Engineering Mathematics - I
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Lecture - 30
Double Integrals (Contd.)

So, welcome back to the lectures on Engineering Mathematics I and this is lecture number 30 and you will continue our discussion on Double Integrals.

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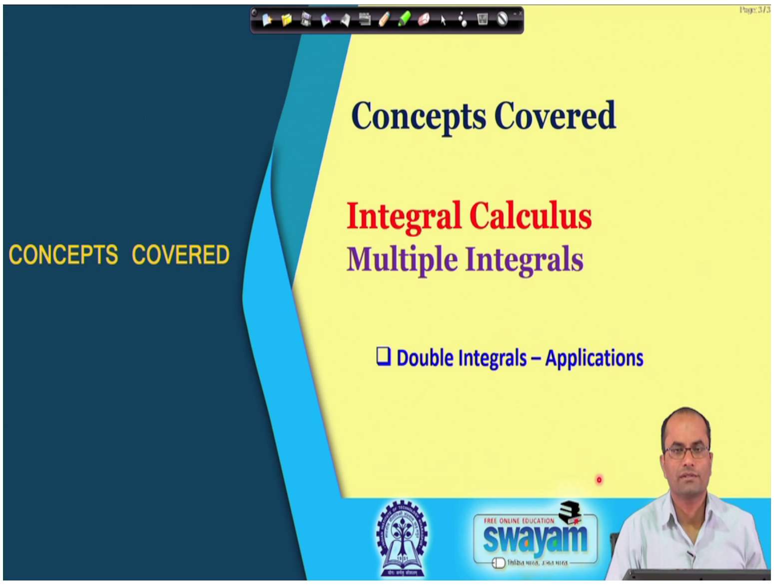

And today in particular we will be talking about some applications of double integral.

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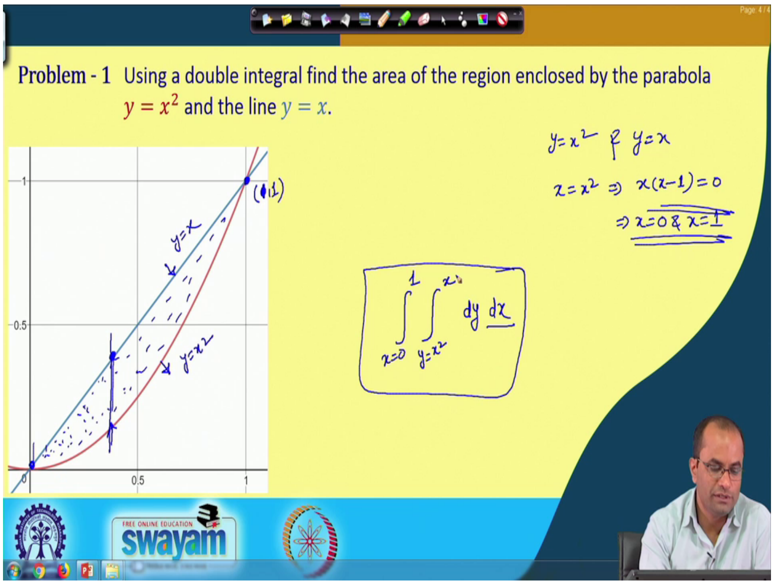

So, let us start with the problem number 1 where we will find the area. So, here using a double integral find the area of the region enclosed by the parabola y is equal to x square and y is equal to x. So, we have two curves here: one is the parabola y is equal to x square, and the other one is the straight line y is equal to x. So, the area here enclosed by these two curves, we want to find using the double integral. And, the idea was which has already been discussed in the previous lecture, that we can simply integrate the function 1 or the constant function 1 over this region and then we can get the area of the region bounded by these two curves.

So, precisely we have these two curves y is equal to x this line, and this y is equal to x square this parabola. And then we have to get this point of intersection which is pretty simple in this case. So, we have y is equal to x square and y is equal to x these two curves. So, y is equal to x is equal to x square and this will be give us the point of intersection. So, x square and minus x is equal to 0. So, we will get x is equal to 0 and x is equal to 1 these two points and corresponding y of course, we can easily get. So, here x is equal to 0 is this point, where we have x 0 and y 0, the another point we have here where the x is 0 and y is sorry x is 1 and y is 1.

So, these are the two points and now we will compute the area of this region enclosed by these two curves. So, we need to find the limits now. So, the limits of this integral; here we have the possibility whether we take first in the direction of x and then in the direction of y it does not matter. So, we will take now for example, in the direction of y first and then in the direction of x. So, in the direction of y we have the limits from this curve to that curve, and these lines will run from x is equal to 0 to x is equal to 1 to go through all this enclosed area.

