Engineering Mathematics – I
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Lecture – 31
Integral Calculus – Double Integrals in Polar Form

Welcome back. So, this is lecture number 31. And today we will again continue with this double integrals, but in particular this polar form. So, the double integrals in polar forms are very useful, and we will see with the help of many examples that if we convert in to the polar forms the integrand, and also the corresponding limits, then this integral becomes much easier to evaluate.

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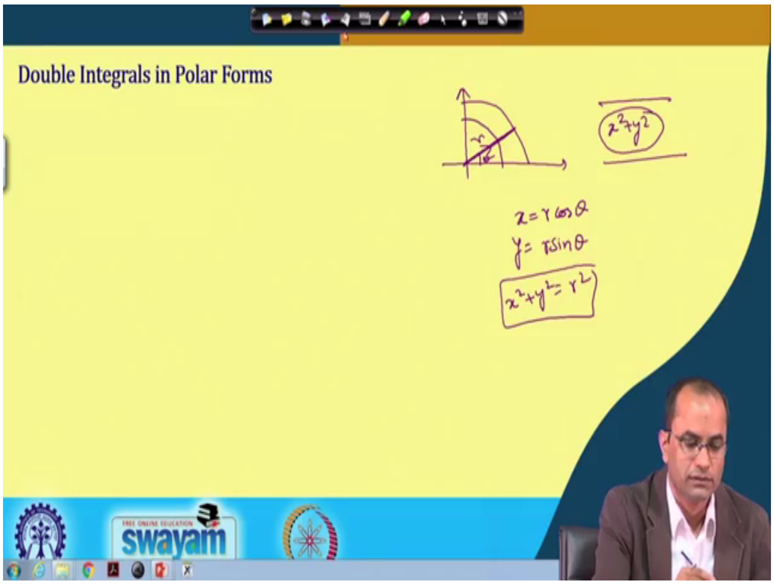

So, the situations we will be considering for the polar form when we have for example the domain which can be described in polar form. So, we have for example, the circular domain or we have some other domain where the boundaries are prescribed in terms of this variable here r and n theta. In those cases, we can easily think of converting the integral to polar form or the integrand itself.

So, when we do see in the integral like x square plus y square form and then again since we know in the polar form the relation from the Cartesian is x is equal to r cos theta, and y is equal to r sin theta. So, this x square plus y square becomes r square. So, this also simplifies in several cases. So, based on the integrand of the integral as well as the domain of integration we will decide whether it is better to go for the polar form. So, let us discuss now.

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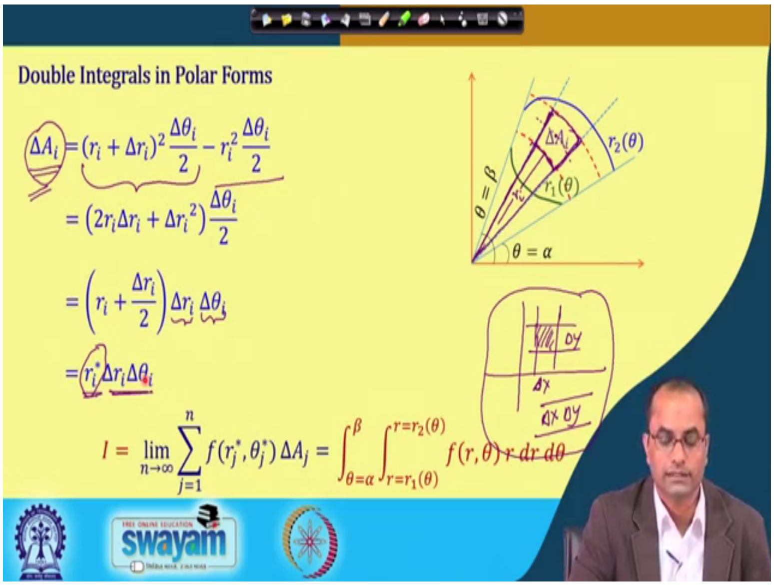

So, suppose we have situation here, the domain is bounded by this curve this is r 2 theta, it is a function of theta; and the another curve which is a function of theta here. So, we are talking about the polar coordinate. So, the distance from the origin to this point will be r, and the angle from this theta is equal to 0 x s will be theta. So, in that situation, whenever domain is bounded by these two curves here r 1 theta and r 2 theta, then we can actually do the integration easily with the help of the polar coordinates, because the limits for this domain here r easy to evaluate because in the direction of r here entering the domain from this r 1 theta and it existing this domain from r 2 theta.

o, in that case and also for the angle theta, we have that theta is going from this alpha angle. So, this is theta is equal to alpha angle, and 2 theta is equal to beta angle. So, we can curve domain with the help of polar coordinating easily, but if we discretize in the a Cartesian form, then they will be many difficulties to cover this domain, because it has this curve which is described nicely by r 2 theta and here r 1 theta.

