Engineering Mathematics – I
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Lecture – 27
Differentiation Under Integral Sign

So, welcome back to the lectures on Engineering Mathematics-1, and this is lecture number-27. And we will be continuing our discussion on integral calculus. And today's topic will be Differentiation Under Integral Sign. So, in particular we will be talking about Leibnitz rule, so that is rule where we apply for differentiation under integral sign. And we will go through also the derivation of this Leibnitz rule and some worked problems.

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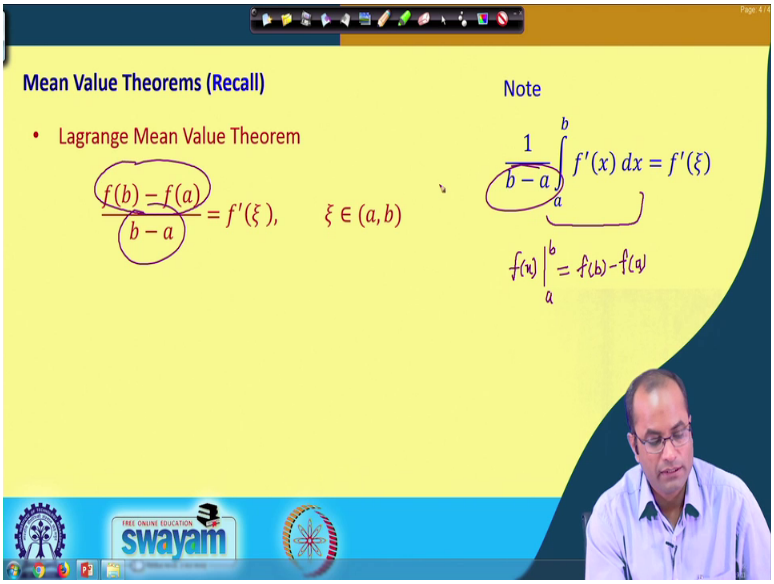

So, before we go to define this rule discuss this rule, we need to recall the mean value theorems. So, we already discuss in previous lectures in differential calculus. This Lagrange mean value theorem, which says that this ratio here f b minus f a the quotient over b minus a is equal to f prime psi, where psi is somewhere between a and b. And here we had this assumptions that f is continuous on the close interval a, b and differentiable on the open interval a, b.

So, having this mean value theorem, we can also include little more based on this Lagrange mean value theorem. So, we have this 1 over b minus a as it is and this f b minus f a term, we can write down as integral f prime x dx. Because, this integral if we solve, so this is nothing but the f x, and then we have limit here a to b, so which will be f b and minus the f a. So, this f b minus f a exactly the same term here, and divided by this b minus a is equal to f prime z.

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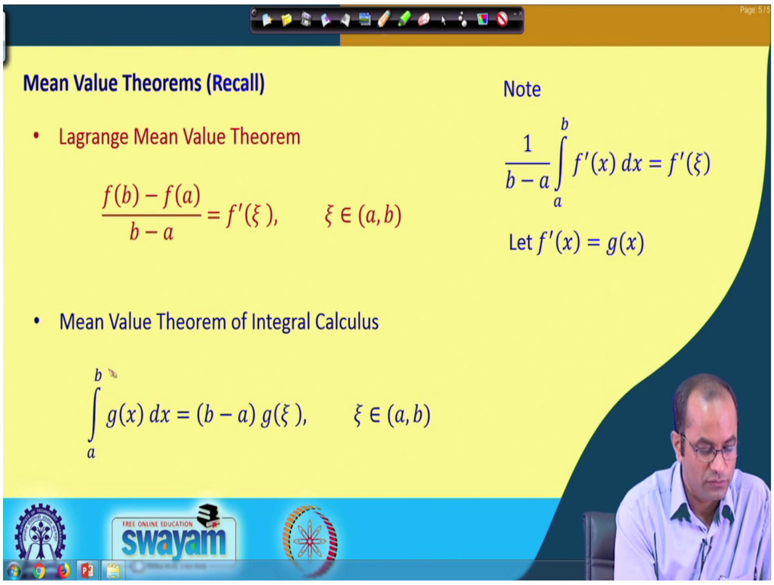

So, now we will make another assumption. We assume that this f prime the derivative of f, this function is denoted by this function g. So, having this now we get another mean value theorem for integral calculus. So, here the f prime will be replaced by g, and we get this rule here so a to b, and this g x dx will be b minus a. So, this from the left hand side b minus a, and g psi so what is this; what does that mean here.

So, if you want to integrate this a to b g x dx, this will be equal to this difference here of the limit so b minus a. And then g psi, so g the integrand is taken at some point in the interval a to b. So, we do not know exactly where is this point, but this mean value theorem says that this integral will be equal to g at some point in the interval, and multiplied by this difference of the limit b minus a.