So, here this curve is y is equal to x square and the above one here is y is equal to x. So, in the direction of y we are entering through this curve x is equal to y is equal to x square and leaving this area, by this curve y is equal to x the straight line. So, here the limits of y will be y is equal to x square to y is equal to x and such lines are moving from x is equal to 0 to x is equal to 1. So, this integral will give us the area of the integral of the region bounded by these two curves.

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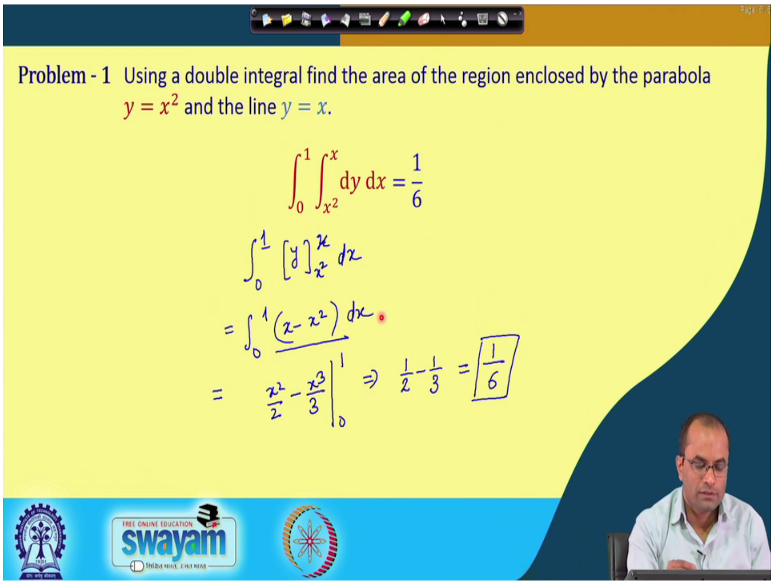

So, we need to compute simply this integral now, which will give us the area of the region enclosed by these two curves. So, here the outer one we keep as 0 to 1 and the inner one with respect to y. So, we will get y and then the limit x square to x and then we have here dx. So, this will be 0 to 1 and y the upper limit is x and the lower limit is x square. So, x minus x square and then we have dx here and now we can integrate the single integral again. So, here we will get x square by 2 and x cube by 3 and the limits will be 0 to 1. So, substituting this upper limit here, the lower limit will make anyway this 0. So, when putting at a limit we have the 1 by 2 and minus this 1 by 3. So, this will give us 1 by 6. This is the area of the region bounded by this parabola and the straight line y is equal to x. So, that is the answer of this integral.

So, we will take another example of little more slightly more complicated. So, here we have instead of y is equal to x for example, y is equal to x plus 2 line. So, in this case again we have one parabola and then the line instead of y is equal to x we have y is equal to x plus 2.

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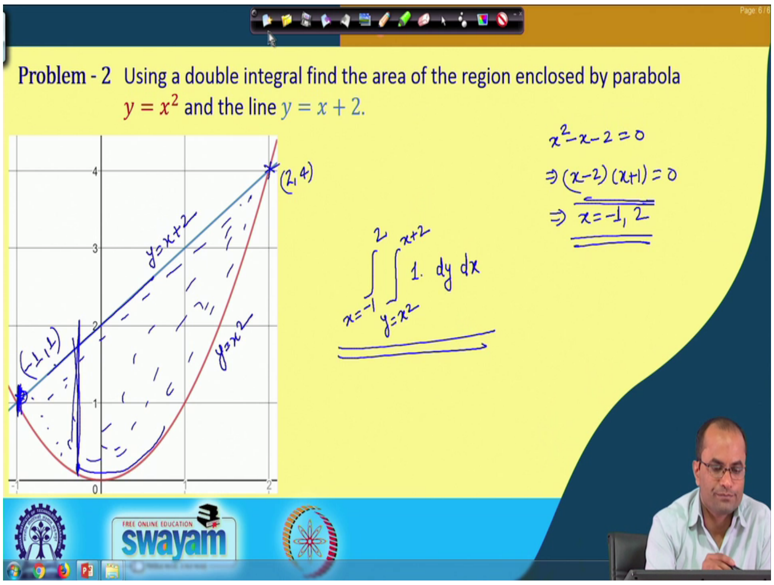

So, this is the situation we have this y is equal to x plus 2 line. So, this intercept here when x is 0, the y value is 2 and when x is minus 1 this is 1 here. So, this point of intersection again; so, y is equal to x square and y is equal to x plus 2. So, we can get just by this solving this equation x square is equal to x plus 2 or x square minus x minus 2 is equal to 0. So, this will give us x minus 2 and x plus 1 is equal to 0.