So, now, how the integral will take the form when we have this polar coordinates. So, suppose we discretize this domain again as similar to the Cartesian one into smaller cells, and let us consider for instance here this cell which we are calling has delta A i. So, for this cell we have the radius from here to this point or any point on this line here it is r i.

So, this distance from the origin to this is r i, and then we take and increment here in r i by r i plus delta r i to reach to the other radius here from thus to this point.

And similarly the angle which we trace to go from here to here, it is described like by delta theta. So, we have the delta theta angle and then this radius from r i 2 r i plus delta i. So, the radius to here is will be r i plus delta r i. So, considering this, and again we have to also know that once we have this circular arc which is angle lets say theta part here or forming angle theta then the area of this region will be pi by sorry theta by 2. So, the angle is theta, theta by 2 and this radius r, so radius r that will be the area of this region here by this circular arc and which is making an angle theta at this point. So, this area will be theta by 2 into r, so that we will be using now here in this consideration.

So, let us try to compute the area of this is small part of this domain which is denoted by delta i. So, in this case, now we have this delta i which is given by the area of this larger region up to this r i plus delta i radius, and we will subtract the inner one. So, we have one this area which is given by r i plus delta r i, because that is radius; and the angle was theta i delta theta i and by 2, and minus we have subtracting now this area here by this one to get the area this delta i. So, the delta i will be given by r i plus this delta i whole square and delta theta i by 2 n minus for the inner part here r i square because this radius from here to here is r i. So, this r i square and delta i by theta delta theta i by 2 again, so that is the area of this small region which we have denoted here by this delta A i.

And now so this we can simplify easily because this will be r i square and delta r i square 2 times r i delta r i, where here again we have this r i square, so that will be cancelled out, and we will get two r i delta r i plus delta i square. And then this common factor from both the terms is delta theta i by 2. So, now, this we can again we write as by taking this common delta r i term.

So, also we can consider this two here. So, we can divide by this two. So, we will get r i from the first term while taking this delta r i outside and dividing by 2, so we will get r i. And from here we will get delta r i by 2. So, this delta A i the area of this region is given by this expression here, which is r i plus delta r i by 2. So, this r i plus delta r i 2 is somewhere in this middle here. So, this is the arc which the circular arc which has the radius r i plus delta r i by 2, and then we have delta r i and this delta theta the increment we have made in the radius and also in the angle.

And we can denote now this here by r i star and then we have delta r i delta theta i. So, the area of this small region now in this case we are getting this radius here r i which we have denoted by r i star that is some this is middle of this two circular arcs we have for this region and then delta r i delta theta i. So, this is let us different than what we have in the Cartesian case. So, when we have done same thing for the Cartesian one, so we were getting this increment in this direction by delta x and in this direction by delta y, and then this area becomes just delta x or delta y.

But in this case when we are talking about the circular or the polar coordinate in this case the area of this region in this domain here delta i is r i star and delta r i delta theta i. While in the Cartesian coordinate, it was simply the product of the increments we have made in the direction of x and in the direction of y. So, here one additional factor this r i has appear. And now because of this, we will get we will define now the integral this i over this domain here which is described in the polar form by this limit.

So, we will do the same. So, the function if a function is given. So, we will compute that function value at a point r i theta j so somewhere in this between. Now, this region r i j point and theta j star point multiplied by this delta A j, similar to the Cartesian one. But now this one since this delta a j s r i times delta r i delta theta, so now this limit will take the or we will write in the integral form now f r theta and because of this delta A j which has come as the r and the product of delta r delta theta. So, this will be r times delta r delta theta.

Ano[ther] now the limits we have to now consider for this domain, but these are given in the polar form. So, it is very easy for the r. We are entering the domain from r 1 theta and existing the domain from r 2 theta, and the theta varies from the angle alpha to beta. So, this is now these are the limits for r and theta and one we should know that this r factor has come as an extra in comparison to the polar coordinate, where we use to have just simply dx and dy. So, here also we have the function and then that will be evaluated r at this coordinate here r theta, and they will be additional factor r and then dr d theta. And to cover the region we have to consider this rangers for theta and r.