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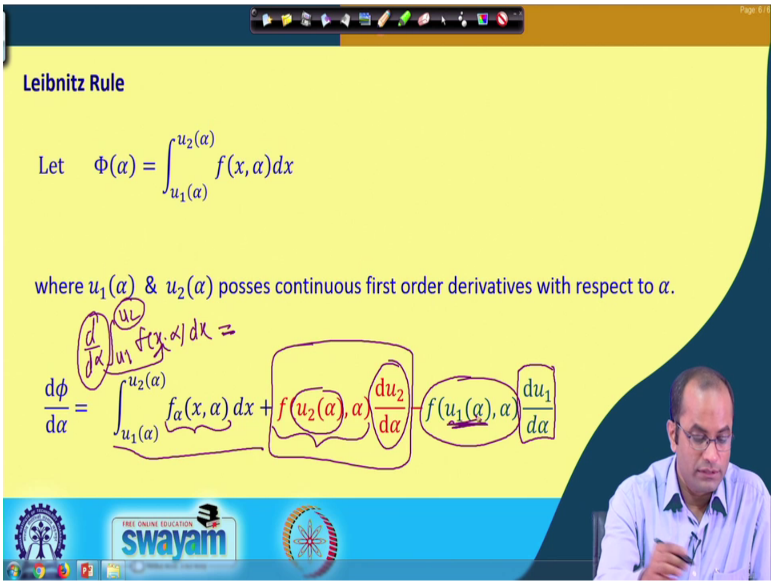

So, with this with these two mean value theorems, we can now proceed for the Leibnitz rule. So, this rule says if we have this integral u 1 alpha, so these are the limits. So, it could be some function of alpha to u 2 alpha and the integrand is function of two variables, so f x alpha dx. So, this we assume that this is a function of alpha, because we are integrating over this x. So, what is this integral? This integral depends on alpha. So, we assume that this is a function of alpha.

And this rule says that if these u 1 alpha and u 2 alpha the function sitting in the limits here as a function of alpha, they have continuous first order derivatives with respect to alpha. So, if this is the case, we can differentiate this phi with respect to alpha or in other words we can take the differentiation of this integral, and the rule says that this d over d alpha will go into the integral.

And we will have the derivative or the differential of this f x alpha. So, the derivative of this f x alpha will be the partial derivative with respect to alpha, because there are two variables here. So, the partial derivative with respect to alpha of this, and then there will be some more terms. So, this rule says that we can take the derivative inside here; we will get partial derivative of f with respect to alpha.

Then plus the derivative of the upper limit here, so u 2 alpha we get here du over d alpha So, the derivative the ordinary derivative, because u 2 depends only on alpha, so d u 2 over d alpha. And then in the integrand here which was f x alpha, the x will be also replaced by u 2 alpha. And then minus now for the lower limit you will have again the derivative of the lower limit d u 1 over d alpha. And the function here, the integrand will be now evaluated at this u 1 alpha, so that is the rule that is easy to remember.

So, when we want to take this, like d over d alpha of this integral here u 1 to u 2 of this f x alpha dx, so this will be equal to the this derivative will go inside the integral so over the integrand. And this will become like partial derivative of f with respect to alpha, so that is the one term.

The second term will have the derivative of the upper limit, and the integrand will be evaluated again at this upper limit. So, this x will be replaced by u 2 alpha minus the same scenario here again. So, the derivative of u 1 the lower limit with respect to alpha, and then the integrand will be evaluated again at this u 1 alpha. So, this is the Leibnitz rule and we will go though this proof also, which is very simple with the knowledge what we have already discussed in differential calculus, we can prove that this d phi over d alpha is equal to this 1.

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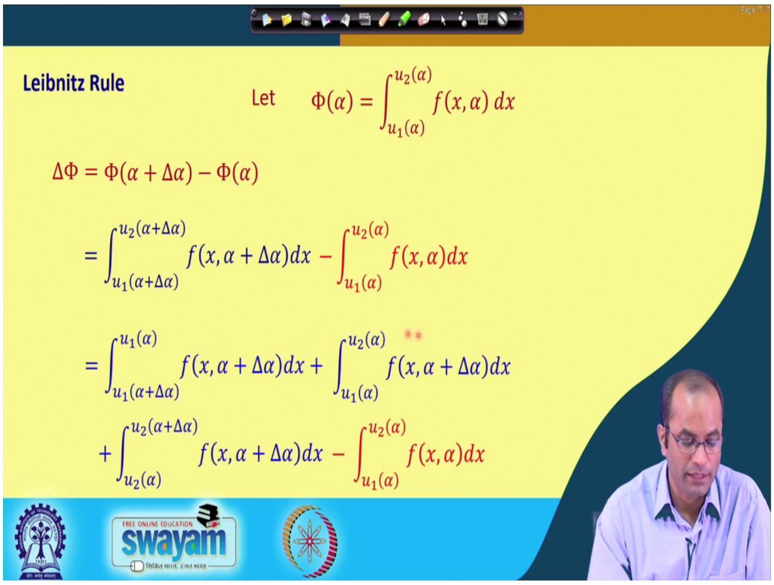

So, here the Leibnitz rule, so we check this integral as a function of alpha phi alpha. And then consider this delta phi, the increment in phi as by taking the increment delta alpha in alpha, so we have phi alpha plus delta alpha and minus phi alpha. So, with this we have here alpha plus delta alpha for alpha. So, here alpha will be replaced by delta alpha, so we will have u 1 alpha plus delta alpha the lower limit. And then the upper limit u 2 alpha plus delta alpha the function f x, and alpha will be replaced by alpha plus delta alpha. So, wherever we have alpha that will be replaced by alpha plus delta alpha and minus this phi alpha, so again the same integral u 1 to u 2 and f x alpha and dx.

