Engineering Mathematics - I
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Lecture - 03
Indeterminate Forms Part - 1

Welcome to the lectures on Engineering Mathematics-I. and today's topic is Indeterminate Forms.

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So, these are the concepts we will be covering today. So indeterminate forms, L'Hospital's rules which is very fundamental principle to determine a such forms and some worked out examples.

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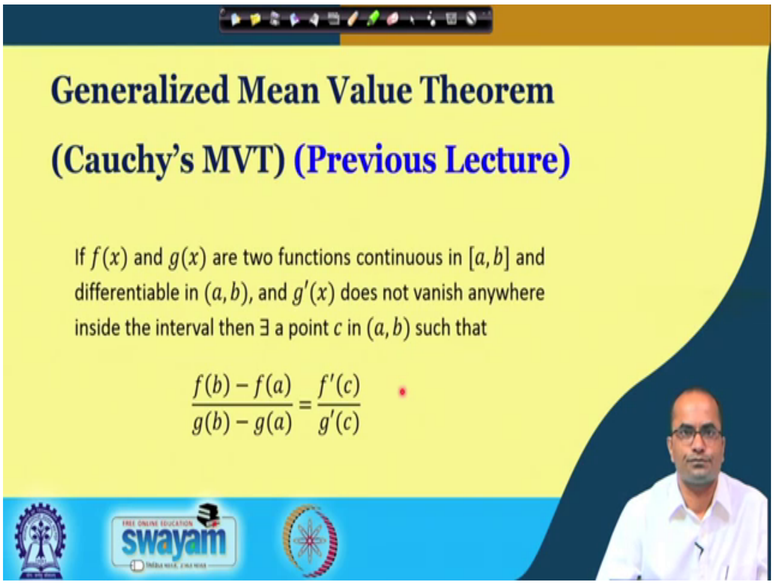

So, before I start to indeterminate forms let me just introduce your recall from the previous lecture, the generalized mean value theorem or the Cauchy mean value theorem which was discussed in previous lecture.

So, there we have seen that if there are two functions f and g continuous in closed interval and differentiable in open interval a b and g prime the derivative of g does not vanish anywhere inside the interval then there exist a point c in open interval a b such that this quotient here f b minus f a divided by g b minus g a is equal to the ratio of the derivatives at that point c. So, this generalized mean value theorem will be used today to prove sum of the results.

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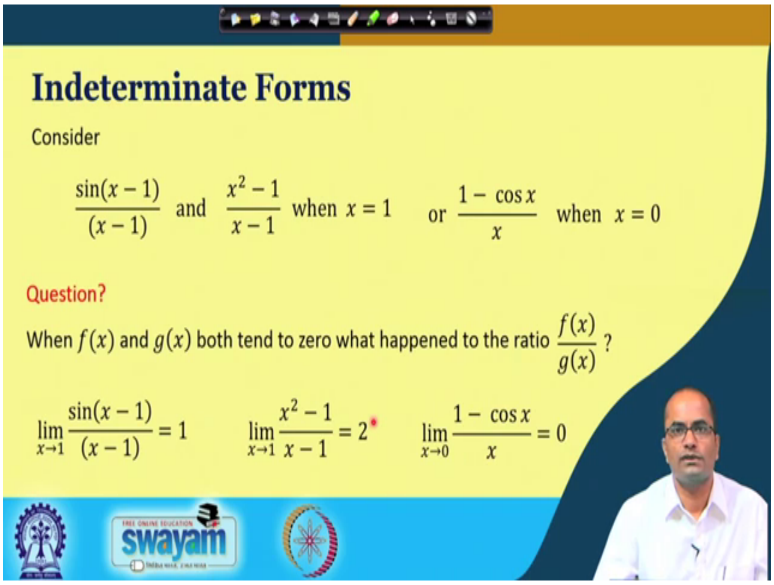

And what are the indeterminate forms. So, for example, if we consider this sin x minus 1 over x minus 1 this ratio of sin and x minus 1 or we can consider like x square minus 1 over x minus 1. So, we want to evaluate these when x is equal to 1. So, if we substitute x is equal to 1 simply we are getting here 0 and divided by 0; similarly here as well we are getting 0 divided by 0. So, in these cases we cannot just simply substitute x is equal to 1 and get the value of these expressions given here or for example, we have 1 minus cos x over x and we want to see that what will happen to this expression when x is equal to 0.