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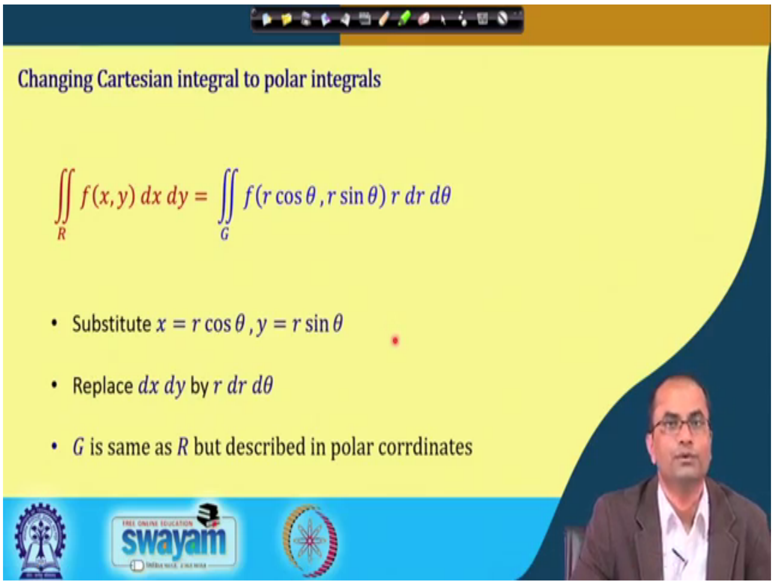

Well, so now we will see here that how to change the Cartesian integral to polar coordinate, though we have already discuss that that how to write the integral in the polar coordinate. So, suppose integral is given in Cartesian coordinate f in f x, y dx dy form over some region r. So, what you will do now to convert into the polar form we will replace this x by r cos theta we will replace this y by r sin theta, and this dx dy will be replaced by r dr d theta, and then the corresponding to r in t theta we have to place the limits that covers the region r.

So, here again f then x will be replace by r cos theta y will be replace by r sin theta. And this dx dy factorial that is important, so this dx dy factor will be replace by this r dr d theta, because the area of the this is small region which we have just seen in case of the polar coordinate it comes to be r dr d theta while in the case of Cartesian it was dx dy. So, having this what is important again the steps are that we substitute x is equal to r cos theta y is equal to r sin theta in the integral, and this dx dy will be replaced by r dr d theta. And this G is basically the same the same region naturally, but now it is described in terms of the polar coordinates. So, we will see with the help of examples now how to evaluate integrals in polar form.

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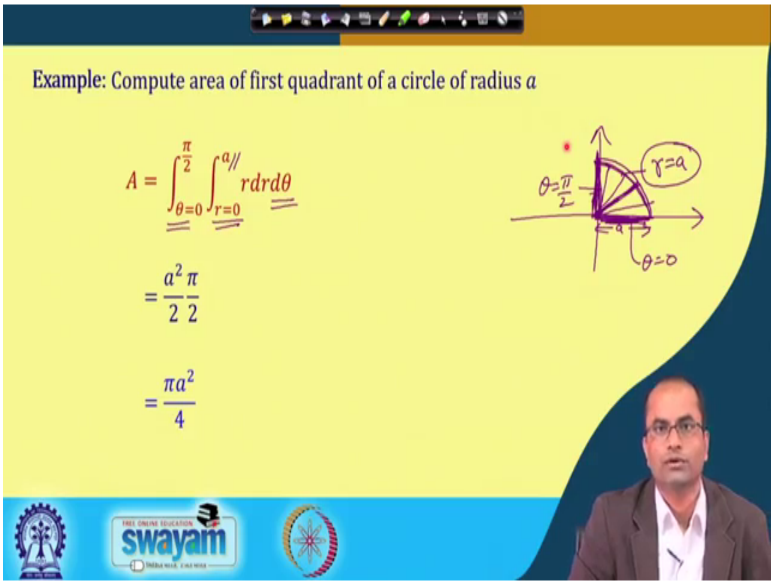

So, now, we want to compute, we want to start with a very simple example compute area of the first quadrant of a circle of radius r, of radius a. So, we have a circle of radius a, and we want to compute the area. So, again for the area we have to just assume that this function f is 1 and then we will get this integral over r dr d theta that will give us the area as we have done in the Cartesian coordinates. So, here this area of the circle will be given by theta 0 to pi by 1 because our first quadrant. So, here this is the circle and the radius is a. So, the limits for theta for instance first. So, theta vary from this 2, it will go up to pi by 2.

