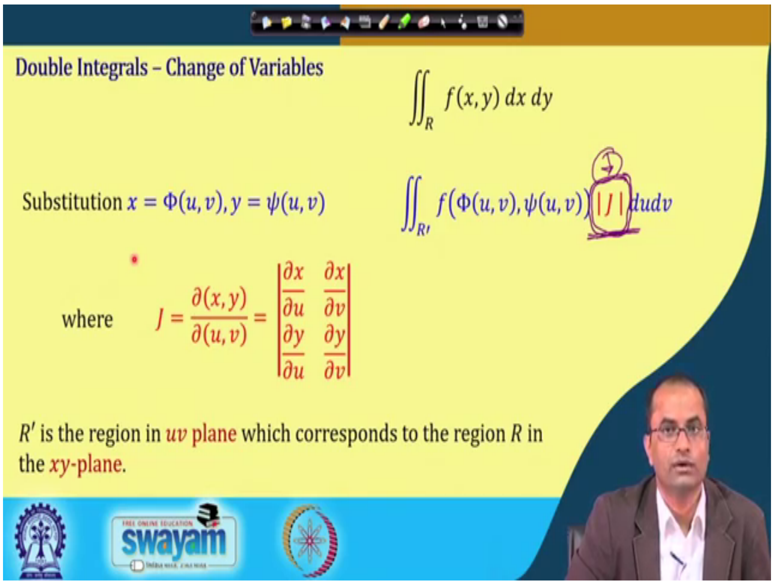

So, if we have a double integral here f x y dx dy, now if you want to make some change of variables. So, let us say x is given by this phi, u v and y is given by some another function u v. So, our new variables are u and v, the older one here were x and y. So, we want to change the variables from x y to u v with this relation x is equal to phi u v and y is equal to psi u comma v.

So, having this we have this integral here over the new region I mean the same region, but for the new variables u and v, so that is what we have used here different notation r prime. And then we straightaway substitute here for x this phi and for y, we will substitute the psi as a function of this u v. Naturally, for the dx dy we will get this du dv, but with this another factor which we have just discussed before also. So, in case of one variable or in case of that polar coordinate you are getting just this as r dr d theta, so there is some factor here this J and we have taken this absolute value also; we are not going through the proof of this because that is a bit involved a topic.

So, now what is this J here? So, the way this J is again kind of derivative which we call Jacobian. So, Jacobian here x y with respect to this u and v which is computed as in the form of this determinant; so the partial derivative of this x with respect to u the first term, here the partial derivative of x with respect to v, now for the y with respect to u and this partial derivative y with respect to v.

And that is this J term here we will compute this one and absolute value will go to the integral with this du dv; so that is the important point here. And then these limits of this R prime, it is nothing but the same region, but now we have to compute over u and v. So, the limits will now follow according to the new variables u and v.

So, with this now we can compute or we can substitute the old variables here x y to some suitable because depending on again the integrand or the region r we may have some other suitable substitution, which makes the integral evaluation easier.

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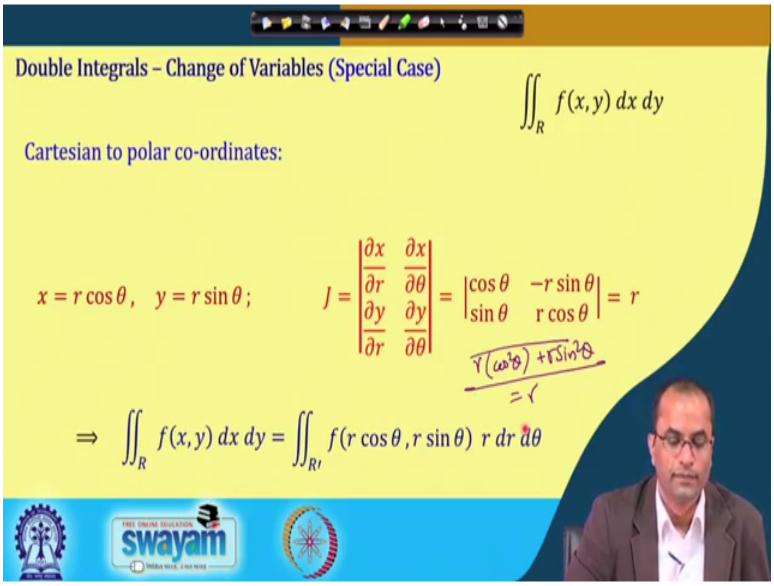

