Engineering Mathematics – I

Prof. Jitendra Kumar
Department of Mathematics
Indian Institution of Technology, Kharagpur

Lecture – 26
Beta & Gamma Function (Contd.)

So, welcome back to the lectures on Engineering Mathematics – I, and this is lecture number 26. We will continue the discussion on Beta and Gamma Function.

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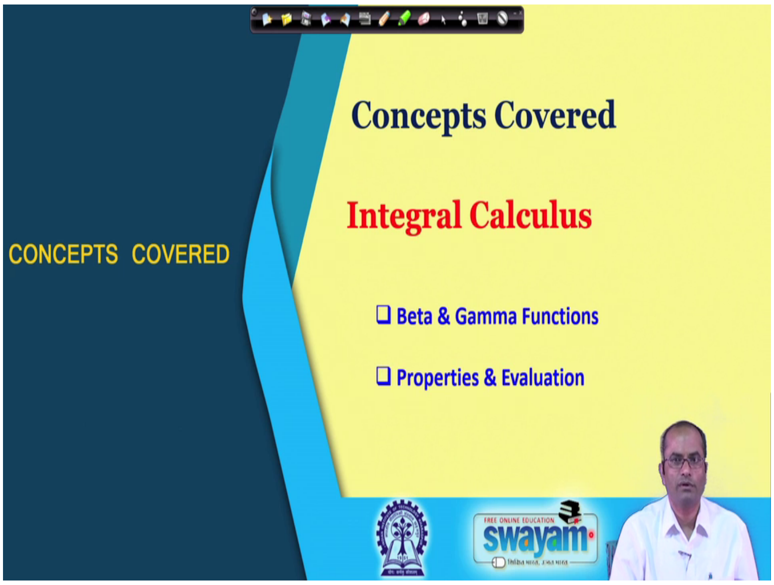

So, these are the special type of examples for improper integrals and today we will be talking about mainly their properties and how to evaluate those integrals. And, in the last lecture we have already seen the convergence of those integrals.

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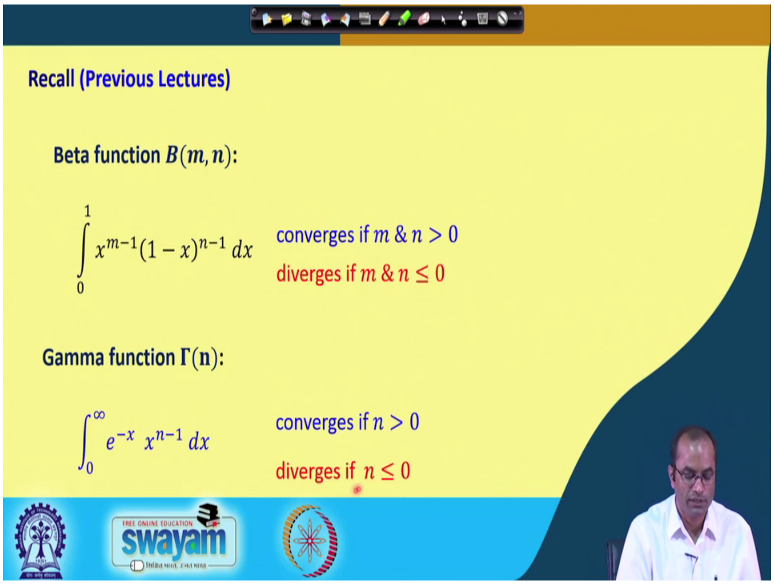

And, if you recall that the beta function was this improper integral 0 to 1 and x power n minus 1 x power n minus 1 dx which converges when this m and n, this number here in the power exponent so, m and n are strictly positive and the diverges this integral diverges when m and n they are less than or equal to 0. And, we have also seen for the gamma function that this integral, this mixed type of improper integral 0 to infinity e power minus x x power n minus 1 dx, it converges if this n is strictly greater than 0 and divergence when n is less than or equal to 0.