So, for theta from 0 to pi by 2, at this line the theta is 0 and here at this line when we reach theta is pi by 2. And r is varying from 0, it is starts from 0 up to this circle here which is r is equal to a. So, the limits of r will be from 0 to the circle r is equal to a, and theta varies from this 0 to theta is equal to pi by 2. So, we need to just evaluate this integral, which will be here r is square by 2. So, r the upper limit is a. So, a square by 2. And then from the outer integral we will get when we integrate with respect to theta so the theta will appear which will gives us pi by 2, so that is that is the area of this part of the circular region.

So, here pi a square by 3 which we have easily evaluated with the help of the polar or the integral in the polar form. But if we do in the Cartesian form, then there will be little it will be little bit complicated because we have to now draw the limits of the x and y and that will contain the circle in the in the Cartesian form. So, the equation of the circle in the polar form is simply r is equal to a. So, we are just to cover this region, we are going from r is equal to 0 to r is equal to a, but in the Cartesian coordinate this will be x square plus y square is equal to a square. So, we have to whether first compute the limit of x or y accordingly we have to get this square root of a square minus x square or y square, so that will make the integral bit complicated ok.

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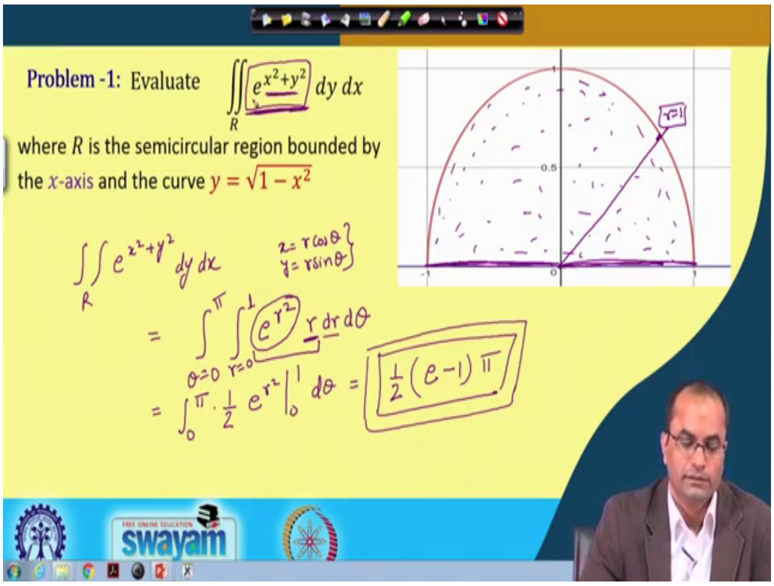

So, moving next, we have this problem now you want to evaluate this integral e power x square plus y square dy dx. And this R is the semi circular region bounded by the x-axis and the curve y is equal to 1 minus x square. So, this curve is nothing but y square is equal to 1 minus x square meaning this x is square plus y square is equal to 1. So, this is the circle. So, we have a circle here. And then the x-axis and we are talking about the semi circular region. So, bounded by this x-axis and this upper part of the circle.

So, if you draw the region here that will be this half circle with radius 1 and then we have here the x-axis. So, this is the region which we want to integrate over this region. So, again since we have the circular region it would be much convenient to use the polar coordinate. Secondly, our integrand here has this term like x square plus y square, so that will or some make it easier for the evaluation of the integral.

So, if we now compute the integral, so that will be over R this e power x square plus this y square and that is given as the dy dx. So, here if you write down in the polar form first of all the integrand, because we are substituting now x is equal to r cos theta and y is equal to r sin theta in the integrand, which will gives us which will give us this x square plus y square as r square. So, we will get here e power r square and this dx dy will be r dr d theta.

So, first let us put the limit of r. So, r will go from 0 to 1 to cover this region. So, r will go from 0 to 1 this is the r is equal to 1. So, r is moving from 0 to 1. And theta is going from so this is theta is equal to 0, and then we will reach to this part here by theta is equal to pi, so theta from 0 to pi and r 0 to 1 e power r square and r dr d theta.