So, if we want to see a particular case that what will happen if we substitute into the polar form; that means, x is equal to r cos theta and y is equal to r sin theta. So, in that case x is equal to r cos theta y is equal to r sin theta. If we compute this Jacobian here, so del x over del r, so del x over del r will be cos theta that is cos theta here and del x over del theta. So, will be minus r sin theta here del y over del r; so this will become sin theta and then y over delta t will become r cos theta.

And now the value of this determinant will be r cos square theta plus r sin square theta and when we take r common. So, this is r cos square theta and again minus plus r sin square theta and then this is nothing but r. So, we have this determinant value as r, and now we can change this integral which was given in the Cartesian coordinate to the polar coordinate and which we have already discussed in the previous lecture that we need to substitute for x and y as r cos theta r sin theta and then dx dy will become r dr d theta. And this region now this r will be r prime will be described in terms of the polar coordinates r and theta.

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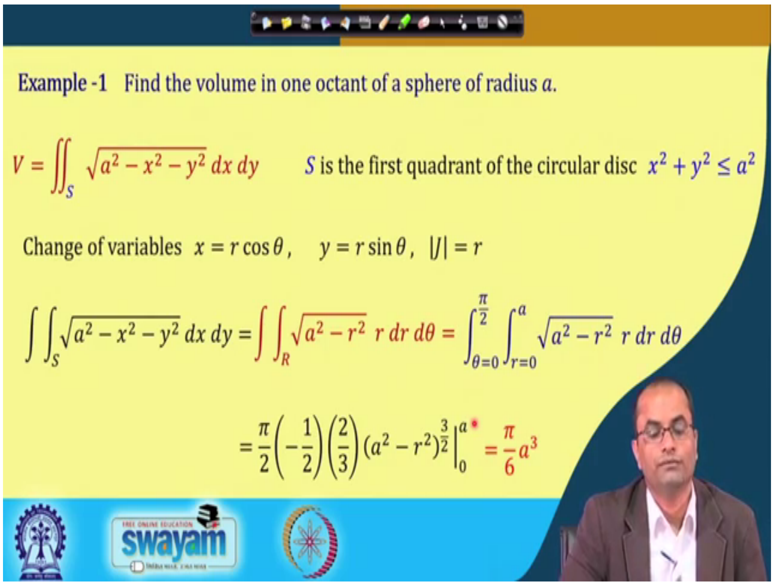

So, with this now we can go through the examples which we will use now the first one to find the volume of one octant of a sphere of radius a; so, another very simple example where we are using the sphere here. So, we know the equation of the sphere. So, its z square in terms of the x square and y square we will get and then here the volume in one octant, so we are considering basically the first octant here. So, this the lift the limit we can easily now compute for this one and our surface here is given by the sphere.

So, we have the integral in terms of this volume that this is coming from the sphere because the equation of the sphere was z square plus this x square plus this y square is equal to a square. So, from here you can compute z square is equal to a square and minus x square minus y square. So, z will be plus here because we are talking about just one octane there. So, this will be the surface or this will be the integrand here a square and minus x square with the square root and then dx dy over this s region. So, that the radius r the circular disk. So, this is in the first quadrant of the circular disk x square plus y square less than or equal to a square. So, that is the circular disk in the first quadrant and where we have this domain for the integral.