So, if we put x is equal to 0 here. So, 1 minus this cos 0 is again 1 so, 0 and divided by 0. So, we have another 0 by 0 form which cannot be evaluated directly by substituting x is equal to 0. So, the question is that when f and g both tend to 0; what happened to the ratio f x over g x. These are the situations which we have considered in these examples in each of them f x and g x both tend to 0. And now we want to see that what will happen to these expressions. And, in today’s lecture we will see that for example, this form here sin x minus 1 x minus 1 when we take the limit as x goes to 1. Because we cannot simply substitute as x is equal to 1 in this expression, but we can talk about the limit.

So, the limit x goes to 1 sin x minus 1 x minus 1 will be coming as 1 whereas, in the second case when x square minus 1 over x minus 1 this 1 can simply get by cancelling this x minus 1 from the numerator, because this numerator one can write like x minus 1 and x plus 1. So, this x minus 1 will get cancel with this x minus 1 and then this limit will be simply 2. This one 1 minus cos x over x 1 can again evaluate we will see later in the lecture that this limit is 0. So, in these all three cases we have seen that these forms were 0 by 0 forms, but their limits are different; in the first case it is 1, here it is 2 and the third one is 0.

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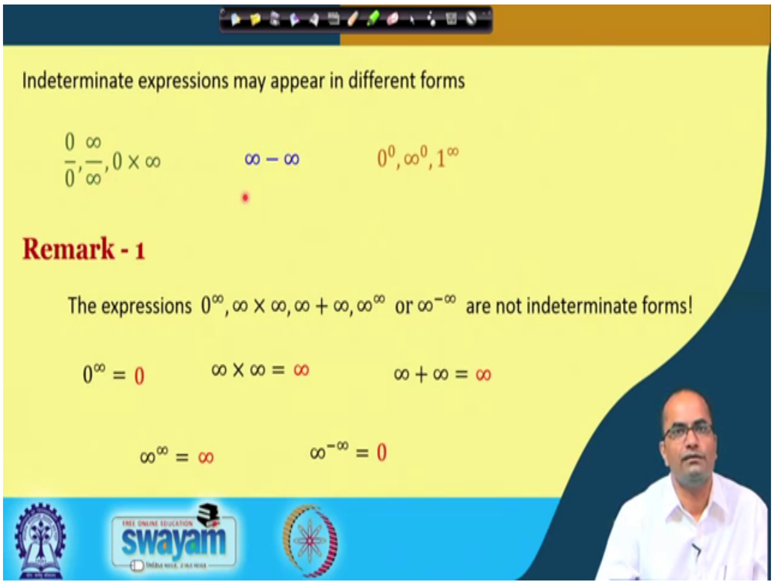

So, what are these indeterminate expressions; we will see now. So, they may appear like in 0 by 0 we have just seen the other possibility is that the numerator and denominator both are infinity. So, we have the form infinity by infinity or the 0 into infinity. There are other indeterminate forms like infinity minus infinity or in these exponent form 0 power 0 infinity power 0 or 1 power infinity. So, all these are the indeterminate form and we do not know what is the value of for example, infinity power 0 or 1 power infinity infinity minus infinity.

So, there was a remark here that these expressions which are different then these which we are calling indeterminate forms. For example, 0 power infinity, infinity into infinity, infinity plus infinity, infinity power infinity or infinity power minus infinity and note that these forms are not indeterminate forms and we can directly find the value of these expressions. For example, this 0 power infinity: what is the value of 0 power infinity it is just 0 and infinity into infinity will the 2 very big numbers when you multiply naturally you will get infinity.

Again here the plus infinity plus infinity again will become infinity and infinity power infinity with the same reason this will be also infinity. And, infinity power minus infinity we can write rewrite it as 1 over infinity power infinity and then infinity power infinity is infinity. So, this will become 1 over infinity which is 0. So, these forms are not indeterminate forms. So, if we find such expressions during the calculations we can directly substitute these values. But, we have to be more careful for such 0 by 0 infinity by infinity is 0 into infinity all these cases. Because, we have to evaluate by some rules those values and it is not clear that; what is the value of infinity by infinity for example.