And now if we look at closely here direct evaluation in the form of this Cartesian coordinates, this was not easier, this was rather not possible to integrate this e power x square plus y square term, but now we got here e power r square and r. So, we can easily integrate this integrand now with respect to r. So, we have theta from 0 to pi, and this will gives us the we will gives a half and then you have 2 r here, so exactly the differential of this r square.

So, this integral will be r square and we have 0 to 1, we have d theta. So, this will be e minus 1 with half. And then this theta as when put the upper limit pi, s, they will be another pi there. So, it will be pi by 2 and e minus 1 that is the value of the integral which was e power x square plus y square and dy dx. So, can using this polar coordinate, this is much simpler to evaluate this.

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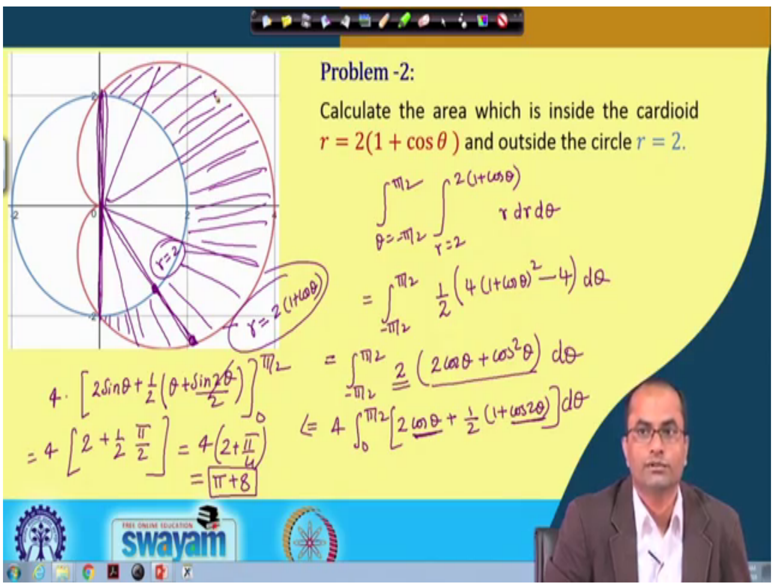

So, let us take another problem here, problem number 2 which will give us now again little more inside. So, here calculate the area which is inside the cardioid r is equal to 2 and 1 plus cos theta and the outside the region r is equal to 1, the circle r is equal to 1. So, there is a circle here of radius this 2, centre 0. So, this is the circle. And then we have this cardioid r is equal to 2 times and 1 plus cos theta. So, we have to first draw the region as usual.

So, in this case we have this cardioid and then we have the circle here of radius 2, and then what is ask here which is inside the cardioid and outside the circle r is equal to 2, so that means, we are talking about this here outside the circle and inside this cardioid. So, this is the region we want to find the area now, and naturally we will use the polar coordinates because this is again much easier.

So, we have we have moving in theta from this point, and then going all the way and the theta will go up to this point. So, theta the limits of the theta we can take like from minus pi by 2 to this pi by 2, minus pi by 2 to pi by 2. And for the r, r is always from this circle which is r is equal to 2 to r is equal to the cardioid. So, the here is r this cardioid here is r 2 1 plus and the cos theta, and this is r is equal to 2. So, the limits of r will be also easier to evaluate and for theta as well. So, the theta varies from minus pi by 2 to pi by 2, and then we have for the r from 2 to 2 1 plus and cos theta and then for the area so our integrand is going to be 1 and then we have r and dr and d theta, so which we can evaluate now easily. So, this is minus pi by 2 to this pi by 2 here r square by 2.

So, r square means here 4 and 1 plus the cos theta the upper limit and then we have minus 4 again for the lower limit. So, this 4 and this d theta. So, we have minus pi by 2 to pi by 2, this 3 we can cancelled out. So, we get two and this one also will be we cancelled out, we will get two cos theta and plus they will be term cos is square theta. So, we have 2 cos theta and cos square theta and d theta multiplied by this factor 2. So, here this integrand is the even one the integrand is even. So, in fact this integral will be four times and 0 to pi by 2, and we have here then 2 cos theta and this also we can convert into cos 2 theta from. So, 1 plus cos 2 theta for this cos square theta, and then we have d theta there so which we can integrate easily. So, we have here the sine term here also will have sine term.

