Engineering Mathematics - I
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Lecture - 29
Double Integrals (Contd.)

So, welcome back this is lecture number 29 on integral calculus Double Integrals.

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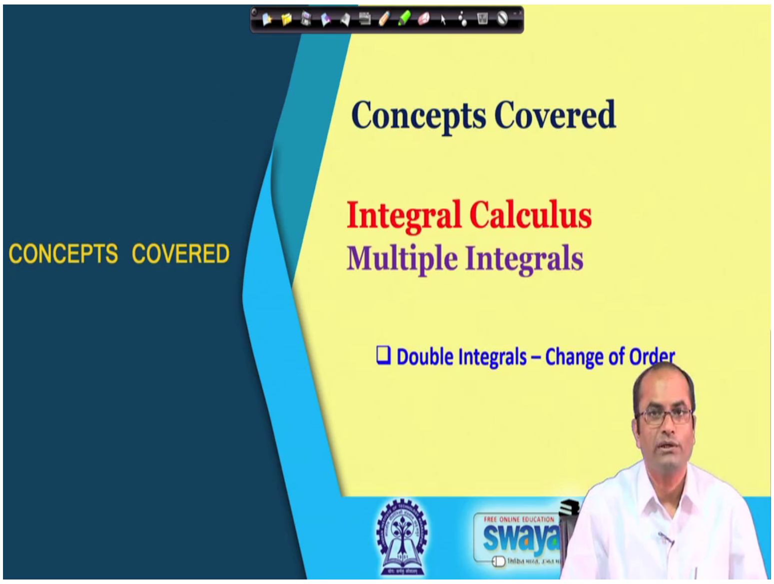

And in particular we will be focusing on the change of order of integration in this lecture.

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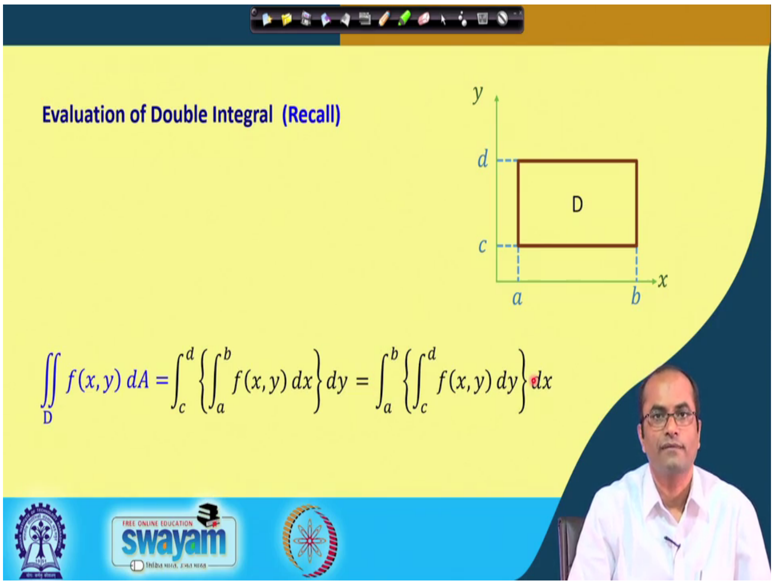

So, just to recall from the previous lecture, we have gone through some simple geometry and for example, if you take the rectangular domain of the function f x then this integral the double integral, we can evaluate by repeating the single integrals 2 time. So, either we can integrate this function over this dx and the limit for x will go from a to b. So, first evaluate the singer integral with respect to x and this will be function of y, which can be integrated and second the step with respect to y. So, the limits of y will be from c to d.

We can change the order; here meaning that we can first evaluate the integral with respect to y, where the limits of y will go from c to d. And then, at the end we can evaluate the single integral with respect to x and the limit will be a to b in that case.

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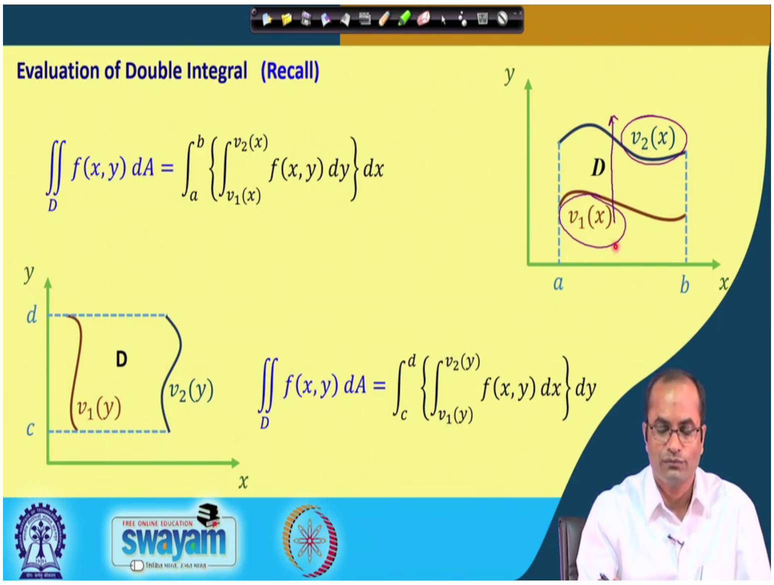

We have also seen some other domain. So, for example, this domain is bounded by the function v 1 and v 2 in the direction of y; and in the direction of y we have these constant limits from x is equal to a to x is equal to b. So, in this situation we have the possibility now that we should first integrate or it is convenient or it is easier to integrate first in the direction of y, and the limit of y will be from v 1 x to this v 2 x and then for the limit of x we will have a to b.