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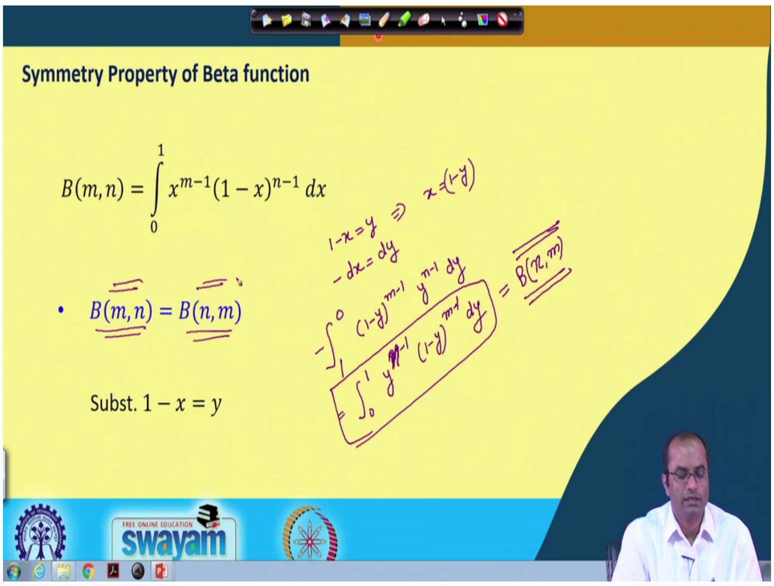

First we will discuss the property one nice property of this beta function which is the symmetry property and this is the improper integral for beta B m, n. And, if we substitute here for 1 minus x is equal to y then we can easily see that the B m, n is equal to B n, m.

So, let us do it here. So, if we substitute here 1 minus x is equal to y ; that means, minus this dx is equal to dy and then this integral; so, then x 0, so, y 1 and then we have here 0. So, x power m minus 1; so, from here the x becomes 1 minus y. So, x power m minus 1; so, we have 1 minus power y power m minus 1 and 1 minus x is y. So, y power n minus 1 and this dx will become dy with negative sign. So, this we can revert now. So, 0 to 1 and we have y power n minus 1 and 1 minus y power m minus 1 dy.

So, this is the again the beta function and which as per our notation this will be denoted by beta. So, this is here 1 minus so, y power n minus 1. So, this is n and 1 minus y power m minus 1. So, per our notation this is beta n, m. So, this beta n, m is equal to this beta m, n. So, we have the symmetry property here. So, does not matter whether we compute this beta with m, n or with beta and n, m. So, that now you will keep in mind now in future discussion.

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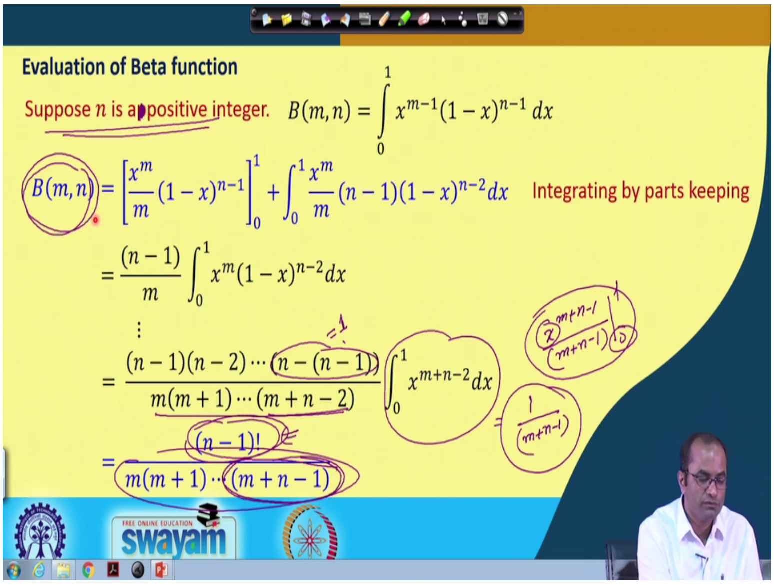

And so, how to evaluate this beta function; the question is the evaluation because these are the special integrals, improper integrals and special care has to be taken to evaluate this beta function. So, suppose here n is a positive number. So, in many cases we can discuss that or we can compute that integral in a very simple fashion. So, first space we are considering here when n is a positive number.