Now, we will do little more manipulation here. So, u 1 alpha plus delta alpha, and we are going to some number this which is actually appearing here u 1 alpha, so up to u 1 alpha this integral, we are going to break now. So, u 1 alpha plus delta alpha to this number u 1 alpha plus then u 1 alpha to some other number this u 2 alpha or the function here u 2 alpha, and then from u 2 alpha to the end of the limit.

So, we have taken this integral as a sum of these three integrals. So, u 1 to alpha plus delta alpha to u 1 alpha, then u 1 alpha to u 2 alpha and then u 2 alpha to the end limit there u 2 alpha plus delta alpha, so that is a property of the limit. We can take many intermediate points and we can break the integral into several integrals.

Then we have here minus exactly that term. So, now we will combine the similar terms, so here we have the same limits u 1 alpha to u 2 alpha here also u 1 alpha to u 2 alpha. So, these two terms you will combine, and we will have then integrand f x alpha plus delta alpha and minus f x alpha. So, these two terms will be combined, and then we will have these two terms as well.

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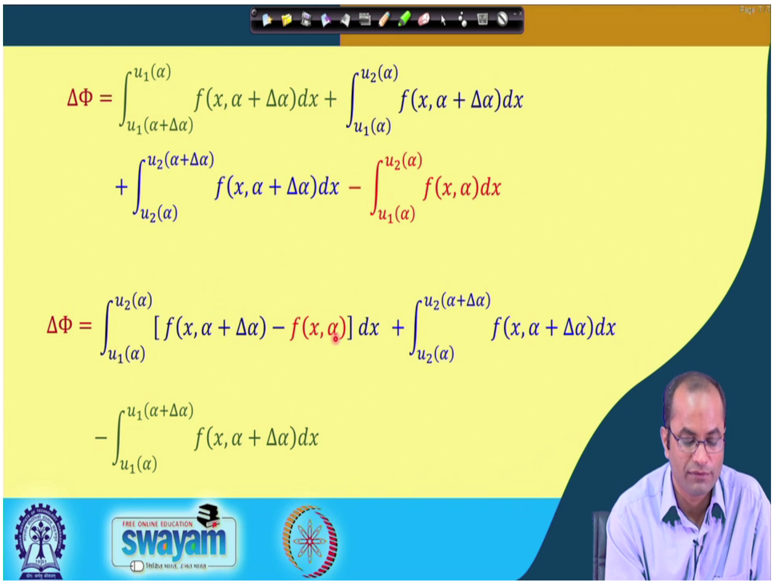

So, combining these two terms, we will get the u 1 to u 2 alpha. And this is the the difference f the integrand f x alpha plus delta alpha minus f x alpha dx plus this term u 2 to u 2 alpha plus delta alpha, and this integrand minus. So, this here the limit we have reversed. So, now u 1 alpha to u 1 alpha plus delta alpha with the negative sign and f x alpha plus delta alpha dx.

So, now we have these three integrals. And now we will apply the mean value theorem. So, here we have this difference of f x alpha plus delta alpha and minus f x alpha. So, the difference is with the second argument alpha plus delta alpha minus f alpha. So, here we will apply the mean value theorem from differential calculus or the Lagrange mean value theorem. And in these two integrals, we will apply the mean value theorem from integral calculus, which was reviewed in the previous slide.

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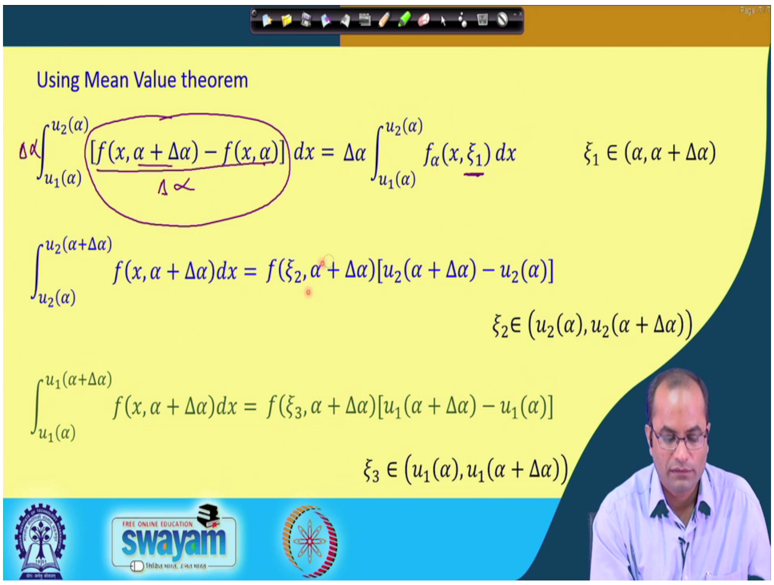

So, using mean value theorem for the first integral, we have this difference. So, we will apply the mean value theorem here, which says that if we divide here by delta alpha and multiply by delta alpha, so we can do so, so divide by delta alpha and then multiply by delta alpha here.

So, this one by Lagrange mean value theorem will be the derivative with respect to alpha, because the increment is in alpha, so derivative with respect to alpha. And f x and some point here psi, which we are calling here psi 1. So, the psi 1 belongs to exactly the interval here alpha to alpha plus delta alpha. And note that this psi 1 will go to 0 and delta alpha goes to 0. So, later on we will be going taking on the limit. So, this psi 1 will approach to alpha as delta alpha goes to 0.