So, in this case this double integral will be first the inner one. So, with respect to y the limits as I said will be from v 1 x to this v 2 x and later on or at the end we can integrate this with respect to x the limits will be from a to b. Or, we have this domain for instance. So, in this case we have to consider first the integration with respect to x because it is easier to set this variable limits for x. So, the limit of x will be from v 1 y to v 2 y and then in the second step we will do the integration with respect to y. So, here we will have this integral the inner one with respect to x, the limits will be v 1 y to v 2 y and the outer 1 will have the limits from c to d.

So, for these simple cases we have seen when we have the rectangular domain it is much easier that we can interchange the limits whether we compute first with respect to x or with respect to y it does not matter. But, when we have such a domains for example, in this figure or in the other one then we should see that which one is convenience. So, in the first one we have realize that, we should first integrate with respect to x and then with respect to y while, in the second case it is much more convenient to first integrate with respect to x and then with respect to y. So, this is one a situation where we can see the convenience of the order here, but there will be other reasons depending on the integrand that which integral whether with respect to y is easier or possible or convenient or with respect to x first. So, that will be the discussion now here.

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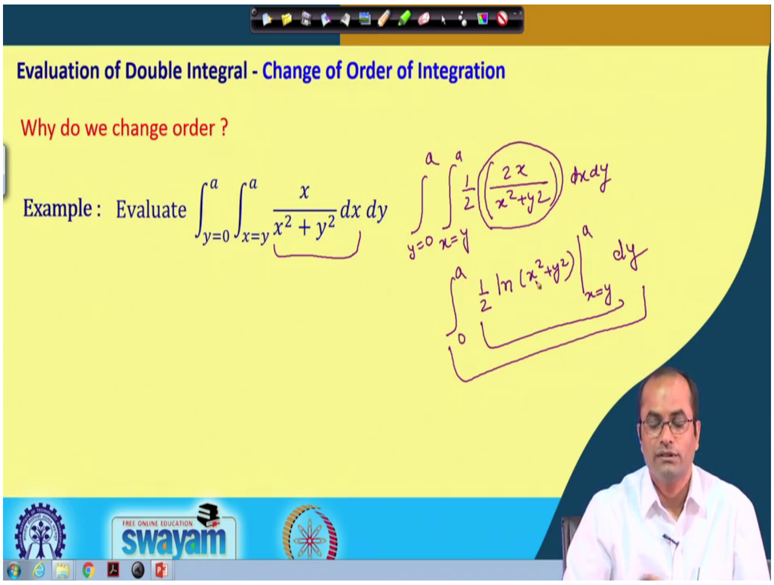

So, for instance we have. So, the question is here that why do we change the order. So, the one reason which will be very clear from this example that suppose we have this integral the inner one x. So, over this x square plus y square with respect to x and the limits for x goes from y to a. So, x is equal to y to x is equal to a that is the limit of the inner integral and for the outer one we have y is equal to 0 to y is equal to a.

The question is now if we differentiate if we integrate here with respect to x first in the given order, then what will happen? We should consider here like 1 by 2 in the integrand and then we have 2 x over this x square plus y square this is our integrand; and then we have the integral here with respect to x from y to a and with respect to y from 0 to a we have dx dy. So, with respect to x this will be the logarithmic functions. So, we will have the ln of x square plus y square the integral of this with the half and then we have to also put the limits now for x from y to a and then the outer one 0 to a and we will have dy.

So, this will become the function of y, now which will be a ln a square plus y square and then here we will have also minus ln with this 2 y square. So, the point is now integrating this function the logarithmic function a square plus this y square or this ln two y square is very difficult, but what we will realize here? If you change the order of integration then this will be much easier to compute. So, if you change the order of integration the first step is going to be always that we have to the sketch the domain of integration.

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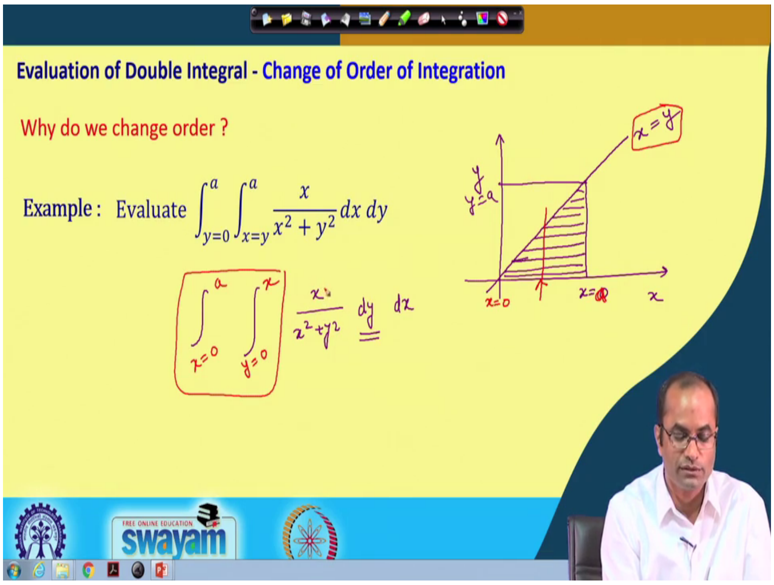

So, in this situation in this case what we have? We have this let us say this is x axis and we have here this y axis. So, this limit of the inner one goes from x is equal to y. So, this is the line here x is equal to y and this goes the inner integral from x is equal to y to the constant limit a. So, x goes from y. So, we starts from this line and it goes up to a. So, let us say this is the point a here.