And if we change now because direct evaluation over this Cartesian coordinate will be definitely difficult because of already the integrand is difficult then the limits again this square roots will appear because of the circular disk and overall the computation will be will be difficult. But if we change to the polar coordinates by this x is equal to r cos theta and y is equal to r sin theta in that case, we have already seen that this Jacobian is coming to be 1.

So, this additional factor which comes with dr d theta and then we have the simple idea that we will change it to the square root here a square and minus this x square plus y square term will become r square and then we have this Jacobian term here with this r and dr d theta. So, we have the integral now converted into r theta form and this additional factor r has come and then we have dr d theta term.

Now, the evaluation is much simpler because this r is sitting here which if we substitute a square minus r square to some new variable t, then we will get precisely this. So, r dr we can convert we can easily evaluate the inner one. So, first the limits here for this R, what will be the limits? We have the circular disc in the first quadrant. So, meaning the theta will go from 0 to pi by 2 only. So, we have this circular disk in the first quadrant. So, the theta will go from 0 to this pi by 2 and r will go from 0 to r is equal to a; that is the radius of the circular disk.

So, now, so we can easily integrate now with respect to r its there is no theta here. So, there you know one the it is just a single integral with respect to r and that will give us here a square minus r square and the power 3 by 1 and then divided by 3 by 2. So, we can multiply by this 2 by 3 from the denominator. So, we have pi by 2 finally, from the outer integral because there no theta in the integrand, so we will get pi by 2 from the outer integral the minus half factor from the integral will come and then the integral will be this 2 by 3 and a square minus r square power 3 by 2. So, this is just a square minus r square power 3 by 2, so half plus 1 and then 3 by 2. So, this 3 by 2 we have taken here, 2 by 3 and a square minus r square power this 3 by 2 and the limits are from 0 to a.

So, if we substitute now these limits, the upper limit will make it 0, the lower limit for this r will make it a square power 3 by 2, meaning the a cube with minus signs this will become plus and finally, we will get the answer here pi by a. So, 6 from 3 and 2 will give us 6 here and a cube will be coming when we put the lower limit. So, the volume of the sphere in this one octant will be given by pi by 6 into a cube the radius of the sphere.

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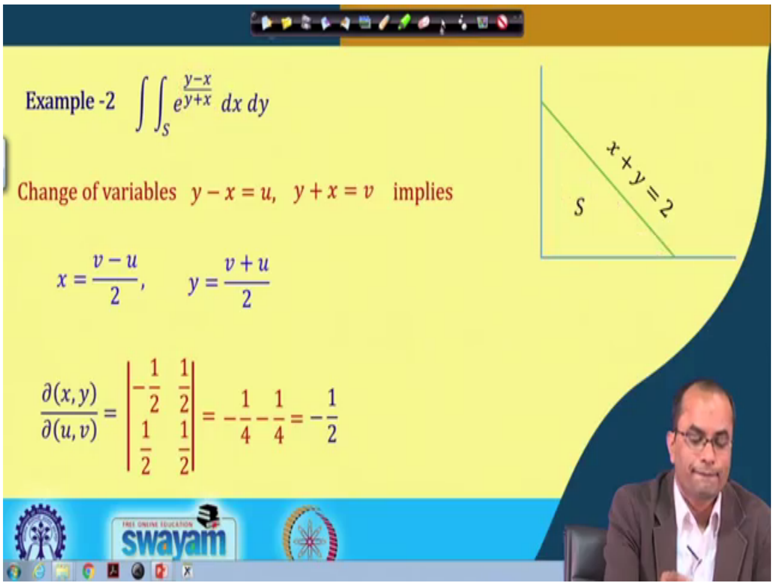

Another example where we can make use of this substitution like here e power y minus x over y plus x is given and the region is this s here which is bounded by these two coordinate axis, so x axis and the y axis here. And then we have the line x plus y is equal to 2 or; so this is the region where we are integrating this function e power y minus x over y plus x. So, the suitable substitution looking at the integrand seems to be y minus x and this y plus x. So, if we substitute, we give a new name to y minus x and also to y plus x then perhaps this integral will become easier to compute.