So, x is equal to minus 1 and x is equal to 2. So, we will have these two points now where these two curves intersects. So, one is precisely this one, the other one at this point here. So, here when x is 2 so, y will be 4. So, 2 comma 4 is this point and then minus 1 and 1 is this point here and we are computing this area bounded by these two curves. Now again the same idea that in the direction of y we are always moving from this curve to this line and in the direction of x will be moving than. So, this is in the direction of y which is always from this curve y is equal to x square to y is equal to x plus 2 and then this line will move from this x is equal to minus 1 to x is equal to 2.

So, this area bounded by these two curves we will compute this integral, the integrand will be 1 and then we have dy dx. The limits of the inner integral y with respect to y will be from this y is equal to x square to y is equal to x plus 2 the parabola to this to this line from this parabola to that line. And now for the x as discussed we will take from minus 1 to 2 minus 1 to 2. So, which we can easily integrate and then we will get the area enclosed by these two curves.

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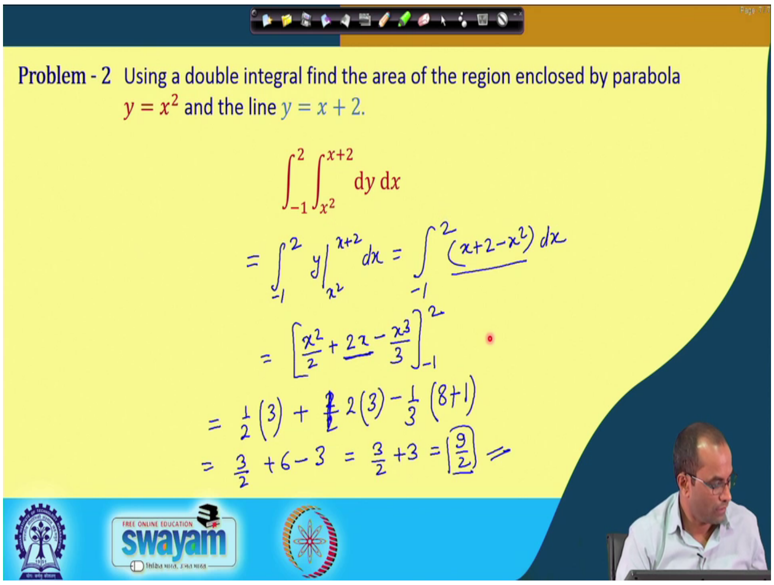

So, this is the integral which we want to evaluate now to get this area. So, let us do that. So, here we have minus 1 to 1 the outer integral and for the inner one with respect to y. So, we have just y there and then here x square to x plus 2 and dx, which will be minus 1 to 2 and this y first x plus 2 and then minus x square dx.

So, now we can integrate. So, here we have x square by 2 plus this 2 x and minus this x cube by 3 and the limits will be minus 1 to 2. So, here we can now put the upper limit and the lower limit at each expression here. So, we have x square by 2; that means, 1 by 2 and this x square. So, 4 and then minus 1 so, it is a minus 3. So, we have minus 3 by 2 plus this 2 and then x square by 2. So, x square will give 4 and then sorry with this already the integrals, we are just putting the value. So, when we put the upper limit here. So, this will be 2 and then minus minus 1. So, this will be 3 and minus is 1 by 3 and we have this 8 and plus again this will be added because this minus 1 power minus 3 will be minus again.

So, this here we get 3 by 2 and plus this 6 and minus 3; that means, 3 by 2 and plus 3 and 6 plus 3 9 by 2. So, this area enclosed by these two curves now as 9 by 2; so, which is also written there.

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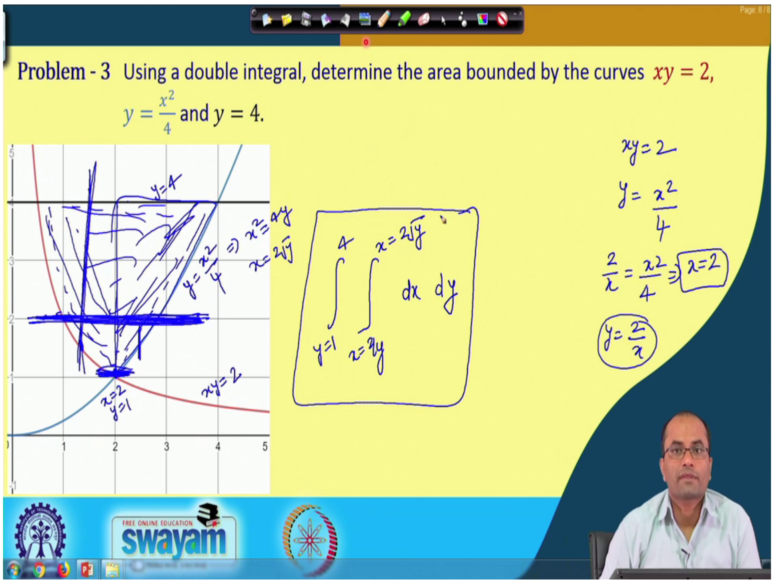

So, now we will take one more example for the area, where we want to again use the double integral and determine the area bounded by the curves xy is equal to 2. So, now, we have 3 curves here x y is equal to 2 and y is equal to x square by 4 and y is equal to 4 this line parallel to the x axis. So, let us look at the graphs of all these curves. So, we have here y is equal to 4 this straight line, we have this parabola again y is equal to x square by 4 and then we have x y is equal to 2.