So, this is x is equal to a and naturally this is also then y is equal to a point. So, this limit here x goes from y to. So, this line to this point x is equal to a; so, this is the domain of integration in our case now, and if you want to change the order of integration. So, this figure will be very helpful now. So, what we are going to have? We are going to have the same integrand naturally and then here dy and dx. So, the order will be change.

So, first we have to fix the limit for y here. So, for the limit of y, now we will go through a line in the y direction and see where it enters and where it leaves the domain. So, it is entering here always for covering the whole domain, it is entering at this y is equal to 0. So, the limit here will be like y is equal to 0 and it is leaving the domain exactly at this line where y is equal to x.

So, from y is equal to 0 to this y is equal to x and the limit for the x will be from this is x is equal to 0 to x is equal to a; this is x is equal to a. So, x is equal to 0 to a. So, in this way we have changed the limit now from x 0 to a and y goes from this 0 to x. And now we can integrate this because with respect to y when we integrate, this is going to be easier we have this is x is constant. So, we have 1 over x square plus y square term.

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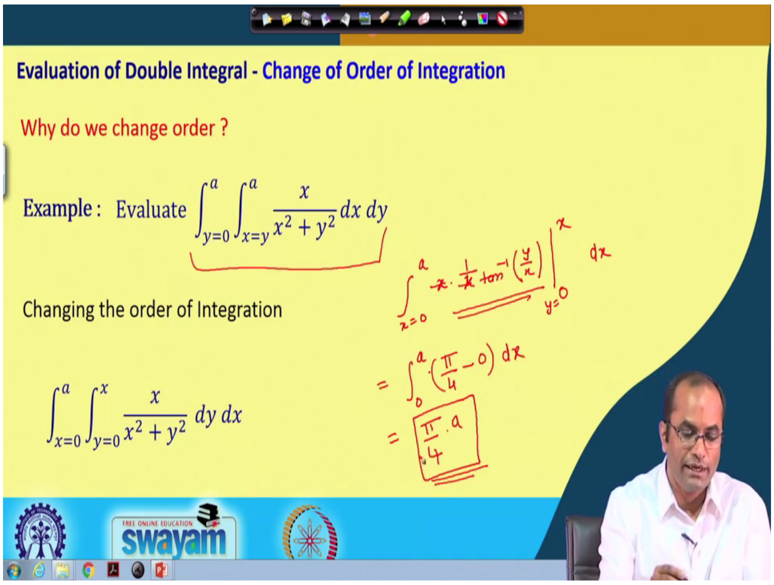

So, let us just go through here. So, what do we have now? We have this after changing the order of integration, we have this y goes from 0 to x and x goes from 0 to a. So, while integrating the inner one first. So, we fix the outer one. So, x from 0 to a and for the inner one we can integrate now. So, 1 over x square plus y square will give x and sorry 1 over x and tan inverse y over x; with respect to y and then we have the limit for y from 0 to x and the outer integral x. So, this x will get cancel and we have tan inverse y over x, which will be 0 to a. So, tan inverse the upper limit when we put tan inverse 1 that will be pi by 4 minus this tan inverse y 0. So, tan inverse 0 will be 0 and then this dx.

So, what do we have here pi by 4 and then this integral 0 to a dx will be just the x and the upper limit will give us a. So, that is the value of the integral. So, this change of order was very useful if you would have not change the order of integration, the direct integral was difficult to compute purchase by changing the order of integration it has become trivial and we got the value as this pi by 4 a.

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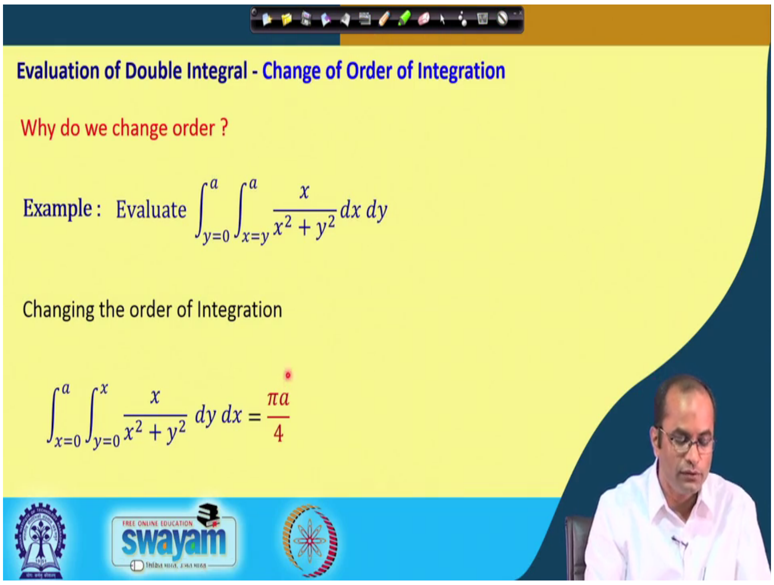

So, the value of this integral is pi by 4 into a.