So, with this now we have the second integral, which was u 2 to u 2 alpha plus delta alpha f x alpha plus delta alpha dx. And now we will use the again the mean value theorem from integral calculus, which says that this here because the integral is over x, so this like a constant in this reference, so integral is over x.

So, there will be a point somewhere in this range or the limits of this integral, where the integral value will be equal to f. So, some point here, so psi this x will be replaced by this psi 2. And alpha plus delta alpha as it is, and then the difference of the limits so u 2 and this minus u 2 alpha, where now this psi 2 belongs to this limits somewhere. So, u 2 alpha to u 2 alpha plus delta alpha, so that was the mean value theorem for integral calculus.

And now for the second integral the same theorem we will apply, so this x will be replaced by some psi 3. So, this function evaluation at the point psi 3 and the difference this of the limits, so u 1 alpha plus delta alpha minus u 1 alpha and again this psi 3 will lie between this u 1 alpha and u 1 alpha plus delta alpha. So, these three integrals will be replaced by these right hand side terms. So, again we will have one integral, and these two terms.

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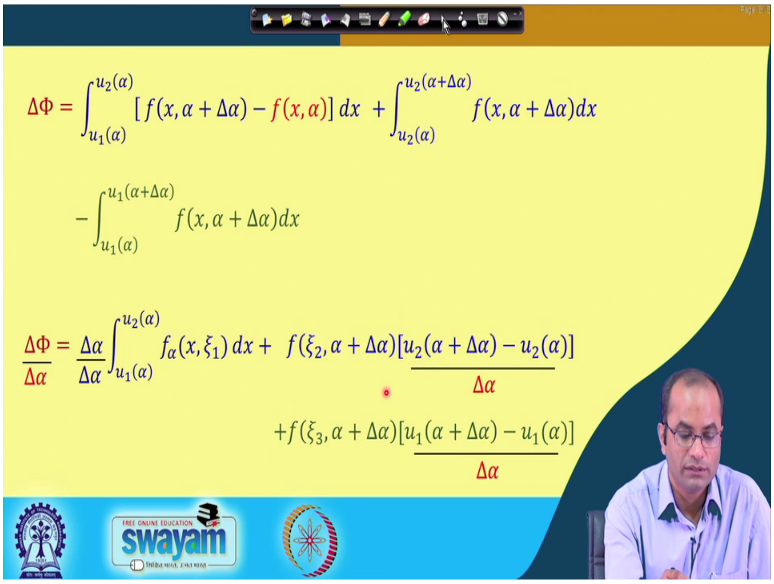

So, this was the original the delta phi in the form of integrals. And now we will replace all here we have applied this Lagrange mean value theorem, then the mean value theorem for integral here also mean value theorem for integral. So, we will get from the first term, because we have applied the Lagrange mean value theorem here. So, we get this f alpha and this d delta alpha will be there.

From the second term, we have applied the mean value theorem from integral calculus and again here as well the mean value theorem from integral calculus. So, these three integrals now, we have new terms here. And we will divide by delta alpha to each of these terms. So, here delta alpha, and here also this delta alpha, here we will divide by delta alpha, and here also we will divide by delta alpha. And now with this delta alpha, so this is delta u 2. So, this is the increment in u 2, when we make increment in alpha by delta alpha. So, this is delta u 2 and this is delta u 1.

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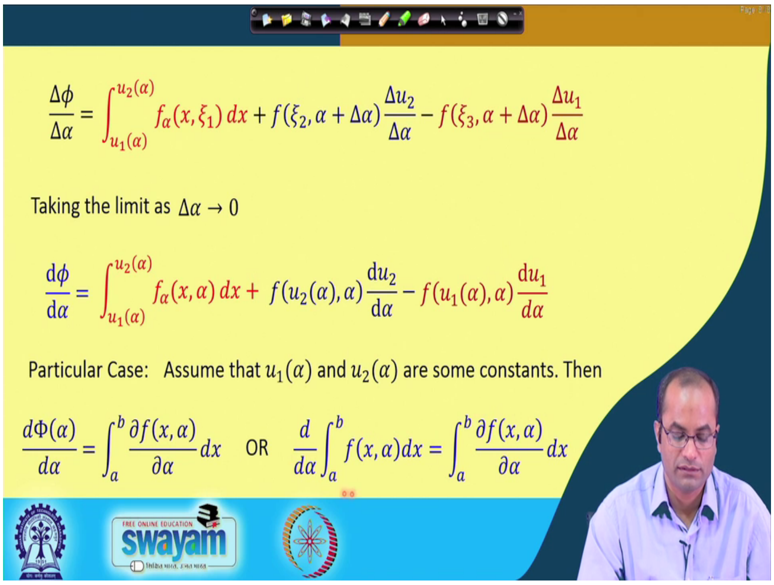

So, with this notation we move now. So, here delta u 2 over delta alpha delta u 1 over delta alpha, and the delta alpha gets cancelled here. So, in this equation now, we can pass the limit as delta alpha goes to 0. So, passing this limit delta alpha goes to 0, this will become d phi over d alpha the derivative.