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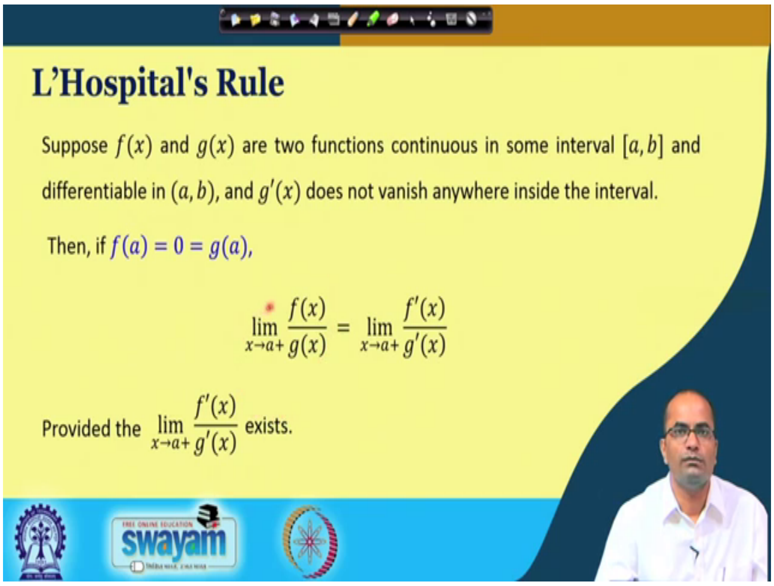

So, there is a concept here the L'Hospital's rule a very fundamental rule for determining such a indeterminate forms. So, what is this here let us go through the, suppose this f x and g x are two functions and continuous in closed interval a b and differentiable in open interval a b and this g prime the derivative of g does not vanish anywhere inside the interval. So, all these conditions are the conditions of the Cauchy mean value theorem or the generalized mean value theorem we have just seen today. And, in addition to those conditions if we have like f a is equal to 0 the function is taking value as 0 at a and the second function g is also taking the value as 0 at this point a. Then we will see in the proof of this L'Hospital's rule that if you want to evaluate the limit as x goes to a, naturally in the setting a from the right hand side the limit the right limit as x goes to a f x over g x this ratio which directly we see here which says f a is 0. So, it is like 0 by 0 form.

But if you take this limit and this rule says that this limit, this limiting value is equal to this limiting value which is the ratio of the derivatives. So, it is a another application of the derivatives and naturally when this limit the limit of the derivatives exist, otherwise this does not make sense if the this does not exist. We will come to this point little later and we will see one example where this limit does not exist, but it does not mean that the limit of f x over g x does not exist. So, this is the rule here that if such limits exists the limit of the derivatives, then we this will be equal to the limit of this ratio of the functions.

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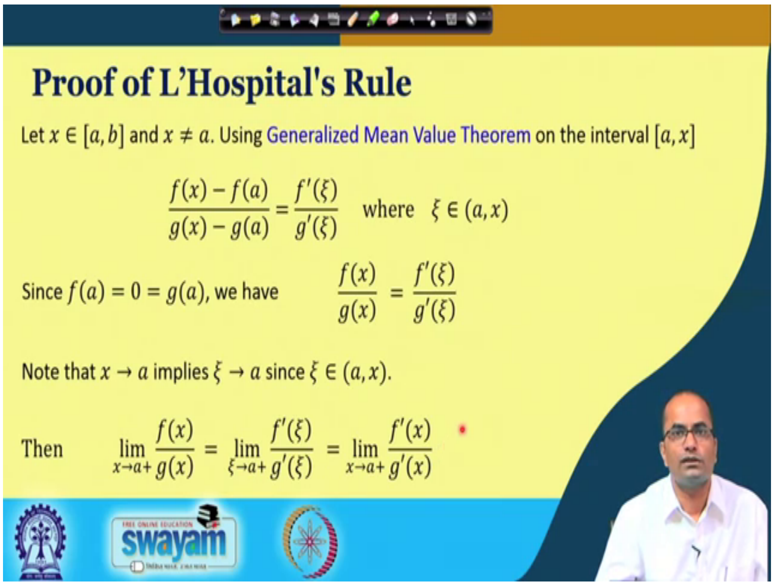

So, the proof is pretty simple if we use the L if you use the Cauchy mean value theorem so, which was summarize today. So, this is the Cauchy mean value theorem when we have two functions f and g then f x minus f a and g x minus g a will be f prime xi over g prime xi and the xi lies somewhere between a and x. So, we have taken a point here x in the interval, x is naturally not equal to a and then we have applied this generalized mean value theorem in the interval a to x ok. So, we know that the value of the function at a; and the value of the function g. So, for both the functions at a is 0.