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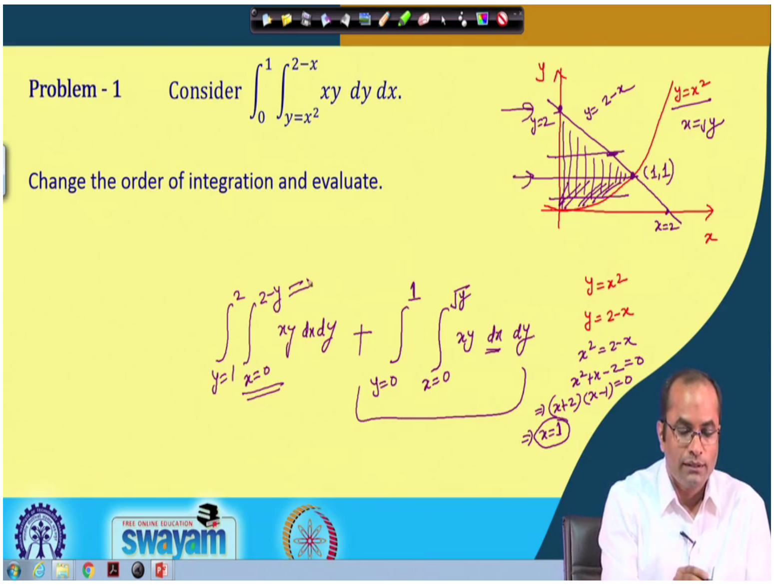

So, we consider few more problems. So, let us discuss this 0 to 1 and y goes from x square to 2 minus x, xy dy dx. And we want to change the order of integration and then we want to evaluate though in this case our integrand is just xy. So, it will not matter whether we first differentiate with respect to y or with respect to x the complicacy level would be more or less the same.

So again to change the order of integration we have to sketch the domain here. So, in this case what do we have? We have this x axis and we have the y axis there and we want to now draw this. So, this y goes from x square and up to y is equal to 2 minus x. So, y is equal to x square is this parabola here. So, we have y is equal to x square and then we have the line. We have the line y is equal to 2 minus x. So, when x is 0 y is 2 somewhere and when this y is 0, x will be 2 again. So, this is the line y is equal to 2 minus x.

So, the region here of the interest is going to be because y goes from x square; so, this parabola 2 to minus x. So, this is our region of integration and now we can easily change the order, but before we need to get what is this point of intersection. So, here y if we put x square is equal to 2 minus x and then we have a x square plus x minus 2; that means, we have x minus 2 and. So, x plus 2 and x minus 1 is equal to 0. So, we have the point here x is equal to 1 and minus 2. So, x is equal to 1 this is x is equal to 1 and y will be also 1. So, this point of intersection here is 1 1 the value of x is 1. And also the value of y is 1 and at this point here the value of y is 2. So, y is 2 and here the x will be 2 and y is 0.

So, we have all the coordinates we can change the order now. So, this was 0 to 1 and now we want to have the order xy and dx dy; so, first with respect to x. So, for with respect to x we have to draw the line parallel to this x axis, and see where it enters and leave the domain. So, in this case it always enters through this x is equal to zero line. So, here we have x is equal to 0 as the limit and then we have the upper limit here which is up to this x y is equal to 1 we have the limit of x always enters through this x is equal to 0 and leaves through this y is equal to x square; that means, x is equal to square root y. So, here x goes from 0 to this parabola which is a square root y, and this is up to the point y is equal to 1. So, y goes from 0 to 1.

So, up to this point we have this and then we will have one more part because when y goes from 1 to 2, then the limits of x will be from 0 to this line; that means, we will have another part here when y goes from 1 to 2 the limits of x will be again from 0 to, but now it exists exit from the line; that means, there x is 2 minus y. So, this 2 minus y and the integrand dx dy. So, we have the 2 parts now; one where y is going from 0 to 1; that means, this domain here this domain and the other one the upper one where y is from 1 to y is from 1 to 2 where the x is going from 0 to 2 minus y.

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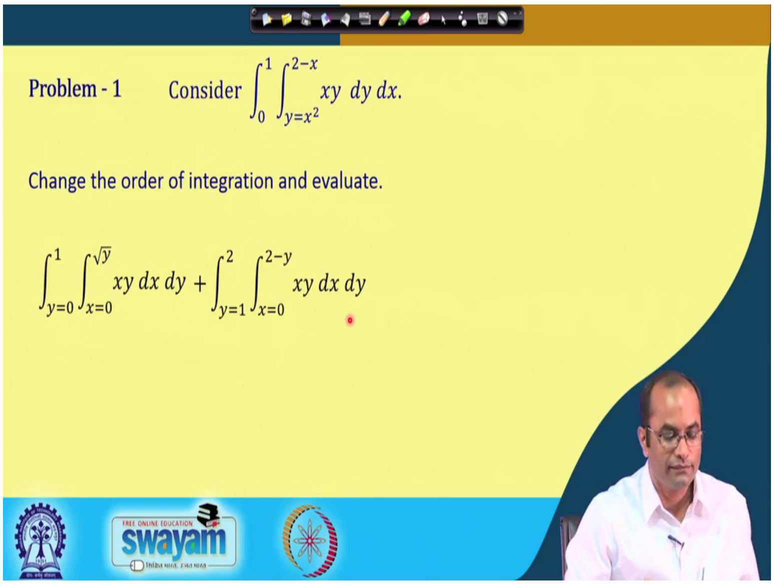

So, having change this order of integration, we have this situation of this 2 integrals. So, we will evaluate these integral separately and then we can find the value of the integral.