And this first term, so the psi 1 will go to alpha, and we have u 1 alpha to u 2 alpha f alpha rest everything is same. So, this psi 1 goes to alpha, when delta alpha goes to 0. Here we have the psi 2 term, and the psi 2 lies between u 2 alpha and u 2 alpha plus delta alpha. So, when delta alpha goes to 0, this psi 2 will go to u 2 alpha. And then here also when delta alpha goes to 0, this will go to alpha.

And here this will become the derivative again the u 2 over d alpha minus the same thing here the psi 3 lies between u 1 alpha to u 1 alpha plus delta alpha, and when delta alpha goes to 0. So, this term psi 3 will go to u 1 alpha, so we have here f and u 1 alpha and this alpha plus delta alpha will be alpha as delta alpha goes to 0. And then this delta u 1 over delta alpha will become the derivative again of u 1 with respect to alpha.

So, we are done with the rules, so which says that this differentiation under integral sign, so when you want to take the differentiation. It will go to the integrand, so the partial derivative of f with respect to alpha, and there will be a term d u 2 over delta so the upper limit, the derivative of the upper limit. And that same limit will be substituted in the integrand, and minus the lower the derivative of the lower limit and the same limit will be substituted into the function. So, a particular case which we often use, so we assume that u 1 alpha and u 2 alpha, they do not depend on alpha, so they are constant. So, in that case the second and the third term here will be cancelled, because d u 2 over d alpha d 1 over d alpha will become 0.

And then we have only the first term. So, if we have the constant limits, we will have only the one term, when we take the derivative or we differentiate under integral sign. So, we will have only the one term, because these two terms will cancel out or usually we write like d over d alpha of this integral. So, this d over d alpha, because these limits are constant, we will go to the integrand. And then we have this differentiation under integral sign. So, we can take this derivative into the integral, but this will be like partial derivative, because f depends on two variables x and alpha.

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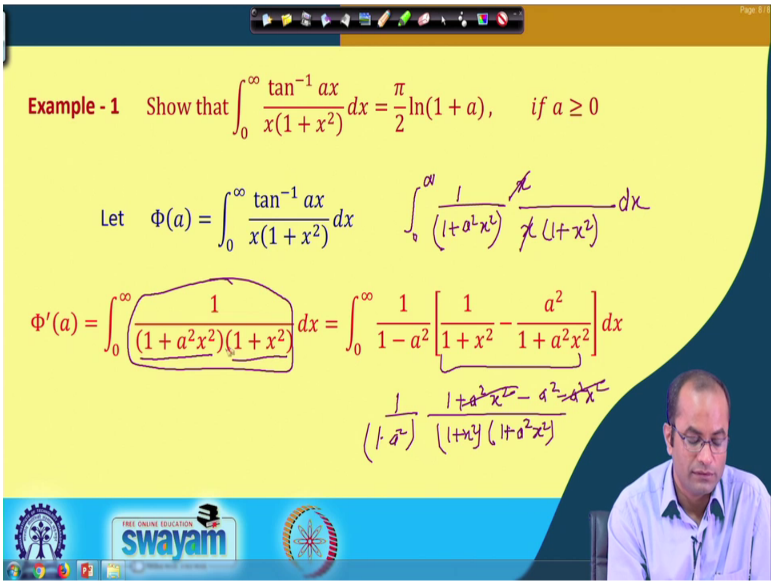

So, now in this example-1, we will show that the integral 0 to infinity tan inverse a x over x and 1 plus x square is dx is pi by 2 the logarithmic of 1 plus a, and a is greater than or equal to 0. So, this idea of this differentiation under integral is very often used to compute such complicated integrals, and they become very simple with the idea of this differentiation under integration sign, because we consider actually in this case this integral the given integral as a function of a, because the a is there as the tan inverse.

So, this is the phi a and 0 to infinity tan inverse a x x over x square plus dx. So, the given integral, we have taken as a function of a. And now we will take the derivative or we will differentiate this with respect to a. So, with the differentiation if we can see some a simplification or the resulting integral, we can easily solve then this going to be useful, because the integral given in this form is its a little bit complicated to evaluate.

But, if we differentiate here with respect to a, then what will happen? So, the limits will be treated as a constant here though one should note that in general this Leibnitz rule or the differentiation under integral sign, which the rule we have proved that is not valid for improper integrals. We need some extra conditions in general, but all these examples that is applicable and we are not going much into the details about the applicability of this Leibnitz rule for improper integrals. So, all the examples considered here that rule is applicable.

So, here treating these constant limits, we can just differentiate with respect to a. So, this tan inverse a x will be 1 over 1 plus a square and x square, and then the derivative of a x with respect to a will be x. So, there will be a term x in the numerator, and there is a term in the denominator. So, this x will get cancelled with that x, so note that. So, if we differentiate here, we are basically getting 0 to infinity. And this tan inverse will be 1 over 1 plus a square and x square, and then we have a term from this a x. So, this x will be there, and then already here it was x and 1 plus x square, and then we have dx term, so this gets cancelled.

And therefore, we got only this 1 plus x square term, and x where x square plus this your 1 plus x square term, so these two. And now we can do the partial fractions here. So, this partial fractions of this 1 over 1 plus a square x square 1 plus x square will be given as 1 over 1 plus x square minus a square over 1 plus a square x square. And then to balance this we need to also multiply by this 1 over a square, because if you see this term is like 1 plus x square and 1 plus a square and x square term here.