So, here as x is approaching to 0 this will go to infinity and same thing when x will go this y will go to 0 and x is going to infinity. So, we have this curve here x y is equal to this is x y is equal to 2 and then we have here y is equal to x square by 4 and this straight line is y is equal to 4. So, these 3 curves and the area bounded by these 3 curves will be precisely this one here.

So, this is the area we are going to compute now with the help of double integral. So, in this we have the possibility though going first through the x axis and then the y axis or the other way round, but in this case if we go first to watch the x axis. So, in this direction if we take the integral first in the direction of x what will happen? We are going from this curve which is x y is equal to 2 to this curve which is y is equal to x by 4, and in this direction we will go from y from this point which we will compute now what is the point of intersection here to y is equal to 4 which is already given.

But, if you first fix in the direction of x; so, if we first fix in the direction of x, then the problem will be that going from this curve to the line always, but up to this point; next to this we will be going from this parabola y is equal to x square by 4 to that line. So, we have to break this integral into two regions one would be this one and the other one would be this one.

So, instead of this if we take first with respect to x that will be easier, because we do not have to break the domain in that case we will go from this curve to this curve always and in the direction of y from this point to that point. So, we need to compute this point of intersection of this x y is equal to 2. So, x y is equal to 2 and the other one is y is equal to x square by 4. So, here we substitute the value of y from there, its a 2 over x 2 over x is equal to x square by 4 and from here we can get that x will be just 2. So, this point here is a precisely x is equal to 2 and the y will be given by. So, this is x is equal to 2 and y is 2 over x. So, y is 2 over x so; that means, 1. So, here y is 1. So, this point here y is 1 and that line here y is 4.

So, in the direction of y we will go from 1 to 4, and in the direction of x we will go from this x y is equal to 2 to y is equal to x square by 4. So, let us write down this integral. So, inner one with respect to x and the outer one with respect to y. So, for the inner one we are moving from this x y is equal to 2; that means, x is 2 over y and their the upper boundary will be given by this x square is equal to x square is equal to 4 y; that means, x is 2 square root y. So, here 2 square root y and for y we will go from 1 to 4. So, that is the area bounded by these 3 curves, we can compute this integral. So, let us do that.

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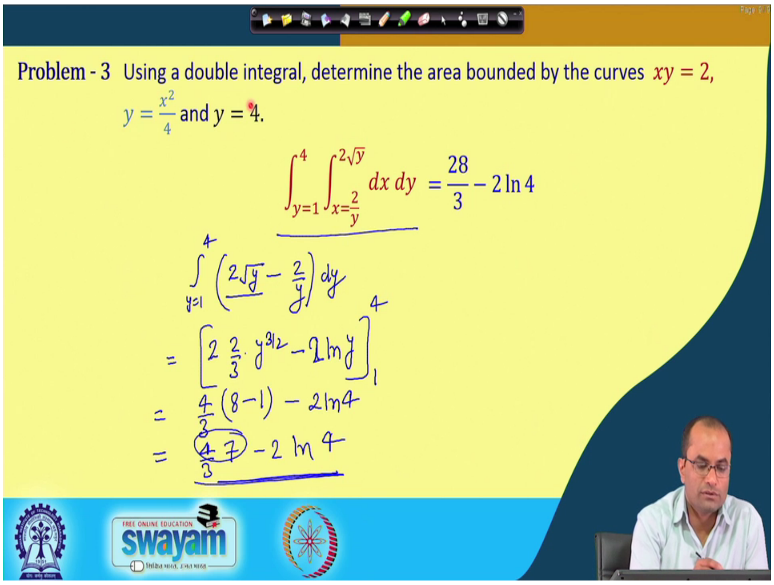

So, here this is the integral we want to compute now. So, for the inner one so, we fix the outer one y is equal to 1 to 4, for inner one with respect to x first. So, we have x and then we will put the limit. So, the upper limit 2 square root y minus 2 over y so, 2 square root y minus 2 over y and dy. So, we can integrate this easily. So, we have 2 and the square root y. So, 2 y power half and then there will be plus 1 and divide by that exponent. So, here that will be 3 by 2. So, we will have 2 by 3 and y 3 by 2 that is the integral of this part and minus this 2 times the ln y and that is the limit 1 to 4.