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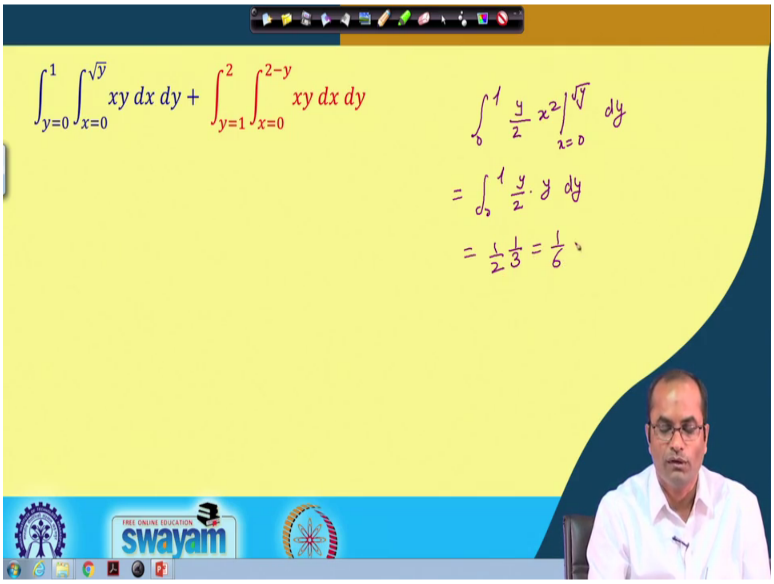

So, considering the first integral here y goes from 0 to 1 and x goes from 0 to square root y so, the first integrals so, 0 to 1 and then here with respect to x. So, y and with respect to x when we integrate x this will be x square by 2, and we have to put the limits for x 0 to square root y. And this is dy here. So, 0 to 1 and we have y by 2 x square will become y again when we put this square root y and minus this 0. So, we have y and this dy. So, we have half y square this will be 1 by 3 y cube and when we put the limit 1 it will be 1 and minus 0. So, we get the value 1 by 6 here.

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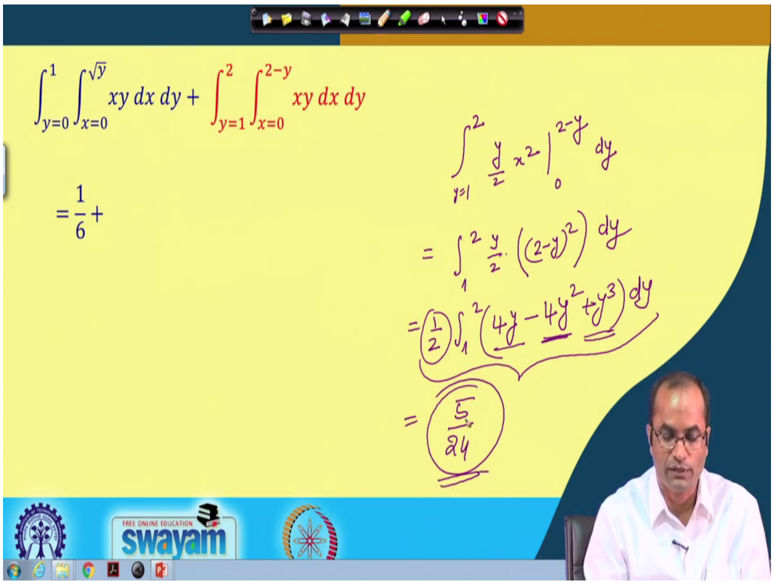

So, the value of the first integral is 1 by 6 and then we can evaluate the second integral again. So, here we have the y from 1 to 2 and the second integral x 0 to 2 minus y. So, we are integrating now for x will be x square by 2 and the limits will be 0 to 2 minus y and then we have dy. So, this is equal to 1 to 2 we have y by 2, and then x square. So, 2 minus y square and then minus 0; so, we have this dy. So, 1 by 2 and this 1 to 2. So, y we have has to be multiplied here.

So, we will get this 4 into y and then minus will be 4 times y from here and then this y square and then there will be y square term and plus this y y cube and then we have dy. So, now, we can easily integrate this 4 y and this is minus 4 y square and then we have this y cube term and with this half. So, this evaluation when we do, this is coming to be 5 by 24 the integral of this. So, 5 by 24 we need to add here.

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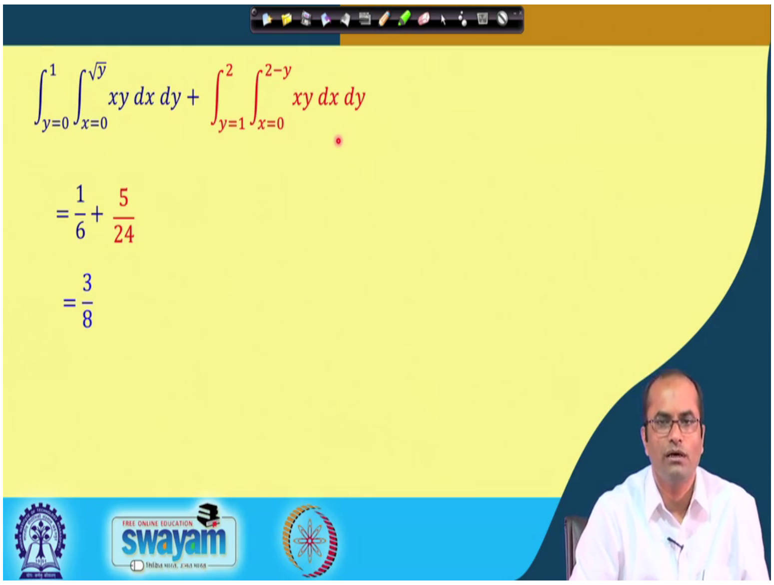

So, this 5 by 24 if we add than this will become as 3 by 8. So, the value of this integral is 3 by 8 and we have used again the idea of change of order of integration which was not necessary in this case indeed it has the integral has become more complicated by changing the order of integration, because we have to now considered 2 domains 1 was from y to 1 and the other one was from 1 to 2.

So, in this particular case it was absolutely not necessary to change the order in fact, the direct evaluation would have been easier; but, here the inverse to learn how to change the order of integration.