So, in this case we consider this beta m, n the given integral and now, if we integrate this by parts and taking that this n is taken as a positive integer so, we will differentiate this part of the function and then this will be for the integration when we do this integration by parts. So, this will be here the integral of this x power m minus 1 which is x power m and divided by this m for this x power minus 1 and now this will remain as it is. So, 1 minus x power n minus 1 and then we have these limits 0 to 1 plus the integral of this x power m minus 1 with the minus sign there, but that will be compensate.

So, here x power m divided by m and then we have n minus 1 and 1 minus x power n minus 2 and the dx. So, this is your n minus 1 and this divided by m. So, we have now what is left. So, n minus 1 and divided by m we have taken this out of the integral. So, the integral becomes 0 to 1 x power m and 1 minus x power n minus 2 and this is dx here.

So, just we have integrated by parts the integral, keeping in mind that this n is taken as a positive integer, so that we will be now differentiating this function wherever needed and this will be used for the integral. So, we have integrated the function and this kept as it is in the first part here then the limits and then plus again the same thing and we have differentiated now this one n minus 1 1 minus x power and minus 2 and this minus x. So, that will be minus 1 again the differentiation, but that minus 1 has been taken care because in this formula here minus sign should appear.

So, we have this because this first integral this will disappear when x approaches to 1 this integral will become 0, when n is a positive integers. So, that means, 1, 2, 3 so, in that case we have a this will become 0 and when x approaches to 0 because of this x this will become 0. So, when n is 1 what will happen? So, for n is equal to 1 this will be any way we can deal it separately, we do not have to consider that case in this way. So, this is naturally when n is greater than equal to 1 because when n is equal to 1, so, we do not have this term and we can evaluate this easily for the given function x power minus 1.

So, here do this again this partial this integration by parts several times until we can remove this exponent here this power n minus 2. So, depending on n we have to different integrate several times. So, in general what you will get here we will keep on doing this.

So, we have n minus 1 in the next iteration we will get n minus 2 and n minus 3 and so on, until we see that this n is becoming here the 1 and then this disappear; so, the n minus n. So, this when 2 was there we got 1 here and when n will be coming at some point of time so, n minus 1 will coming here.

So, in that case this will become now n minus n and then same thing will happen here. So, we have m and then in the next situation this will be m plus 1 then m plus 2 and so on and we will we can see the pattern here. So, when 2 was there the 1 is coming in this term the 3, then 2 will come and so on. So, 1 less is we are adding here. So, we have n minus 1, so, n minus 2 will be coming here with m.

So, we have in the denominator these terms mm plus 1 and 2 m plus n minus 2 and in the numerator this is going up to one. So, this here and gets cancelled with n and this is the 1 term. So, this is nothing, but n minus 1 n minus 2 and this product continuous up to 1 and then this integral become 0 to 1 and x power. So, because here the power is increasing here it was x m minus 1 after integration once by power we got this m here, so, naturally this is same as what we have in the denominator the last term. So, this power here of this x will become m plus n minus 2; so, m plus n minus 2 dx.

And, now this is the integral with no other term 1 minus x term disappear completely. So, we have only one term in the integral and this is easy to integrate. So, what we get after this integration this will be x power m plus n n minus 1 because we will add 1 and then this will be divided by again with this term and then we have the limit 0 to 1. So, when we put this x is equal to 1 we will get 1 over n plus n minus 1 and that is a reason we have taken here this product to m plus n minus 1 that is a that is the next term when we add 1 there. So, we have the continuity here we have m, m plus 1 up to the same plus and minus 2 and then the last term m plus n minus 1.