So, we changed the variable from y minus x we substitute as u and y plus x we substitute as v, and then out of these two we can compute also x and y in terms of a v for example, if we add 2 we will get 2 y is equal to u plus v, so many y is equal to u plus v by 2. Similarly, if we subtract this one from this one we will get x is equal to v minus u by 2 as x and y will be v plus u by 2. So, we have x and y in terms of u and v from the relations and we can get the Jacobean term because that is needed for these transformation in the new variable in the new integral.

So, here the del x over del u and del x over del v we have to compute. So, this del x over del u is as minus half which is coming here and del x over del v will be half the second term then with respect to u for y. So, del y over del u will be half and del v over del v will be also half. So, we have this Jacobian here which comes to be minus 1 by 4 and minus 1 by 4, so it is a minus 1 by 2. So, the Jacobian value in this case is minus half.

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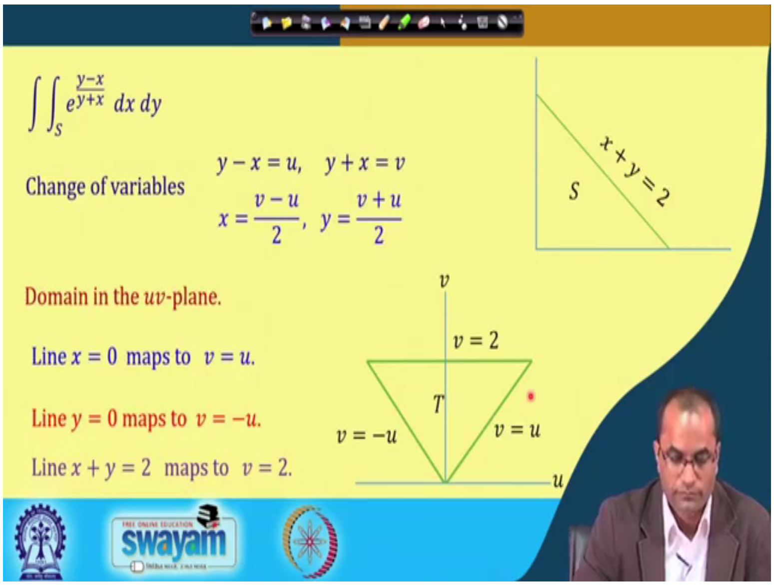

So, now this change of variable; we have to see now the domain the new domain here corresponding to the new variables u and v.

So, the domain for u and v plain would be because line x is equal to 0 here. So, we have this line x is equal to 0, this is x this is x is equal to 0 and we have this y is equal to 0 and we have this line already x plus y is equal to 2. So, having this we have to see the new domain now in terms of variables u and v. So, this line x is equal to 0 what it implies here? x is equal to 0 implies v is equal to u.

So, this implies v is equal to u. So, we got one relation in v and u which will give which will give us the boundaries of the new region. And the line y is equal to 0 will give v is equal to minus u, so that is another boundary of the new region v is equal to minus u. And the third line here is x plus y is equal to 2, so x plus y x plus y is equal to 2 means the v is equal to 2. So, we have this maps to this v is equal to 2.

So, with these 3 again v is equal to u, and v is equal to minus u, v is equal to 2, we can construct the new domain; so this is in terms of v and u. So, we have a v is equal to u line, we have v is equal to minus u line and then v is equal to 2 line. So, this region now is the new one for our integration, ok. So, now we can evaluate the new limits based on this new region in terms of u and v.