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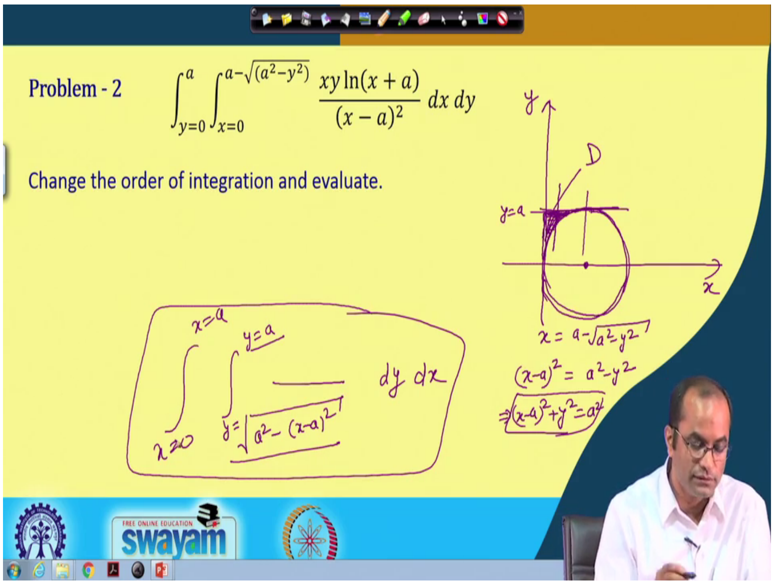

Now, this is problem number 2, where we will again change the order of integration and evaluate and indeed in this case change of order of integration is necessary because if we just considered first with respect to x, this integrand is pretty complicated for integrating with respect to x, but if you change the order. So, you will have here dy first we will be integrating with respect to dy and in that case we have only this y there. So, by changing the order we can see directly looking at the integrand, that we can easily integrate with respect to y this given integrand.

So, for changing the order we have to now draw this order of integration or the domain of this integration here. So, we have the x axis we have here y axis and x goes from 0. So, x goes from 0 to this point. So, what is this? x is equal to a minus square root a square and minus y square. So, if we bring this x minus a and take whole square, then this will be a square minus y square. And this implies this is x minus a square and plus this y square is equal to a square. So, this is the circle which we can draw now first. So, x is equal to a. So, this is the center a and 0 is the center and a is the radius again. So, this is going to be somewhere here. So, we have the circle of radius a and now if we look at the integral limits. So, for x it goes from 0 to the circle. So, from 0 to the circle and it in the direction of y it is exactly going to y is equal to a that is the radius of this circle. So, this is small region here that is the order of that is the region of this integration. So, this region is the order is the region of the integration this d, which we have to now see when we change the order of integration what will happen. So, if we change the order. So, first we have to fix the limits for dy.

So, the integrand will remain as it is and then later on with dx. So, changing for y we have to now draw a line parallel to the y axis and see where it enters the domain and where it exit the domain. So, for y it is now entering through the circle at any point here to cover the whole domain; so, entering through the circle. So, what is the question here?

So, that will be y is a square and minus x minus a whole square that is the lower limit of y, and it exist from y is equal to a. So, this is y is equal to a the upper point. And now for the x direction it is fixed now from x is equal to 0 and to x is equal to a, because these lines move from x is equal to 0 to x is equal to a. So, that is the outer limit of the integral. So, we have change the order here easily and now the inner one goes from y is equal to a square minus x minus a square to y is equal to a and the inner one goes from x is equal to 0 to x is equal to a.

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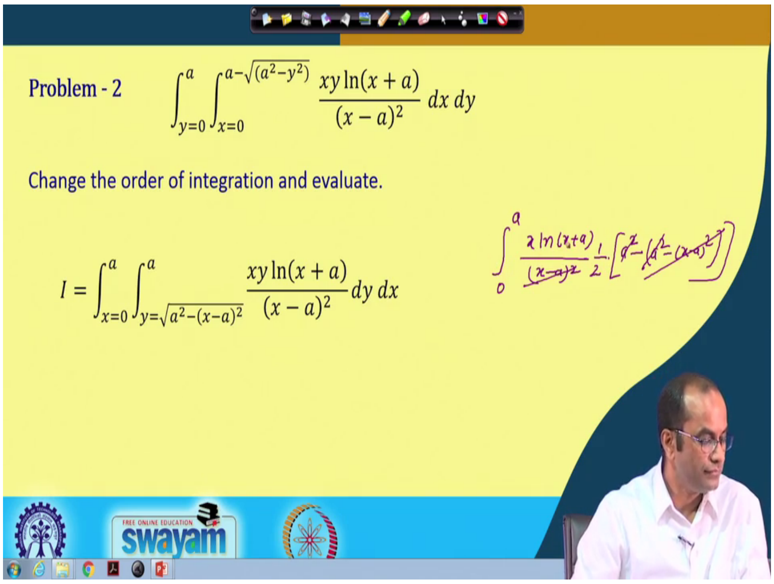

So, we have this for the inner one with respect to y and then we have also for with respect to x. So now, we can easily integrate the inner one. So, integrating this one with respect to y we will have just the y squared term there. So, the outer integral will be 0 to a, and then the integrand which is if I leave y. So, we have ln x plus a and then here x minus a whole square and then this is y square by 2. So, now, putting the limit for y square; so, we have a square and minus this y square, which will be a square minus x minus a whole square.

So, a square minus a square will cancel out then we have x minus a whole square and that whole will go out with this one. So, we are half x ln x plus a; we have half x ln x plus a.