So, here we have 4 by 3 and then this 4 the square root 2 and then power 3. So, we will get 8 there and minus this 1. So, the lower limit you will get this minus 2 ln the upper limit 4 and when we put the lower limit here for y we will have ln 1 and that will be 0. So, in this case we have now the 4 by 3 and this is 7 minus 2 ln 4. So, 28 by 3 minus 2 ln 4 that, is the value of this integral. 28 by 3 minus 2 times the ln 4 that is the area bounded by these 3 curves. So, we have seen these 3 examples, where we have computed the area bounded by the curves or sometimes these two curves or the 3 curves. So, in this case it was enclosed by a 3 curves and we had another application which we have already discussed that is the volume of the solid.

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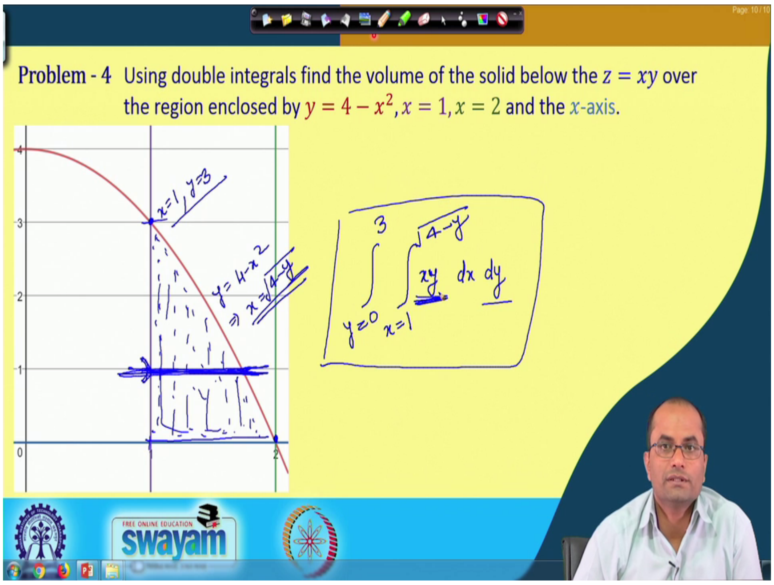

So, here the using the double integrals, we want to find the volume of the solid below this z is equal to x y. So, we have this z is equal to x y surface over this region which is enclosed by this parabola, again y is equal to 4 minus x square and x is equal to 1 and x is equal to 2. So, these are the 3 curves again and the x axis the forth one is the x axis. So, we have to first see what is the region here in the domain, and then we have the function surface here z is equal to x y. So, we want to find the volume below the surface and over this enclosed area in the xy plane. So, we have to first sketch the region as usual. So, we have the parabola and then these 3 lines.

So, we have this parabola which is other direction now y is equal to 4 minus x square. So, when x is 0 the y is 4. So, the vertex is here. So, we have this y is equal to 4 minus x square that is the parabola, we have x is equal to one this is the line x is equal to 1 and we have the line x is equal to 2. So, this is the line here and then we have the x axis. So, the region bounded by all these 4 curves is this one. So, this is the region bounded by the 4 curves though this y is equal to. So, x is equal to 2 does not play a role here because this is precisely passing through this point of intersection of this parabola and this line.

So, even if we remove also this x is equal to 2 the region enclosed will be the same. So, now, we also need to get this point here. So, that is pretty clear when x is 1. So, when x is 1 the y is 3. So, this x is 1 and then y will be 3 at this point by this y is equal to 4 minus x square and that this point here the x is 2 and the y is 0. So, now, we need to just put the limits and the function will be xy. So, this automatically this double integral with the integrand this xy will give us the volume of the solid below by the surface and above this xy plane. So, to get this these limits here again we have both the possibilities we can go first in the direction of x or we can go in the direction of y first. So, let us go in the direction of x and then for the inner integral and then this line will move from this to that point. So, for dx and the dy will be the outer one the integrand now will be xy because we are going to get the solid if we put this instead of xy 1 again we will get the area of this region, which is shown in the figure.

So, now, for the limits of x, we are now moving from x is equal to 1 this is the line x is equal to 1. So, always we are entering here to the domain from x is equal to 1 and the exit point is this a parabola y is equal to 4 minus x square or from there we can get this y is equal to 4 minus x square. So, we can also write that x is equal to 4 minus y and the square root. So, for the inner one x we are moving from 1 to the square root 4 minus y and for the outer one for the y, we are moving from this line to that line; that means, y is equal to 0 to y is equal to 3. So, this is the integral we want to now evaluate and that will give us the volume of the solid below this that surface z is equal to x y which is the integrand of this integral.