So, if we compute this now, so we will get 4 times and the limit. So, later on so 2, this is sin theta and plus this half we have for one th theta and plus again or the sine 2 theta by 2, and then the limits 0 to pi by 2. So, this is 4 2 and the sine pi by 2 will be 1, again sin 0 will be 0, here we have half theta will give us pi by 2, and then sin this will be pi and minus sin 0 0. So, in either case this will be 0 when we put the limit. So, we have here the 4 times and 2 plus pi by 4 or pi plus 8, so pi plus 8, so that is the area here of this region which is drawn in the figure. And with the help of again the polar coordinate this was very easy to evaluate.

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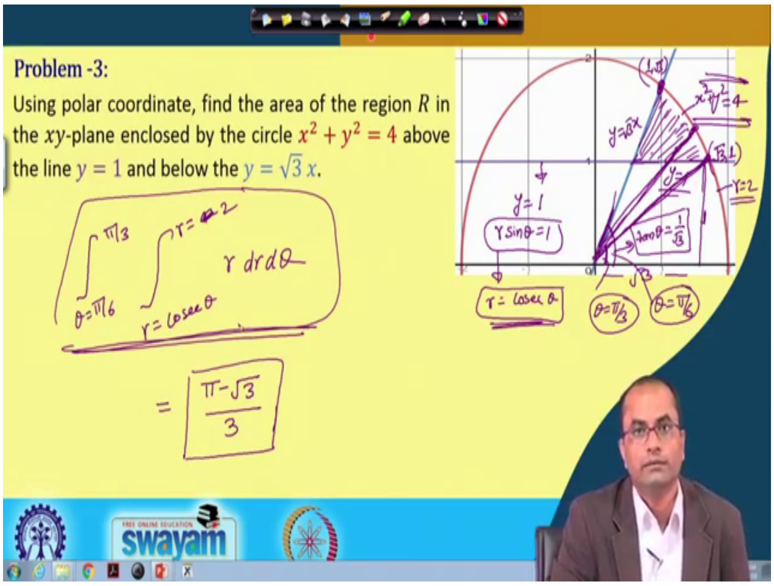

So, coming to the next problem; problem number 3. So, using the polar coordinate will find the area of the region R in the xy-plane enclosed by the circle x square plus y square is equal to 4 above the line y is equal to 1 and below the line y is equal to a square root 3 x. So, we have two lines y is equal to 1 and we have y is equal to a square root 3 x, and then the circle here x is square plus y square is equal to 4. So, that is the situation. The circle and then we have this line here y is equal to square root 3 x and then we have the line y is equal to 1, y is equal to square root 3 x, and then this is the circle which is given by x square plus y square is equal to 4.

Now, enclosed by the circle above by this line and below by this line; so, this is the region which we want to now find the area using the polar coordinate. So, in this case, now we have to see the limits for R and also for theta. So, we are moving in the direction of r. So, from this point or for theta from this point and we have going up to that point. So, we need this here and we need that, so these two angles.

So, first we need to compute these points where these lines are meeting. So, this is y is equal to one line, and then we have the circle here x square plus y square. So, y if we put 1, x you will get as a square root 3 and 1 that is the point here. And in this case since y is equal to square root 3 x if we put there, we will well get x as 1 and y as a square root 3. So, with this now we can compute easily this angle because this is the vertical distance is 1, and here is a s square root 3 for the first angle this one. So, tan of this angle, tan of theta will be 1 over a square root 3, and we can get that theta s this pi by 6. So, this is theta pi by 6.

Similarly, for the other one, ten theta will be square root 3, that means, this theta will be pi by 3. So, we are moving from pi by 6, 2 pi by 3 for theta. And for r, we are going from this line to the circle. So, this equation of the circle is r is equal to 4. So, we are going up to this r is equal to 4, and from this line the equation of this line is y is equal to 1. So, if you put r sin theta is equal to 1 from here we can get the r. So, r is the cosec theta. So, we have now everything ready to compute, so that integral will be the limits for theta from pi by 6 to the pi by 3.

And for r we are going from this line the equation of this terms of this theta will be cosec theta, and then we are going up to r is equal to 4, r is equal to 2. So, the radius is 2. So, this is the circle is r is equal to 2. So, here r is equal to 2. And then we are finding the area so no integrand, and we have r and dr d theta. So, we have to just integrate the simply integral. And after this evaluation, I am a skipping this part, we will get pi minus this is square root 3 over 3. So, the important is drawing the region and putting the limits of the integration.