So, when we multiply there would be 1 plus a square and x square, and this will be minus a square minus a square and x square. So, this will be cancelled. So, we have 1 minus a square and to balance this we need 1 minus a square there. So, this will be also cancelled by this, and we will get exactly the integrand of this integral. So, this is fine, so with this partial fractions.

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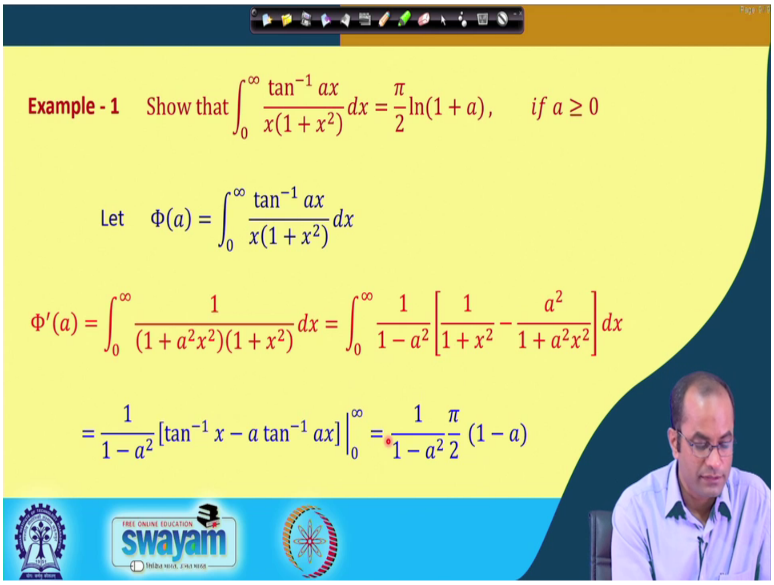

And now we can easily integrate this. So, the first integral is 1 over x square, which is tan inverse x. And in the second case also we have tan inverse a x and that divided by a. So, this will be 1 over 1 minus a square, the first integral gives tan inverse x and minus, so 1 over a and this a square will give us a, and tan inverse a x, and the limit 0 to infinity.

So, when we put this infinity tan inverse x, when x goes to infinity this will be pi by 2 and minus here also, it will be pi by 2, so minus a pi by 2. And when we take the limit, the lower limit as x goes to 0, both will become 0. So, we will have 1 over 1 minus a square, and pi by 2 we can take common and we will get 1 minus a. So, this is the integral, and this 1 minus a here we have 1 minus a square. So, this can also be cancelled out, and we will get here 1 plus a. So, we get this differential equation that d f d phi over d a or phi prime a is equal to pi by 2 and into 1 over 1 plus a.

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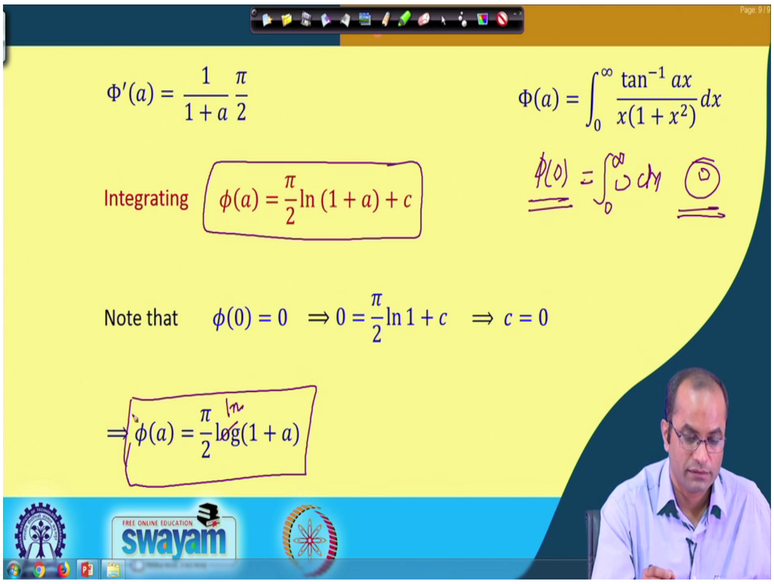

So, we got this derivative here, and that is the always the process this which makes it easier, because now we can easily integrate this differential equation. So, when we integrate this, we will get phi a and this will be ln, so the logarithmic of 1 plus a with pi by 2 and plus some constant, constant of integration an arbitrary constant. But, that constant also is easy to evaluate, because we should know that this phi a was this tan inverse a x over x and 1 plus x square dx and the phi a, so the phi a value phi 0, so at 0.

So, we will take the point where we know the integral value easily, so this is 0 to infinity and tan 0, so this will be 0. So, here we have phi 0 is 0, this information is given which we can use here to evaluate this constant c.

So, with this phi 0 is equal to 0, so here 0. And then this is 0, so ln 1 is 0 again, so we get the c is equal to 0. So, we got this constant as 0, and then phi a becomes pi by 2 ln 1 plus a, so phi a is pi by 2 and this ln. So, this log is just the natural logarithmic ln and 1 plus a, and this what we want to prove. So, we got this value here by using this Leibnitz rule or the differentiation under integral sign very quite easily.