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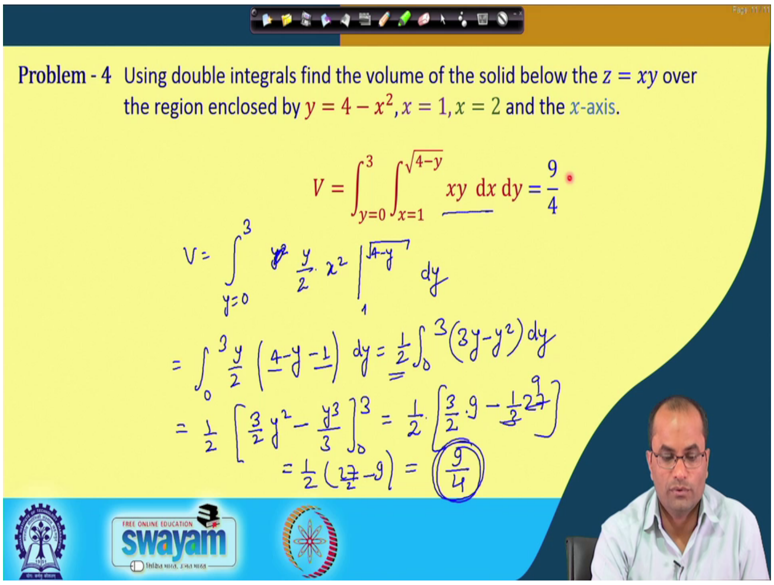

So, now let us evaluate this integral. So, here this was the limit we have just seen. So, x from 1 to this is square root 4 minus y and y is 0 to 3 and we want to integrate this with respect to x first. So, this v the volume with respect to x we have to integrate first so; that means, this will be x square by 2. So, y sorry this is y and by 2 and we have x square and these limits from 1 to 4 minus y and the outer integral is y. So, this is 0 to 3 and y by 2. So, here x square so, the 4 minus y and minus 1 dy which will be 0 to 3. So, here this is 3 minus y. So, 1 by 2 outside and then 3 y minus y square that is the integrand and dy.

So, we have 3 y minus y square dy and the integral over y from 0 to 3 and this vector half there so, half the integral 3 y. So, 3 by 2 y square and minus y cube by 3 the limits 0 to 3. So, 1 by 2 and then that is post the upper one here 3 lower one will make anyway 0. So, 3 by 2 and this y square so, here 9 minus 3 and then 27. So, which is a 9 again so, here 27 by 2 and then minus 9. So, this is 1 by 2 and 27 by 2 and this minus 9 there. So, this is 18 and then we have a 27 there. So, we get 9 and then this is by 4. So, we get the value 9 by 4 of this volume which is below the surface y is equal to x y and above the region close by the parabola and these 3 lines. So, that is the answer of this integral as 9 by 4.

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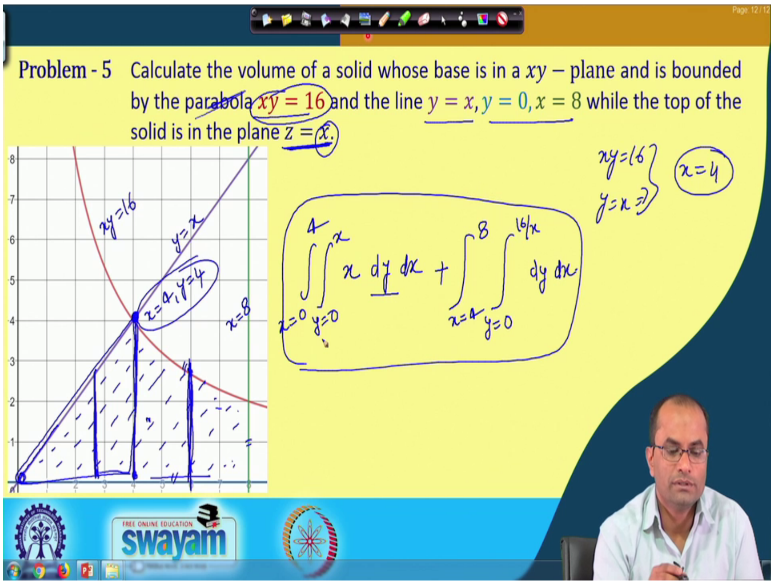

So, we move further to discuss another example where again we will compute the volume of the solid whose base is in the xy plane, and it is bounded by the parabola. So, this is not the parabola. So, bounded by the curves here x y is equal to 1, y is equal to x and this y is equal to 0 and y is equal to 8. So, in this case again we need to draw these curves here we have the 4. So, among them 3 are the lines and then this xy is equal to 16 with similar kind of a graph we have in the earlier figure. So, to have this now this is the situation we have this xy is equal to 16 and then this is the line y is equal to x, we have y is equal to 0 and we have x is equal to 8. So, these are the 4 curves. So, again let me just mention. So, this is y is equal to x and we have x y is equal to 16 and we have in this case x is equal to 8 and the region enclosed by these 4 curves.