So, this here with this expression left hand side will become f x over g x. So, what we have here the f x over g x is equal to f prime xi over g prime xi and this xi belongs to a to x interval. So, now, you note that if this limit here we take as a x goes to a and since this is xi belongs to the open interval a to x. So, if we x goes to a naturally the xi will also go to a. So, this is what the next here. So, if we take the limit here x goes to a f x over g x and then this will be equal to the limit xi goes to a plus because the xi belongs to the interval a to x.

So, the xi will go to a plus. So, from the right side and this derivative here f prime over g prime. And now we can replace just this xi by some other name or the most suitable is x in the setting here. So, what we have seen that this limit x goes to a plus this ratio f x over g x is nothing, but the limit x goes to a plus f prime over g prime. So, this is the proof of this L'Hospital's rule using the generalized mean value theorem.

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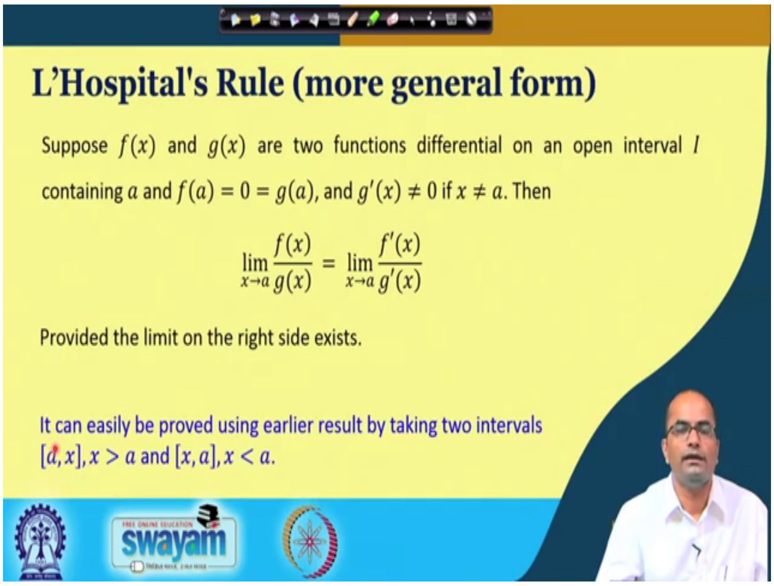

There is a slightly more general form of this L'Hospital's rule, because if we note here that we have taken x to a from the right side because we had taken this interval for the functions a to b. So, this is a more general form this if f and g are two functions differentiable on open interval I and naturally they are also continuous on the xi containing this a and this f a is 0. So now, a is somewhere inside the interval not at the boundary. So, if we have f a is equal to 0 is equal to g a and g prime x is not equal to 0 in this interval, if x is not equal to a; x is equal to a anything can happen then we do not need such restrictions on g prime. But, other than this x is equal to a the g does not a vanish.

So, in this case also one can easily prove that limit x goes to a now there is no left or right concept here. So, the limit simply x goes to a f x over g is equal to the limit x goes to a f prime over g prime x provided the limit on the right hand side exist. And, the proof is similar to what we have already done before because, now we can consider two intervals here in this I. So, a to x when x and taking x greater than a and we can also consider another interval x to a when x is a smaller than a.

So, in these two intervals we will apply the previous result which will establish there that x goes to this a plus in this case is equal to limit x goes to a plus. And, then when we apply that result to this interval and we will get that x goes to a from the negative sides of both the limits from the right side. And from the left side we will get the same result which will conclude that the limit x goes to a f x over g x is equal to the limit x goes to a f x over f prime over g prime x.