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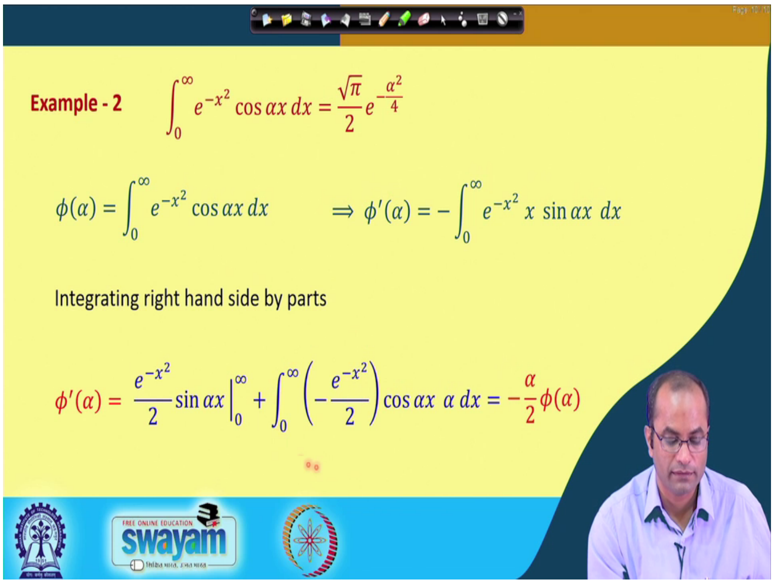

So, another example, where we will be calculating evaluating 0 to infinity for minus x square cos alpha x dx, so again the same process. So, we will take this as the phi of alpha the function of alpha, and then get the derivative of this phi with respect to alpha. So, here this will become with minus sign, so minus sign alpha x. And this alpha x with respect to alpha will give us x, and that is exactly the point, because this was not easy to differentiate partially.

But, now since x has appeared here with e power x square, and now we can easily differentiate. So, we can easy integrate this term, and differentiate this term, so we can apply the rule of partial of integral by parts. So, by doing so, we will get this phi prime alpha, here integration by parts, we will get this e power minus x square over 2 with minus sign, and but this minus sign will make it plus, so that is the integral of this 1.

And the sin remain, so sin alpha x and then the limit 0 to alpha then there will be a minus sign, and this minus minus will become plus. We have 0 to infinity, so 0 to infinity again the integral of this e power minus x square x, which is minus e power minus x square over 2. And this sin the derivative of this sin alpha x will be cos alpha x, and then the derivative of alpha x will be alpha and dx.

So, we have now this here when x goes to infinity, this will go to 0. And when x goes to 0, because of the sin this will go to 0. So, first term will vanish. And then the second one, we have this minus sin with half and this e power minus x square cos alpha x the same integral, which we have started with phi a. So, this will be like minus alpha by 2 and this phi a. So, minus alpha by 2 and this phi a and this same integral.

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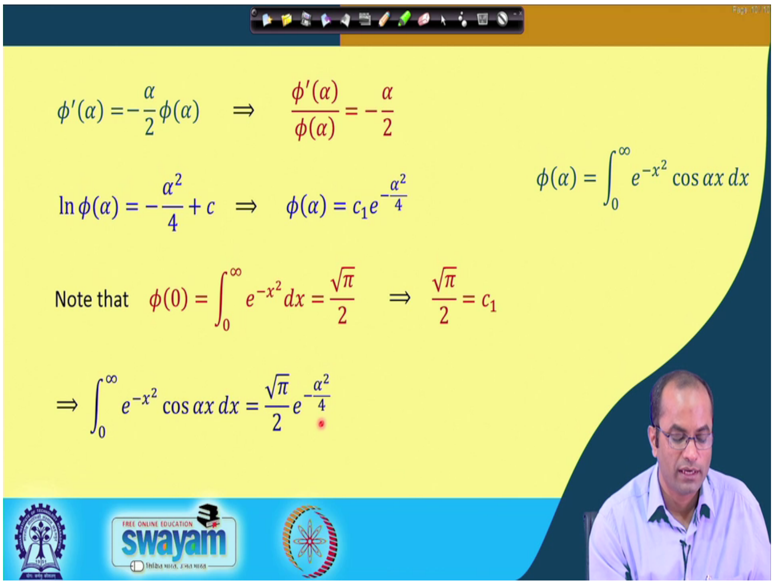

So, we get this differential equation phi prime alpha is equal to minus alpha by 2 and phi a, so which is very simple to integrate. So, we can take this phi alpha to the left hand side, and here the d alpha will go to the right hand side and integrating this. So, we will get this logarithmic of this phi alpha, and is equal to minus alpha square by 4 plus some constant and this phi alpha, so we will take the exponential this side.

So, we will get exponential minus alpha square 4 into e power c, and that e power c we can assume another constant. So, we have c 1 e power minus alpha square by 4. And now we note that this phi 0. So, the phi alpha was e power 0 to infinity for minus x square cos alpha x dx. So, phi 0 will be 0 to infinity e power minus x square dx. And that e power minus x square dx, this integral we have seen in previous lectures as well.

And in next lectures, we will also prove that this is equal to square root pi by 2. So, this value of this standard integral is square root pi by 2. So, we know the phi 0, so by this we can compute now the c 1, so the c 1 will be square root pi by 2. Because, when we put alpha 0 here, this e power minus alpha a square by 4 will become 1, we have c 1 right hand side, and this will be square root pi by 2. So, this implies that 0 to infinity e power minus x square cos alpha x dx will be equal to square root pi by 2 and e power minus alpha square by 4, and that is the value of the integral we want to evaluate.