So, this is not the region here this the region by this 4; so, 1 the 2 and the third and the 4. So, the enclosed by this we have precisely this region here and now we will compute the volume of the solid which is above this region here and below the bounded by the xy is equal to 1. So, whose base is this and the plane here z is equal to x yeah. So, this is the function this is a surface z is equal to x. So, our integrand is going to be x now. So, again we have the situation that we can either go in the direction of y first or we go in the direction of x first, but in either case we have to break the domain because even if we go first in the direction of x. So, up to this point here we have a different situation and after that it goes from y is equal to x and exit from x is equal to a.

Or, if we set in the direction of a y first; so, we have again these two regions one is this one which is from the entries from y is equal to 0 to this curve which is given by y is equal to x and then from 4 to 8 we have this different curve. So, what is this point here? The point of intersection that is also important. This is y is equal to x and this is xy is equal to 16 and y is equal to x. So, out of this we will get x square 16. So, x is 4. So, this is exactly x is equal to 4 and naturally this y is equal to 4. So, that is the point here. So, we can let us take now first in the direction of y, and then in the direction of x. So, we need to have two integrals now; so, in the direction of y first and then in the direction of x.

So, and integrand is going to be x. So, in the direction of y; however, ranges for this first region which is this triangle y goes from 0 to this curve which is x and then for x we will move from this point to this point; that means, 0 to 4 the another one then 4 to 8 our x will vary from 4 to 8 and now for the y its again the same curve y is equal to 0 here, but now there it is x y is equal to 16. So, this y is equal to 0 to 16 over x and then we have again dy and dx.

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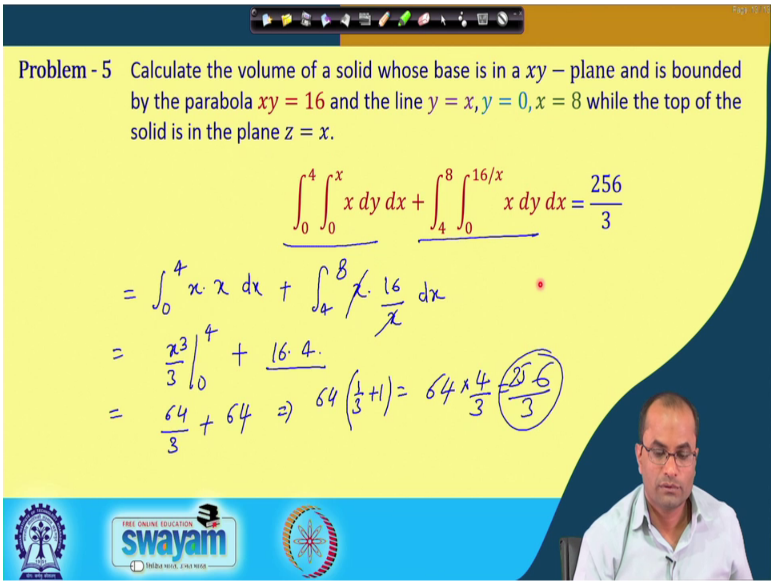

So, this integral we will compute which is given here. So, again this is a simple each of the integral is simplest the first one is 0 to 4 and then with respect to y. So, x remain as it is and with respect to y we will get x there. So, we have dx here we have 4 to 8 and then this x and again y there. So, upper limit will give us 16 by x and minus the 0. So, dx. So, these are the two integral the first one is x square. So, we will get x cube by 3 and the limits 0 to 4 and then here we have the 16 and then x; so, 8 minus 4. So, we will get 4 the integral value there. So, here 64 by 3 and plus 64. So, this is 64 if I take common this is 1 by 3 plus 1; so, 64 into 4 by 3 so, 16 246 and by 3.

So, 256 by 3 is the volume of the solid below this plane z is equal to x and bounded by that region xy is equal to 16 and y is equal to 6, y is equal to x y is equal to 0 and x is equal to 8. So, that is a value 256 by 3.