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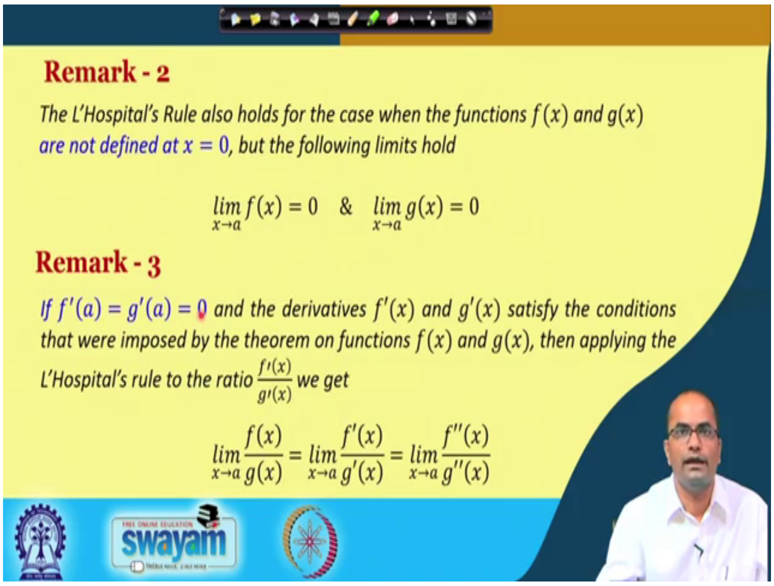

So, another important remark so, this L'Hospital's rule also hold for the case when the functions f and g are not defined at x is equal to a. So, what we have taken in the previous two results that f a is 0 and g a is 0. So, those at that point the function values was 0, but the same rule one can apply if for sample function these two functions are not defined exactly at a, but their limits are 0. So, the limit x goes to a is 0 and limit x goes to a of g x is 0. So, in this case also we can apply the same result what we have established earlier another remark.

So, if we realize that the first derivatives are also 0 at a and the derivatives f prime and g prime they satisfy the conditions that were imposed earlier on functions f and g mainly the continuity differentiability then applying the L'Hospital's rule to this ratio. So, we can do we can apply the L'Hospital's rule again to this f prime over g prime because the similar situations happening now for f prime and g prime.

Because they both are 0 and then the rule says that this limit here f prime g prime will be equal to the double derivatives, the ratio of the double derivatives of f and g as x goes to a. So, the this is again more generalized form that this limit f over g can be evaluated by the limit of the ratio of f prime g prime, but if these two f prime and g prime become 0 as x goes to a or x is equal to a then we can again apply the L'Hospital's rule.

So, the same limit will be equal to the limit of f the second derivative divided by g, the second derivative as x goes to a or we can continue this further if for example, the f prime sorry f double prime here. So, the double derivative of f also vanish at x is equal to a and this double derivative of g also vanishes at x is equal to a. So, we can further apply this L'Hospital's rule together limit of this f double derivative divided by g double derivative.

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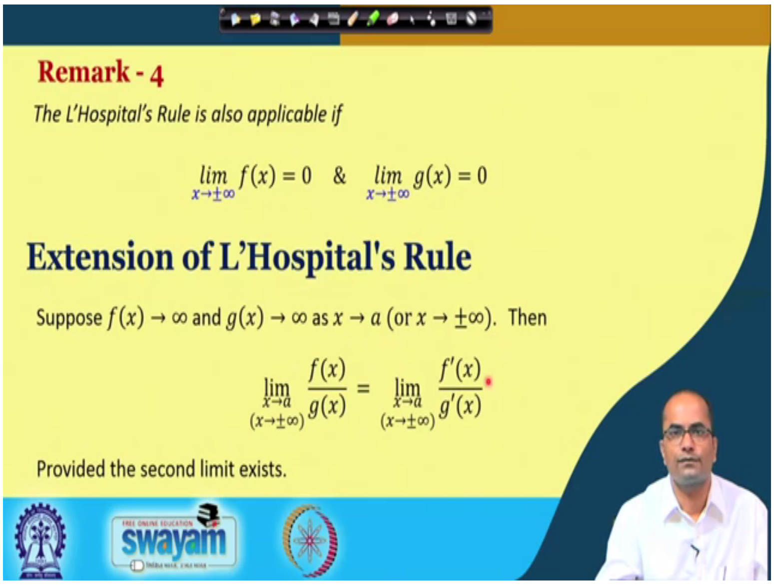

So, L'Hospital's rule is also applicable. So, another generalization here that not necessarily that x goes to a we have just discuss that x goes to a was some finite number, but one can also apply this result when limit x goes to plus infinity or x goes to minus infinity. So, this is a very general rule which we are not proving here for example, this infinity case, but one can apply the L'Hospital's rule their too.