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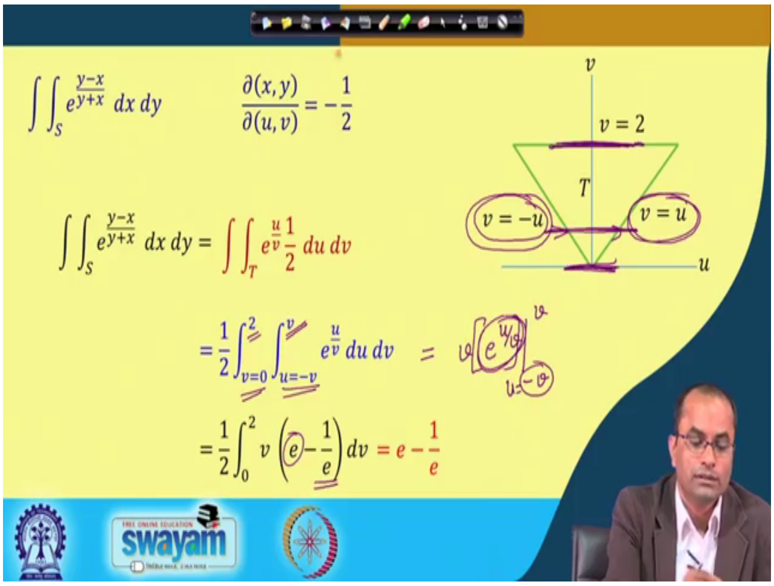

So, remember the Jacobean was minus half and we are going to compute this integral e power minus y minus x over y plus x dx dy, and our substitution was y minus x was u and y plus x was v and this is the new domain given in terms of v and u.

So, having this integral we can now substitute exponential power this was u and this was v. So, e power u over v and this half because of the Jacobian. Again, note that this was minus half, but in the integral we will take always the absolute value. So, this is coming as half here and then du dv which we want to integrate over this new domain t in terms of u and v. So, we have to put now the limits for u and v. So, first if you put the limit of u here, limits of u always from this line to the other one, from this line to the other one.

So, for u we are entering the domain at this line and exiting the domain at that line; that means, v is equal to minus u from here and oh sorry u is equal to minus v from here and u is equal to v from there. So, we have u is equal to minus v and we have u is equal to v and now such lines are moving from this v is equal to 0 to this v is equal to 2 line. So, v is equal to 0, 2 is equal to 2, that is a simple limits for this new domain. And now you can easily integrate this e power u over v with respect to u, so this will give a power u by v and with one v sitting in the numerator.

So, v and then this e power u over v when we put the upper limit v, so, it will give us e power 1, so that means, e minus the lower limit here. So, again just to make it clear we have this integral here e power u over v and we are putting the limit minus v to v for u. So, when we put the upper limit here or v this will become e which is written here and when we put the minus v, it will give us 1 over e term.

And now this is a constant, so here with respect to v this will gives a will give us v square by 2, so and this is 2 they are the upper limit, so 4 by 2. That means, 2 and that will also cancel out from this 2. So, finally, this integral is nothing but e minus half, ok.

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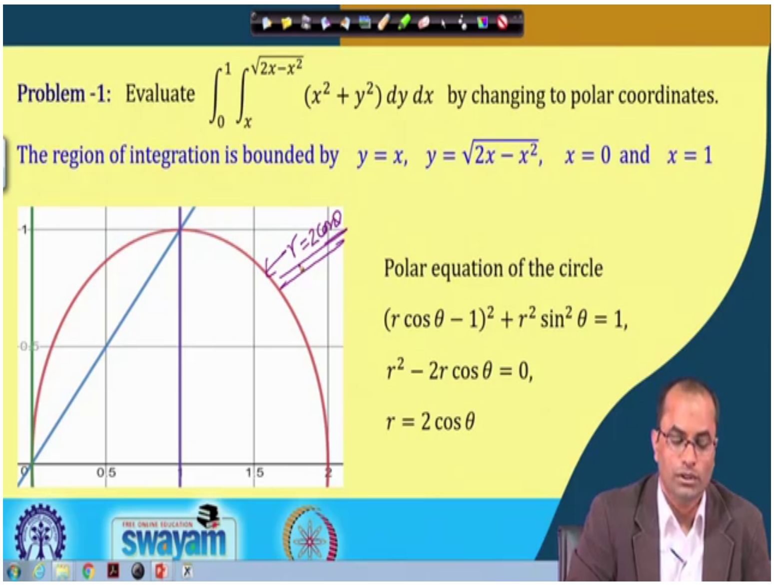