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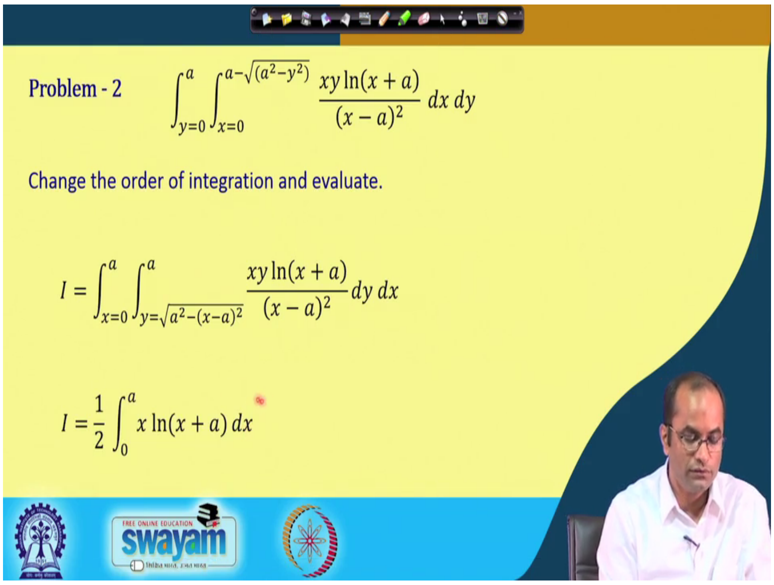

And this is exactly the result of the integral the inner integral, and now we can evaluate this outer integral as well.

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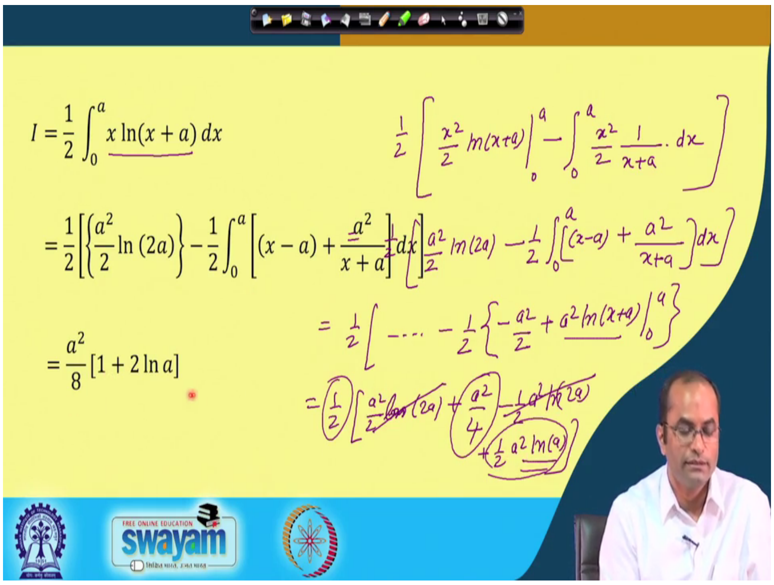

So, this is again is here. So, we have to apply the idea of this integral by parts. So, we will do that. So, here half as it is and then we have here the x, which will become x square by 2 ln x plus a for the integration by parts 0 to a and then minus again 0 to a this will become x square by 2 the integral of x and then the differentiation of this ln x plus a will be 1 over x plus a and then we have this dx.

So, half and then upper limit x; so, a square by 2 and ln 2 a minus the 0 because of this x and then we have minus sign with half and this 0 to a this x square. So, I am adding here plus a is minus a square and then plus a square. So, this x square minus a square will be become x minus a x plus a. So, x plus a will get cancel. So, you will get x minus a here and then there was a term plus a square. So, this x plus a and then dx. So, we have half this first term, then minus this half and then we can integrate this now. So, x minus a square by 2 and the upper limit will give 0 and the lower limit will give minus a square by 2.

So, here we have the a square and this will be ln x plus a and the limit from 0 to a for this part. So, what do we get? Half the first term which is a square by 2 and this ln ln 2 a and we have minus minus plus a square by 4 here and then here, a square with minus half term a square and ln 2 a and then 0. So, this will be plus half and then a square and ln a. So, this a square by 2 ln 2 a will get cancel and we will have this term a square by 4 and a square by 2 here.

So, a square by 4 we can take common. So, this will become a square by 8 and we will have 1 plus this 2 times ln a that will be the answer. So, this will be a square by 8 and this 1 plus 2 times this ln a that is the value of this integral.

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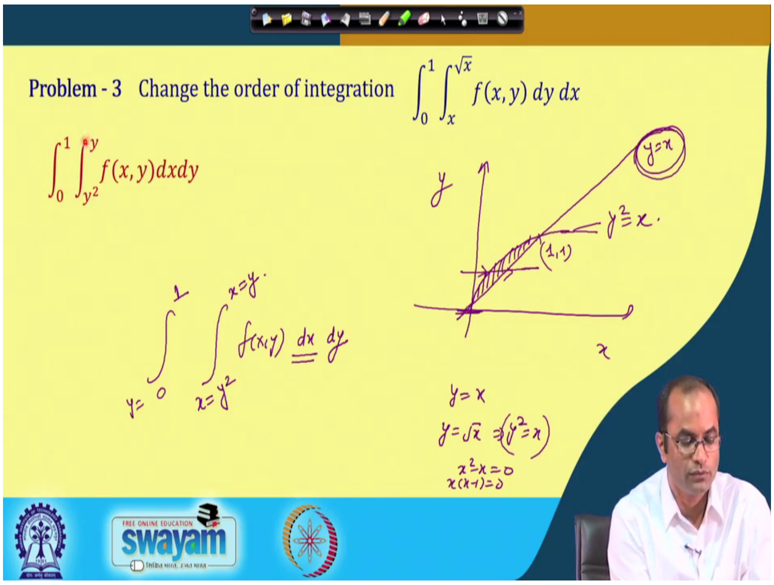