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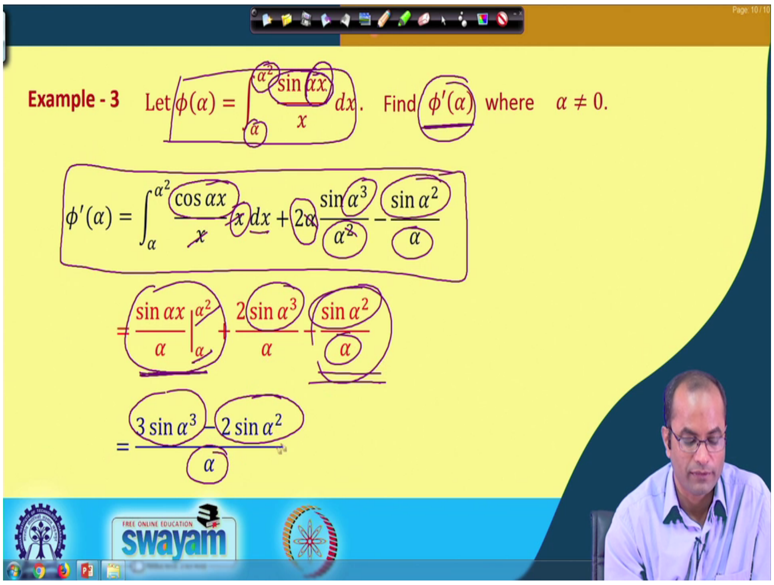

So, the third example is again very simple. So, we have this phi alpha is equal to alpha to alpha square, and sin alpha x over x dx, and we want to get this phi prime alpha, where alpha is not zero. So, it is a direct application of this differentiation under integral sin.

So, if we differentiate this under integral sin, naturally this we will take the derivative of this sin alpha with respect to alpha. So, we will have cos alpha x, and then with respect to alpha the differentiation of this alpha x will give us x there and then dx plus the derivative of the above limits, so that is 2 alpha.

And then in the integrand, we will substitute this alpha square for x. So, we have sin alpha cube for this alpha x over alpha square and minus the derivative of this alpha with respect to alpha, which is one here and then sin alpha x. So, x will be replaced by alpha. So, sin alpha square, and then here alpha.

So, we have applied this Leibnitz rule to this integral. And now here this x, x gets cancelled, and we have this cos alpha x term. So, this cos alpha x the integral will be sin alpha x over alpha, the limit alpha to alpha square plus this 2 alpha. So, this alpha also get cancelled here 2 over alpha and sin alpha cube, and here we have sin alpha square over alpha.

So, now when we put the upper limits, we will get sin alpha cube by alpha here also we have 2 times sin alpha cube over alpha, so that will be 3 times sin alpha cube over alpha. And then this minus from here from the first term, we will get sin alpha square over alpha, and then from this term also this sin alpha square over alpha. So, we will get minus 2. So, here also the minus sin alpha square over alpha here also sin minus sin alpha square over alpha. So, this will become minus 2 sin alpha square and divided by that alpha, so that is the value here of this integral 3 sin alpha square and minus 2 sin alpha square over alpha. So, this was the direct application of the Leibnitz rule.

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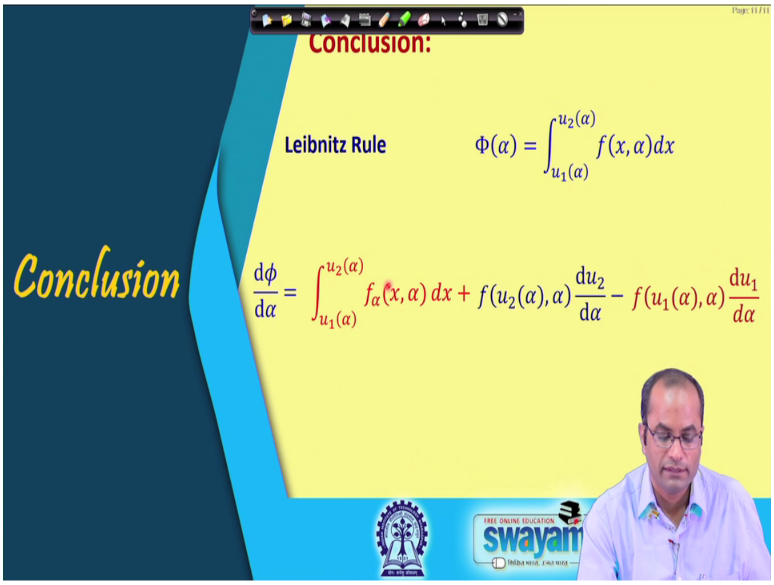

And coming to the conclusion so, we have learned this Leibnitz rule, which is useful for differentiating the integrals. So, if we have the integral, we want to take derivative with respect to alpha there. So, this rule says that we can take this derivative term to the integral integrand, which means that we will have the partial derivative of f with respect to alpha plus the derivative of the upper term, so du 2 over d alpha. And then in the integrand, we need to replace this x by the upper limit minus the derivative of the lower limits, so du 1 over d alpha and in the integrand we will replace this x by u 1 alpha, so that is the Leibnitz rule.

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These are the references we have used to prepare these lectures.

And thank you very much.