So, another problem where we want to integrate this x square plus y square by changing two polar coordinates and this is natural because the integrand is also supporting that changing to polar coordinate will help and perhaps here as well, because it is a circle here. So, this region of this integration the given integral is given by y is equal to x line. So, here we have y is equal to x line, and y is equal to square root 2 x minus is x square the upper boundary here for. So, y is equal to square root 2 x minus x square and then for x, x is equal to 0 and x is equal to 1.

So, this the enclosed region by these curves will give us the region of integration for this integral. And what is exactly this one; this is y square is equal to 2 x and minus x square. So, this is y square and plus x square minus 2 x is equal to 0. So, we have y square and plus x minus 1 whole square. So, we get minus 2 x, we get x square and get 1, so that will be compensated by the 1 there. So, this is x y square and plus x minus 1 whole square and is equal to 1. So, a circle of radius 1 and center at 1 0; so it is a shifted center at this 1 0 and the radius now 1 there.

So, this is we can draw now. So, this is the circle which we have the sifted 1. So, the center is at 1 there and now y is equal to x line, so this is y is equal to x line, y is equal to x line and we have the circle here y is equal to 2 x and minus x square and this is x goes from 0. So, this is x is equal to 0 line to x is equal to 1 line. And the region enclosed by this one, so from x to the circle, so this is the region of integration which we are using here and we have to now cover with the help of the polar coordinate.

So, again for the polar coordinate for the limits of the theta because from this line and now we are going to this one; so here it is x is equal to y line. So, this is in terms of the theta it is like pi by 4, so 10 pi by 4 is 1 because this is if this is 1 and this is also the same 1, so 10 theta will be 1 by 1; so it is a pi by 4. And then we are going up to this point here this line that is pi by 2. So, the theta is varying from pi by 4 to pi by 2 and for r we are always from 0 to this circle. So, 0 to this circle; so, we are going from 0 to that circle. So, for the limit of r is also simple, but we have to get now what is the equation of this circle in terms of r and theta.

So, the polar equation of the circle; so the given equation was y square and this sorry x minus 1 square plus y square is equal to 1. So, x r cos theta minus 1 whole square plus r square and this is for y square and is equal to 1. So, from here if we simplify this, we will get r square and minus 2 r cos theta is equal to 1, because this 1 will get canceled in r square sin square theta and r square cos square theta will give us this r square. So, from here we get this r is equal to 2 cos theta. So, here the circle is r is equal to 2 cos theta. So, the limits of r is also clear now, r is going from 0 to 2 cos theta.

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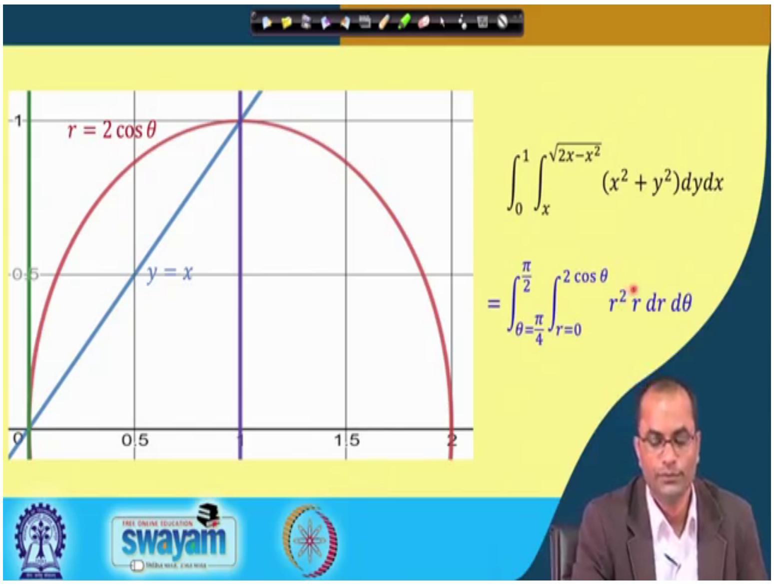