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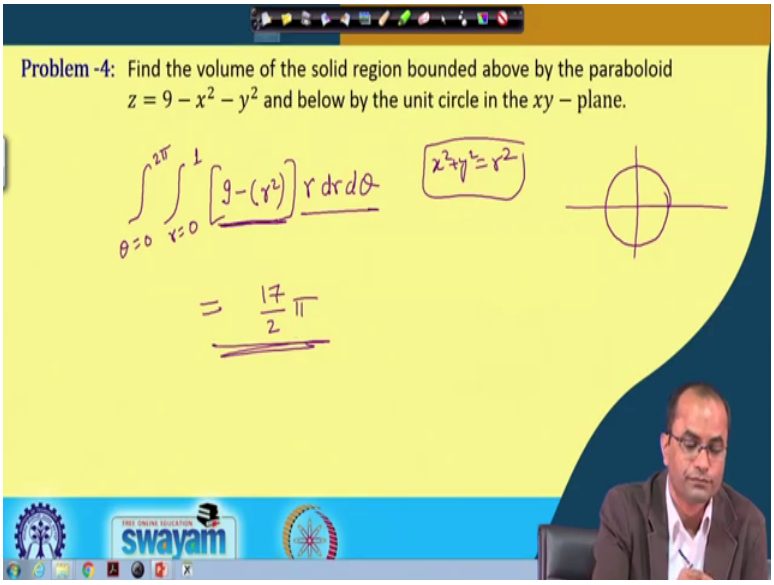

Moving now to the next problem, you want to find the volume of the solid region bounded above by the parabola, paraboloid z is equal to 9 minus x square minus y square and below by the unit circle. So, above this unit circle, we want to get the volume of this paraboloid. So, here can we have the unit circle. So, we know the region now and we can easily draw the limits.

So, the r will go from 0 to 1, we have unit circle for the limits of the theta it will be from 0 to 2 pi, and we have the integrand 9 minus this x square plus y square which will be become r square in the polar coordinate and then we have r dr and d theta, so that is the integrand here, and then this r dr d theta. So, this x square plus y square in polar coordinate that becomes r square. So, we got this 9 minus r square and r dr d theta which again it is a very simple to evaluate. And this will be coming up 17 by 2 into pi the evaluation of this integral for these limits for these limits ok.

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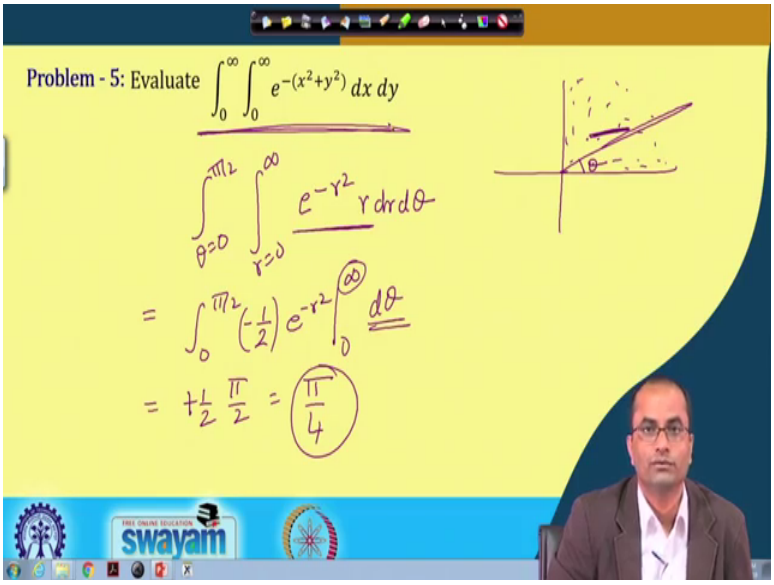

The last example another important example which with the help of the polar coordinates we can evaluate, so 0 to infinity, 0 to infinity and then we have so that means we are talking about this first quadrant here in the whole up to infinity, and then e power minus x square plus y square dx dy. So, for this region here now in terms of the r and in terms of the theta, so the theta varies from 0 to pi by 2, and r varies from 0 to infinity and e power minus this x square plus y square will become r square and then dr d theta.

So, which is again we have then before also similar example this will be half e power minus r square minus half and e power minus r square, and then we have 0 to infinity and this d theta. So, minus half and this e power infinity when r approaching to infinity this will be 0 e power 0 will be 1. So, you get one here and then this theta integral over this zero pi by 2 will give a pi by 2, so we have pi by 4. So, the value of this integral 0 to infinity and 0 to infinity is pi by 4, which is very simple when we use the polar coordinate, but it is very difficult if you try to do on the Cartesian coordinates.