And, when we put 0 there so, this will be 0. So, this is the outcome here out of this integral out of this integral we will get just 1 over m plus n minus 1. So, this term here we have incorporated. So, m plus n minus 1 and the numerator here because this was the product from n minus 1 to 1 and that is the definition of the factorial we use. So, this is the notation now factorial n minus 1 we have use for this product n n minus 1 n minus 2 up to 1.

So, this beta m, n when this n is a positive integer we got that this is factorial n minus 1 and then here this product m m plus 1 2 m plus n minus 1. So, same thing we can do when we assume this m to be positive and noting that this beta is a symmetric function which we have just seen before. So, there is absolutely no issues, you will get the similar formula, this m will be replaced by n.

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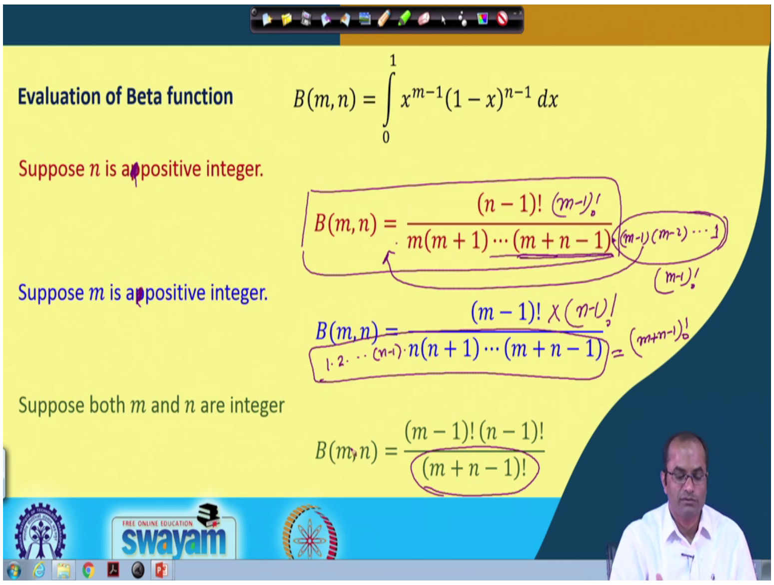

So, in this case we have just seen when we have taken any positive number. So, when we have taken any positive integer we got this beta m, n as factorial n minus 1 and this product of this m terms. And, we can do the same similar calculations when m is a positive integer and in that case also we will get a similar result. And, the result will be just we can replace this m by n we will get this expression here with m minus 1 factorial and then these terms with n and n plus 1 and the same term m plus n minus 1.

If both are integers m and n both are integers so, in this case we have you can either take this case for example. So, when here m is also integer, so; that means, we have m m plus 1 and so on this product again appearing of these integers with the difference of 1. So, if we multiply in the in the numerator and in the denominator term also m minus 1 m minus 2 and up to 1 so, basically this side here; so, what we are getting that 1, 2, 3 and so on m minus 1 m and this product goes up to m plus n minus 1.

So, here also this was nothing, but the factorial m minus 1. So, this was factorial m minus 1 term. So, we have multiply with factorial m minus 1 term and then this also becomes from 1, 2 m plus n minus 1 the product. So, this is becoming again the factorial m plus n minus 1. So, we have n minus 1 there and factorial n minus 1 and in the dominator here we will get m plus n minus 1 factorial. Same thing we can observe if we take this formula for example, and in this case we have to multiply by this n minus 1 factorial because we have to now multiply here also by n minus 1 factorial; that means, 1, 2 and n minus 1 and then we have n.