So, now the extension of this L'Hospital's rule to the infinity by infinity form. So, suppose this f x goes to infinity and g x goes to infinity as x goes to a or x goes to plus minus infinity similar to the earlier case. But, the now the differences that we have instead of f x goes to 0 g x goes to 0 they both are tending to infinity. And, in this case also we have the same rule that this limit of the ratio of these two functions will be the limit of the ratio of their derivatives; when this f x and g x goes to 0 provided this right that the limit at the right hand side here this exists.

So, limit f prime over g prime exist. So, what is the general rule now if we include those all results what we have discussed so far, that they are two they are could be two forms. So, either 0 by 0 form or infinity by infinity form. In either case whether x goes to a or x goes to plus minus infinity the limit of the ratio of the two functions will be equal to the limit of the ratios of their derivatives.

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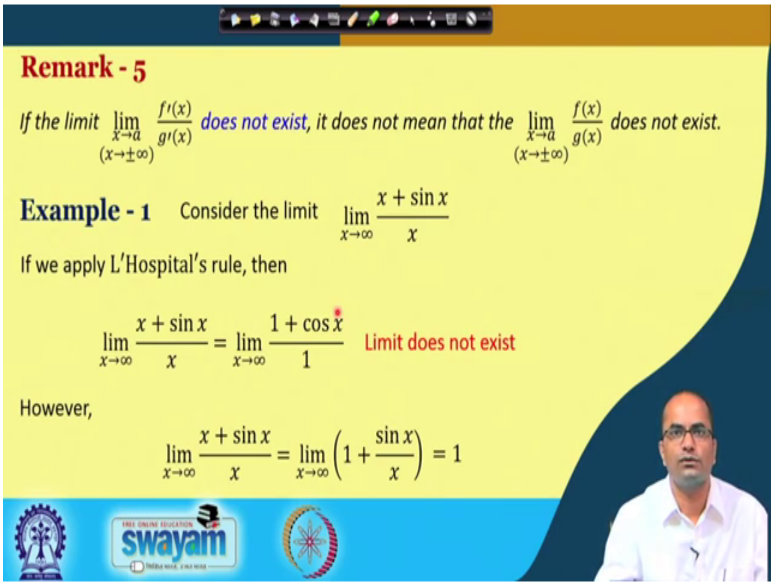

So, this is another important remark which I have mentioned before. So, if this limit here does not exist, if the limit of the derivatives does not exist; it does not mean that the limit of f divided by g does not exist which we can see by the simple example. So, if we consider this limit x goes to infinity x plus 1 divided by x. So, in this case what is happening if we just see what is the form here; so, x goes to infinity plus here something finite. So, the numerator is infinity and divided by again x goes to infinity. So, we have the infinity by infinity form and in this case if we simply apply the L'Hospital's rule what will happen.

So, here if you take the derivative of the numerator it is a 1 plus the sin x will become cos x and the limit of the derivative of this x is 1 and the limit x goes to infinity. So, here the limit x goes to infinity 1 plus cos x since this cos x when x goes to infinity is not defined. So, basically this limit here 1 plus cos x and as x goes to infinity is not defined. So, this limit does not exist, but if we evaluate this in some other ways like x goes to infinity x plus sin x and we rewrite this as 1 plus sin x over x. So, we divide this x here to x and then sin x and separate it. So, we have x over x plus sin x over x meaning this 1 plus sin x over x and now we can directly evaluate now this limit; so, 1 plus the sin x over x.

So, when x goes to infinity. So, if this x goes to infinity here and this sin x something finite is sitting there. So, something finite and divided by infinity this will go to 0. So, the second part here sin x over x as x goes to infinity will go to 0. So, we have 1 plus 0 means this limit is 1. So, if we would have concluded here by applying the L'Hospital's rule, because this limit does not exist. And, we could have claimed that the limit x goes to infinities x plus sin x over x does not exist, but that would have been a wrong conclusions. So, that is what in the rule every time we have written provided the limit of the ratio of the derivatives exist. So, that is very important.