So, this integral which was originally given we will be converted to this r square here, for x square plus y square and dy dx or dx dy will become the r dr d theta limits of r from 0; from 0 to this circle 0 to the circle; that means, 0 to 2 cos theta. And then we have theta is equal to pi by 4 to pi by 2, so which we can easily compute this one now.

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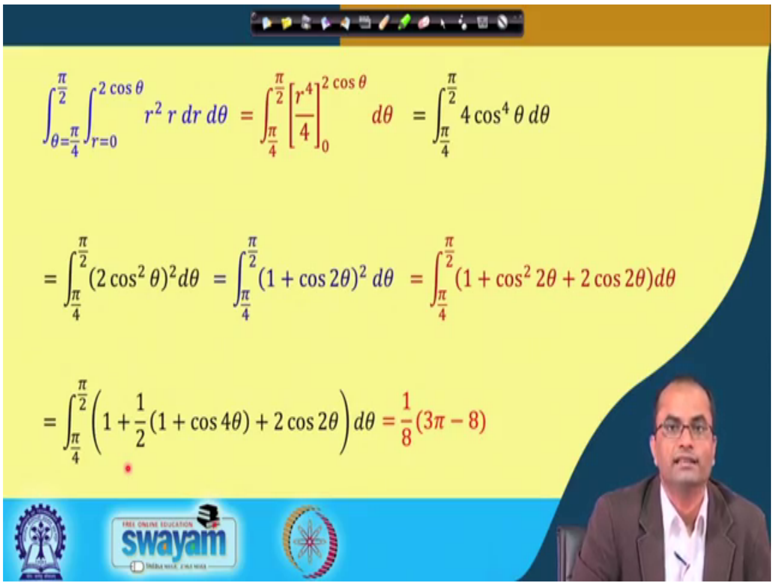

So, r from 2 to cos theta and theta from pi by 4 to pi by 2; so this is r cube means r 4 by 4 and the limits from 0 to 2 cos theta. So, the upper limit minus the lower limit; so upper limit will become then this 4 is there, so this will be 16 and then 4. So, we will get 4 cos

4 theta d theta and the limits of theta here pi by 4 to pi by 2 we have to simplify this power here. So, this we have written 2 cos square theta whole square and 2 cos square theta we can convert into this 2 theta form. So, this 2 cos square theta will be 1 plus 2 cos square theta by this relation that cos 2 theta is nothing but 2 cos square theta minus 1.

So, with this relation we have converted into 2 theta angle and then this will be again or we have to now simplify further. So, 1 plus this cos square 2 theta the whole square and 2 times the product, so 2 times cos 2 theta. So, again this square term has come up, but again we can convert into this double angle. So, this will be this cos square 2 theta will be half and 1 plus cos 4 theta now.

And now we have everything here which we can integrate; so 1 and this 3 by 2. So, this will be the constant term then we have the cos 4 theta and then we have here cos 2 theta which will be coming as sin 2 theta and cos here also sin 4 theta, and then we need to put the limits there. Finally, we will get with this simplification as 1 by 8 and 3 pi minus 8.