So, this continues now to this one and this will again become this factorial m plus n minus 1 which is considered here m plus n minus 1 factorial. So, this is the formula when m and n both are integers. So, we have this beta functions can be written in terms of the factorial; so, factorial m minus 1 factorial n minus 1 and divided by factorial m plus n minus.

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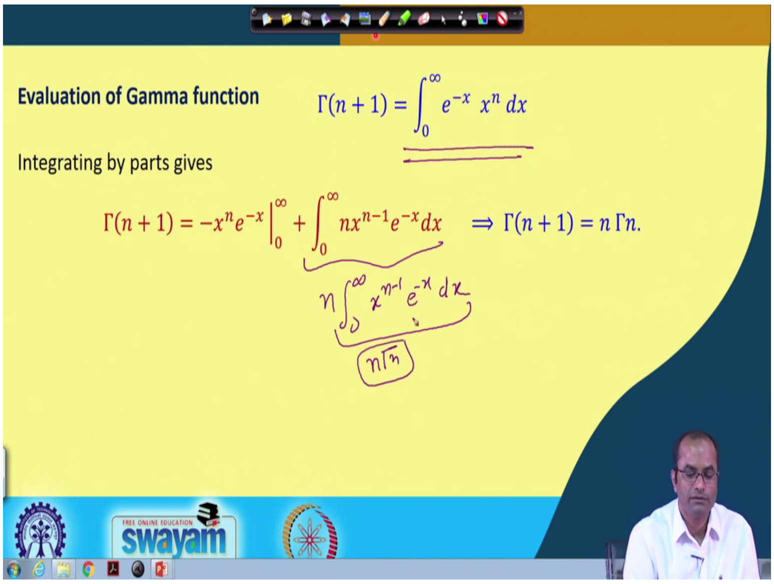

Now, evaluation of the gamma function: so, can we have this integral 0 to infinity power minus x x power n dx and you will use the same idea that just integrating by parts we will get at nice relation of this gamma n plus 1. So, when we integrate by parts so, naturally now we will differentiate the x power n minus 1. So, we have e power minus x 1 the integral of e power minus x will be same minus e power x, x power n will remain as it is and then we have this limit here 0 to infinity.

So, limit 0 to infinity plus integral so that minus minus will become plus here. So, integral and then this integral of this e power minus x which is minus e power minus x and that minus has been already taken care. x power n minus 1 when we differentiate this we will get n x power n minus 1. So, this first term when x approaching to infinity because of this exponential this will go to 0, and when x approaching to 0 because of this x, n positive because otherwise it does not make sense we have already seen the convergence. So, this will become 0, the first term.

So, what do we get now out of this relation that this gamma n plus 1 is equal to; so, this here if we take this n outside the integral so, this integral is nothing, but n and then we have 0 to infinity x power n minus 1 minus e power minus x dx and as per the definition of the gamma function what are we getting just n is replaced by n minus 1, so; that means, this here this integral is nothing, but gamma n.

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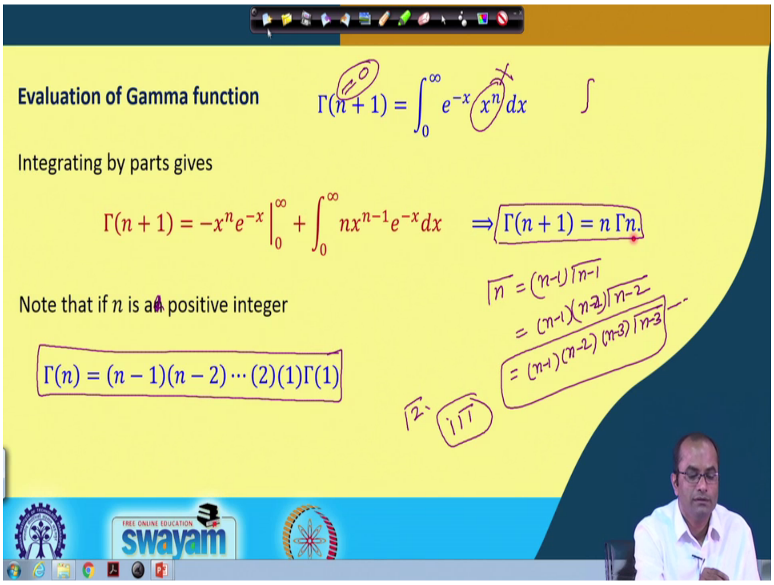