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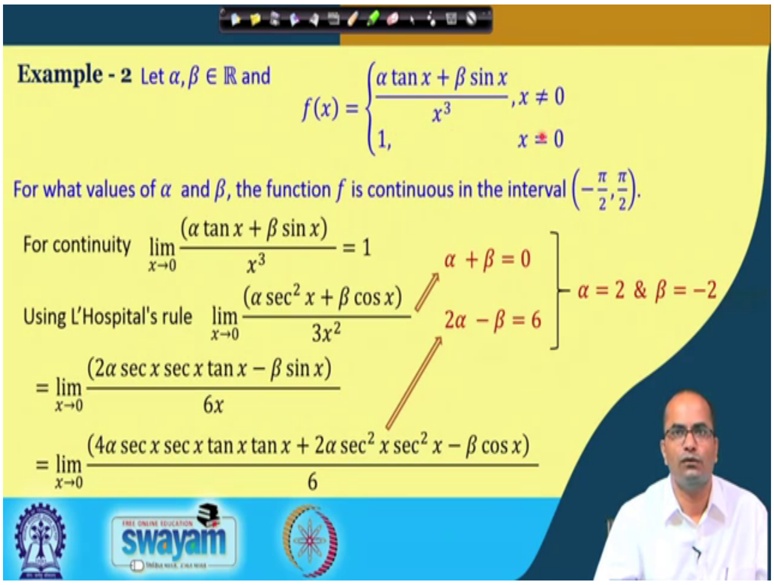

Now, one example here so, let this alpha beta in R. So, they are the real number and we have this f x is equal to alpha tan x plus beta sin x over x cube when x is not equal to 1 and the value is 1 at x is equal to 0. So, in this case we want to find for what values of alpha and beta the functions f is continuous. So, the function f is continuous in the interval minus pi by 2 to pi by 2. So, now for the continuity what do we need? So, for the continuity of this function this limit of this alpha tan x plus alpha sin x over x cube should be 1, because at x is equal to 0 the function is defined as 1. So, rest everywhere the function is continuous the 1 the problem is at x is equal to 0.

So, here for x not equal to 0 we have this nice function defined over minus pi to pi the tan. So, it is a continuous sin is continuous x cube is continuous. So, the function is continuous, the only problem it could create at x is equal to 0. So, we are now setting here that if limit this x goes to 0 alpha tan x plus beta sin x over x cube is equal to 1 then this function will become continuous. So, out of this condition we will compute alpha and beta. So, for what values of alpha and beta this expression here or this limit here is equal to 1. So, now let us compute this limit here alpha tan x plus beta sin x over x cube. So, when we take x goes to 0 the tan 0 is 0 sin 0 is 0 and x cube is also 0.

So, we have basically the 0 by 0 form. So, let us apply the L'Hospital's rule to this expression. So, if we apply L'Hospital's rule alpha and tan x will become sec square x plus beta sin x will become cos x and divided by 3 x square. So, the x square x cube when we take the derivative will become 3 x square and we take the limit here x goes to 0. So now, if you realize what is happening to this function now here. So, we have the alpha and then x goes to 0 this sec x which is 1. So, here you have alpha plus the beta cos 0 is also 1. So, we have here alpha plus beta divided by 3 x square and then x goes to 0. So, here we are getting this 3 x square is going to 0.

So, we have alpha plus beta divided by something which is going to 0. Now, the only possibility to move further or to have this limit as 1 will be when alpha plus beta is equal to 0. Because then we will get 0 by 0 form and we can further apply the L'Hospital's rule. But in this case so, what we can set to move further that this alpha sin square x plus beta cos x divided by 3 x square to have this limit as 1, we can set that alpha plus beta is equal to 0 because when x goes to 0 then we can move further and apply the L'Hospital's rule again. So, applying this L'Hospital's rule again so, we got already one condition on alpha plus beta which is equal to 0 and now if we apply. So, again so here the derivative of alpha the sin square so 2 alpha sin x. So, 2 alpha sorry sec x and the derivative of sec x will be sec x tan x minus beta because cos x will give you minus sin x.