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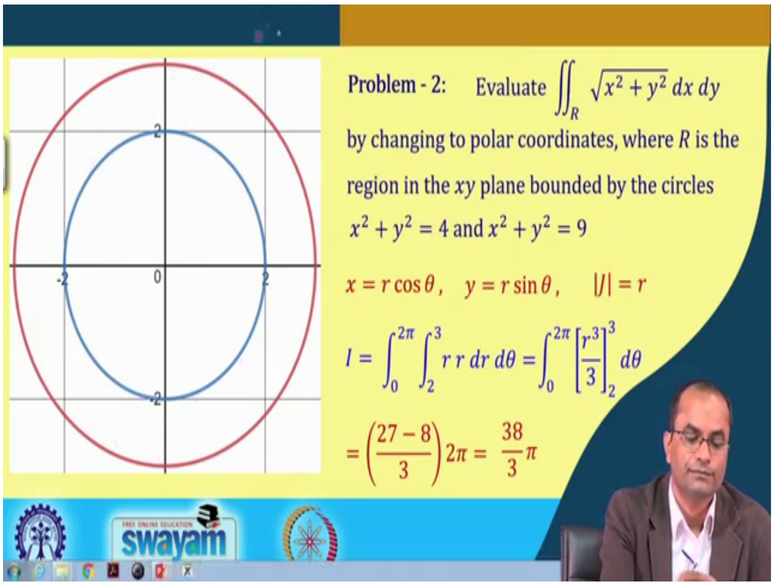

The last example which we want to evaluate here this integral over this x square plus y square the square root dx dy by changing to polar coordinates and this r is the region in the xy plane bounded by the circles here x square plus y square is equal to 4 and x square plus y square is equal to 9. So, we have 2 circles, with these 2 radius; so one is x square plus y square is equal to 4. So, with radius 2 and this with radius 3.

And we want to evaluate this given region here. So, again certainly in this case are changing to polar coordinate will help us because of the integrand and as well as because of the region here which is bounded by these two circle. So, by changing to polar coordinate this is much simpler now to get the limits because putting x is equal to r cos theta, y is equal to r sin theta, z will be r and then this integral will be 0 to 2 pi the whole circle and then for r here limit will be from 2 to 3.

So, r is going from always from 2 entering the domain when r is equal to 2 and leaving the domain when r is equal to 3. So, we have the limits for r, we have the limits for theta, and now we can easily compute this as 0 to 2 pi here r square. So, that will give us the r cube by 3 and 2 to 3 these limits which we can simplify this and we will get 38 by 3 into pi as the value of this integral.

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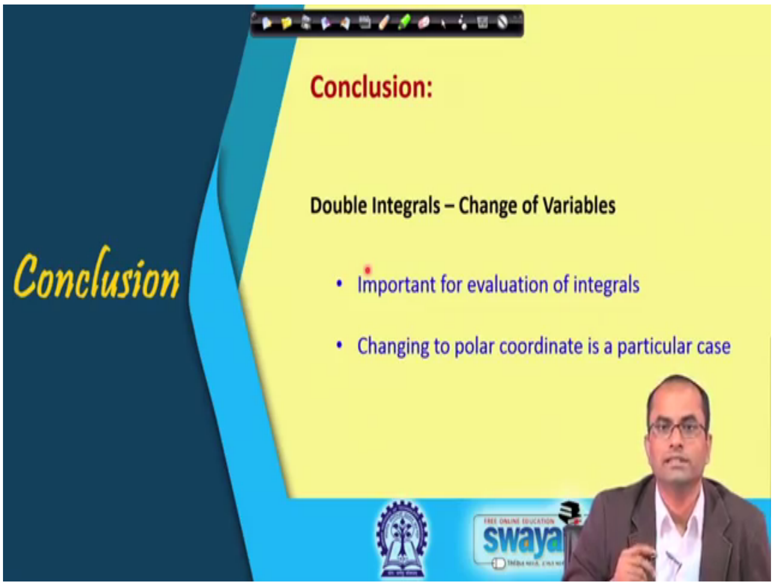

So, what we have seen now this that this change of variables is another important aspect of evaluation of double integrals and changing to polar coordinate was a particular case. But we have seen the other cases as well where by substituting two different variables makes the domain of the integral easier and also the integrand easier.

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So, these are the references used for preparing these lectures.

Thank you.