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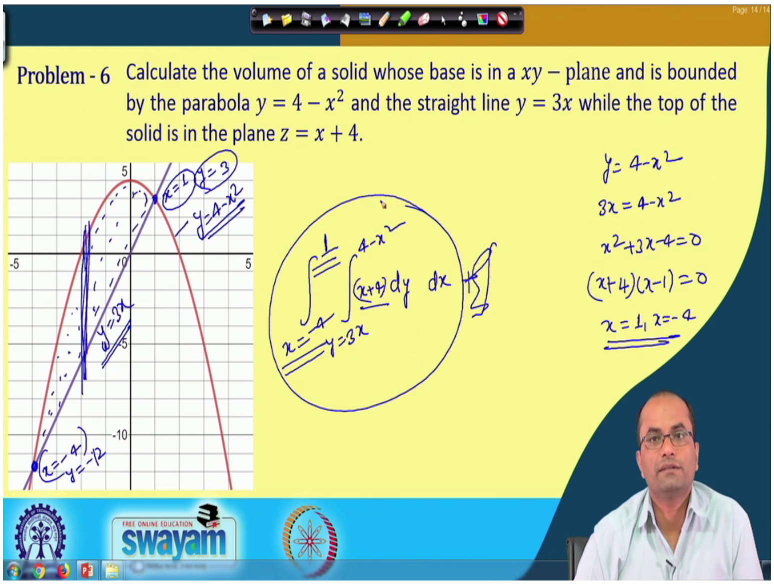

And now the last example of today's lecture; so, we want to compute the volume of the solid whose base is again the xy plane, but now we have the parabola here y is equal to 4 minus x square, we have the straight line y is equal to 3 x and on the top we have the plane z is equal to x plus 4. So, again the similar problem; so, the integrand will be x plus 4, and the region will be computed from this y is minus x square and y is equal to 3 x. So, these are the lines here. So, the curve so, y is equal to 3 x. So, we have y is equal to 3 x and this parabola is y is equal to 4 minus x square we have seen earlier also similar regions.

So, now this is the region where we want to get the volume below the z is equal to x plus 4 plane. So, on this we need to compute now what is this intersection points. So, for this we can now have y is equal to 4 minus x square and y is 3 x. So, I will put 3 x there and then 4 minus x square. So, we have x square plus 3 x minus 4 is equal to 0 or x minus 4 and so, x plus 4 and x minus 1 is equal to 0. So, x is equal to 1 and x is equal to minus 4. So, these are the two points. So, here we have x is equal to 1 and correspondingly why we can get from this y is equal to 3 x and here we have x is equal to minus 4. So, the corresponding to this y will be minus 12 and here the y will be 3 x so, 3.

So, these are the points and now we want to integrate. So, we have to write the integrals and again the question that we either we can integrate first in the direction of x or in the direction of y. So, let us integrate first in the direction of y because that will be easier. So, in the direction of y it always goes from it enters through this line and exit through this parabola so; that means, this y and then dx. So, the y goes from 3 x to this parabola, y is equal to 4 minus x square and the limits of x will be from minus 4 to this x is equal to 1.

So, this integral we need to compute and the integrand will be x plus 4 and plus we have to yeah that is the only integral because we have cover the whole region here, this line is always entering through this line and exit from that parabola and then for x we have also taken from minus 1 minus 4 to 1. So, we need to just integrate the simple integral to get the values.

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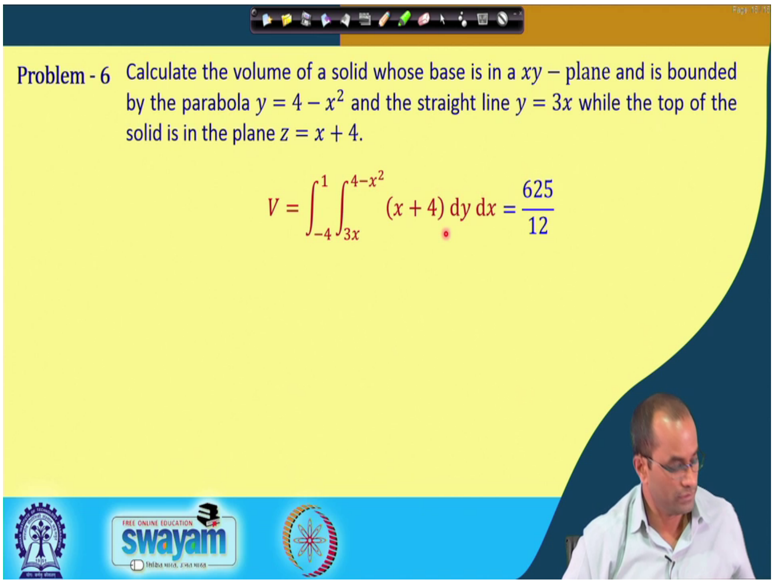

So, in this case this is the volume in terms of the double integral, which we can again this is simple integrand we can integrate. And, after the calculations here the values are coming as a 625 by 12 it is a simple integral one can easily compute.

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So, what we have seen today that applications of this double integrals mainly the computation of area and also the computation of volume. In later lectures we will also see another application where we can compute the surface area as well. So now, we have computed only the area of the domain of the function and then, later on we can also compute the surface area of the given surface using the double integrals. But this was a straight forward application; the computation of the area and the computation of the volume which we have cover today.

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These are the references which are used to prepare these lectures.

Thank you very much.