So, it is a minus beta sin x and divided by 6 x and now if we check what form we are getting now here. So, this tan x will make this 0 here sin x will make the 0. So, we are getting 0 in the numerator and divided by 6 x which is again 0. So, we are getting 0 by 0 form. So, we can apply the L'Hospital's rule once again to this expression. So, here limit x goes to 0. So, here we have this sec square x and tan x. So, the sec square x will give 2 times sec x sec x tan x. So, then it becomes 4 x 4 alpha and sec x sec x tan x and the tan x remain as it is plus this 2 alpha sec square x and then tan x will become again the derivative sec square x minus beta sin x will become cos x and divided by the 6 here because this was 6 x and derivative is 6.

So, now if we check again what is the value here? So, this sec x 1 tan x will be 0. So, this expression will become 0 and then here when x goes to 0 this is like 2 alpha and then minus beta. So, this in the numerator we are getting 2 alpha minus beta and divided by 6 and the limit x goes to 0. So, 2 alpha minus beta by 6. So, to have this value as 1 we need to set that 2 alpha minus beta is equal to 6. So, another condition we got that 2 alpha minus beta is equal to 6. So, if you solve these 2 equations alpha plus beta is equal to 0 and 2 alpha minus beta is equal to 6. So, we will get that alpha is equal to 2 and beta is equal to minus 2.

So, for these values of alpha and beta this function will become continuous or in other way this limit here alpha tan x plus beta sin x over x cube will become as 1. The function was defined as 1 at x is equal to 0.

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So, these are the references which we have used to prepare these lectures. So, the integral a Differential and Integral calculus by Piskunov and this is Volume 1; the Kreyszig Advanced Engineering Mathematics and also the Thomas' Calculus.

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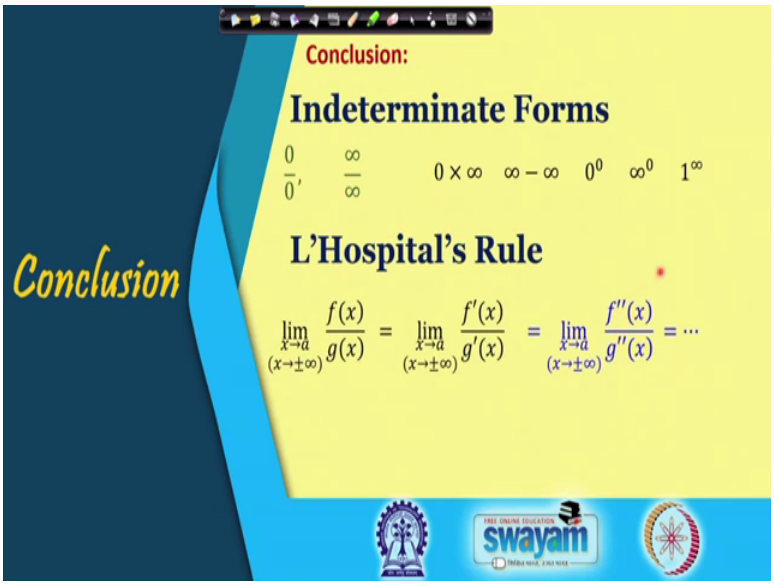

So, what did we learn today these indeterminate forms, they may take these several forms like 0 by 0, infinity by infinity, 0 into infinity, infinity minus infinity, 0 into 0 power 0, infinity power 0 1 power infinity. So, what we have learn today how to compute such limits when we have the 0 by 0 or infinity by infinity form. And, the L'Hospital's rule which was useful to compute this limit was that whether we have the 0 by 0 or infinity by infinity form here for the ratio f over g x. We can apply this rule which says that this limit will be equal to the limit of the ratios and if again this f prime and g prime they both becomes or takes the form 0 by 0 or infinity by infinity then we can again apply the rule. And, then we will get this limit is equal to the limit of the ratio of the second derivatives and so on we can continue further till we get the limit.

But that important point was that these rule is valid when, when those limits exist we cannot conclude if those limits here of the derivatives do not exists then we cannot conclude that the original limit does not exist. So, this rule is a very useful rule. In the next lecture we will learn now how to deal the other forms; for example the 0 infinity, infinity by infinity and so on.

Thank you.