Now, this problem we want to change the order of integration. So, changing the order of integration in this case again let us quickly draw the region, and we have this x here and we have y. So, y goes from x to y is equal to square root x which is y square is equal to x. So, we have the parabola we have this line here. So, y is equal to x line; y is equal to x line and we have this y is equal to y square is equal to x the parabola. So, they will intersect first of all at x square minus x is equal to 0. So, x and x minus 1. So, x is equal to 1 they will intersect. So, this parabola will have this. So, this is y square is equal to x and now if you look at. So, this y goes from the line to the parabola. So, this was the given order of the integration, and now we want to change the order. So, what we will have now the f x y and we will have dx and dy there.

So, the we will fix first the limit of x. So, we need to draw a line here in the direction of this a parallel to x, and we was see there where it enters the domain and where it exit the domain. So, here the entry point is again the parabola. So, the x is the limit will be from y square to and this is leaving the domain exactly at x is equal to y. So, here the x is equal to y and in the direction of. So, this was the point the 1 1 point and in the direction of this y now. So, such lines are moving from y is equal to 0 to y is equal to 1. So, that is the change of order of integration. So, for the limits of y will be 0 to 1 and limit of x will be y square to y and then f x y dy.

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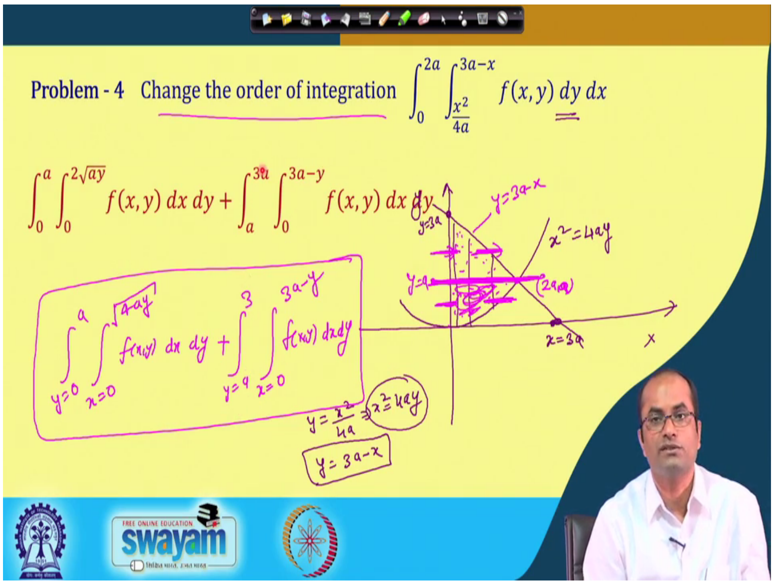

So, one more example that is the last example for this lecture; so, we have now again we want to change the order of integration and our limit for the y goes from x square by this 4 a to 3 a minus x. So, again we need to draw the region here. So, x axis and then we have here the y axis. So, this y goes from x square over 4 a or this x square is equal to 4 a y, and then we have y there at the upper limit 3 a minus x. So, this is the line here. So, let us draw first this line and that will be x 0. So, we have a 3 a point here y is equal to 3 a point.

And when the y is 0 so, x is 3 a. So, here x is equal to 3 a. So, this is the line and now we have this x square is equal to 4 a y. So, that is the parabola x square is equal to 4 a y. So now, here if we look at the limits for y; so, y goes from the parabola to the line. So, this is the direction for the y from the parabola to the line, and now we want to change the order of integration; that means, in the direction of x we have to draw the draw a line parallel to the x axis and again we have that situation which we have discussed earlier, that there will be 2 domains because of this problem here.

Before this line so, in this domain in the lower domain here, we have the entry point exactly at x is equal to 0, but the exit is the parabola. And in the upper part of this domain here, we have the entry point this x is equal to 0 the same, but the exit point here is again this line. So, there are 2 different regions we have to consider now if you want to change the order of integration.

So, for this first part we will have we want to have now the dx dy. So, for this x here x goes from 0 x goes from 0 and to this parabola which is a square root this 4 a y or square to square root a y and then in this case we have to also see what is this point here which will come as 2 a into 0. So, 2 a into a. So, up to this x y is equal to a. So, from 0 to y is equal to a, and then the plus from a to 3 a. So, y is equal to a to 3 a now for the range of this x is this line to that line. So, here x goes from 0 again and to this line which was y is equal to 3 a minus x. So that means, x here is 3 a minus y and then we have this f x y and dx and dy.

So, this is the order the change of order of integration we got this 2 integrals because we need to divide now the domain into 2 parts because the line this parallel to this x axis was not following from 1 to the another one, but in this lower domain we had from this x is equal to 0 to this parabola while in the upper domain here when y is greater than a, then we have the situation that it is entering again at x is equal to 0, but it will exit now from this line y is equal to 3 a minus x and that is the reason we have to break the integral into 2 parts. So, this is the result which we have written already here from a to 3 a and 0 to 3 a minus y.

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So, coming to the conclusion what is important when we change the order of integration? First of all this is very useful very convenient in many situations, when we observe that the given integrand is not possible to integrate in that given order. So, by changing the order of integration it becomes easier to integrate, but for that the sketching of the region of integration is very important and then putting the limit of the integration after sketching the domain it becomes much easier.

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So, these are the references we have used for preparing the lectures.

Thank you very much.