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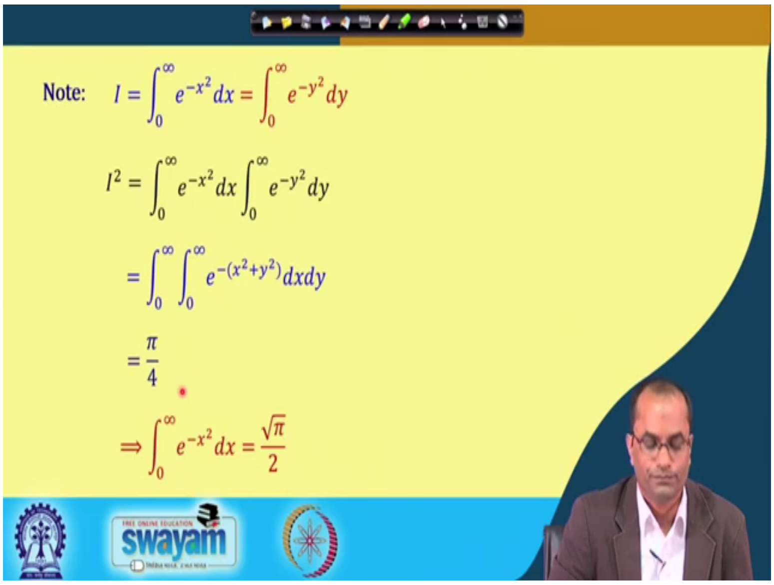

Here just not based on the above example if you consider this I here 0 to infinity e power minus x square dx which is equal to another integral I am considering terms of y. So, just the integral variable changes name to y now, but does not matter the value will be the same. And we consider the product of these two integrals, that means, 0 to infinity power minus x square in 0 to infinity this e power minus y square dy. And this we can also write as the double integral e power minus x square and plus y square dx dy.

And we know the value of this integral just previously computed, it is a pi by 4, and this I square is pi by 4. So, we can get this I the value of this integral which we have you several times or we will be using also several times this value 0 to infinity power minus x square dx, the value is square root pi by 2 or when we take from minus infinity to plus infinity, the value here comes to be square root pi, so thats another important integral which with the help of the double integrals, and in particular with the help of the polar form of the double integrals, we can compute this 0 to infinity power minus x square dx in a very simple form.

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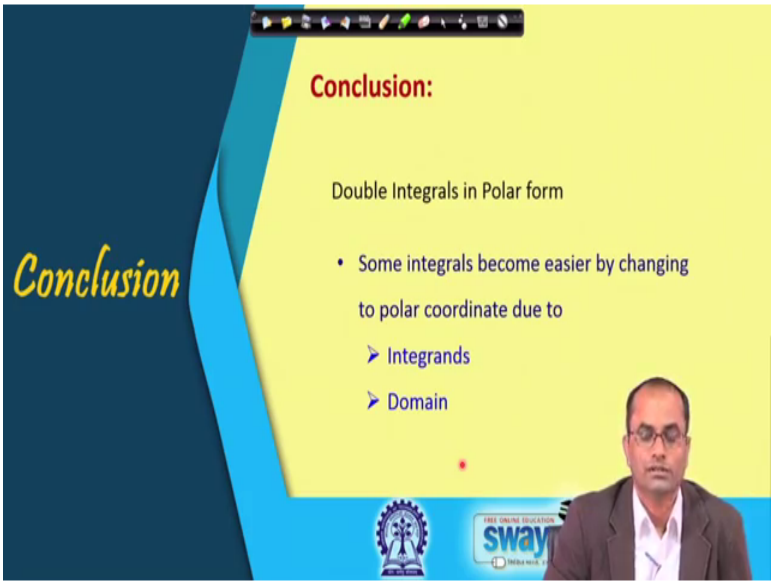

So, the conclusion that is double integrals in polar forms are very important, and mainly the some integrals like we have seen just before becomes very easier. We can evaluate easily by change into the polar coordinate. And that based on the integral integrants and also the domain, we can easily decide that which form is better whether the Cartesian or the polar form we have seen through some examples.

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And these are the references which we have used to prepare these lectures.

And thank you very much.