So, we got this relation that gamma n plus 1 gamma n plus 1 is equal to n gamma n and that is a very useful relation which will be used to evaluate this gamma functions. If n is a positive integer if n is a positive integer, what will happen? So, when n is a positive integer this we can easily valuate in this case because as per the definition or as per this reference relation what do we get gamma n, we can write down in terms of like gamma n will be n minus 1 gamma n minus 1.

So, here if I if I use here gamma n and we use this relation so, this will be n minus 1 and gamma n minus 1. Again for gamma n minus 1 we can use that relation. So, we have n minus 1 and this is n minus 1 and gamma n minus 2 and we can keep on repeating this.

So, n minus n minus 2 gamma n minus 2; so, n minus 2 then here n minus 3 gamma n minus 3. So, this we will continue until we get this 2, 1 and then gamma 1. So, at some point we will get like gamma 2 there then which we can write gamma 1 gamma 1 and so on.

So, we will get this when n is a positive integer the simplification we will get that the gamma 1 we can easily compute now and now we will end up with because this was a positive integer and we are just reducing by 1, so, we will end up with this gamma 1 naturally. So, this gamma 1 if we see from this integral so, when we substitute there n is equal to 0 if we substitute here n is equal to 0 in the integral we have gamma 1. So, n is equal to 0 means this term will disappear.

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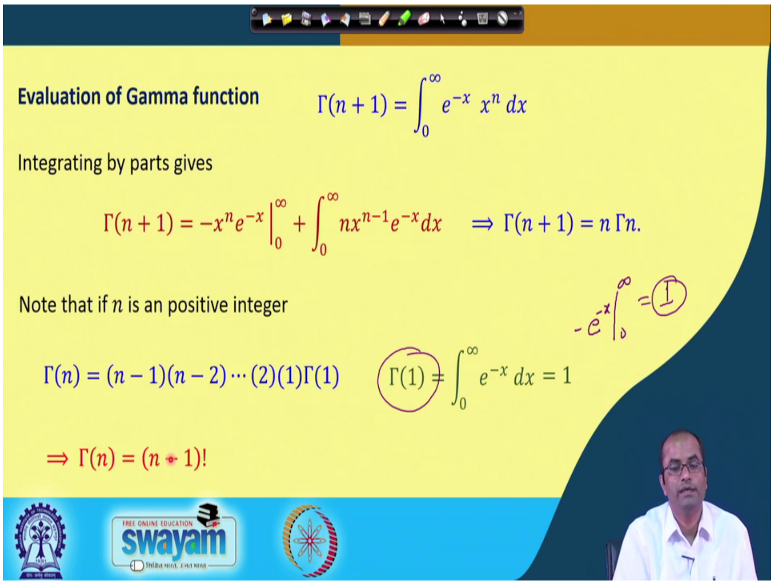

So, we have simply the integral 0 to infinity e power minus x dx that a gamma 1 and this integral we can evaluate easily because exponential minus x integral is minus exponential x and we have the limit here 0 to infinity. So, when x approaches to infinity we will get the 0 and minus minus plus x approach into 0 you will get 1. So, this will be 1. So, the value of this gamma 1 is the 1.

So, having this value now, so, we have gamma n is equal to 1, 2, 3 the product up to n minus 1 which items of factorial we can write down; that means, gamma n is nothing, but the factorial and minus 1. So, that is a nice relation we got that such integral which was improper integral, but we have a nice relation in terms of the this factorial when n is integer at least in this simple case. So, the gamma n is coming to be n minus 1.

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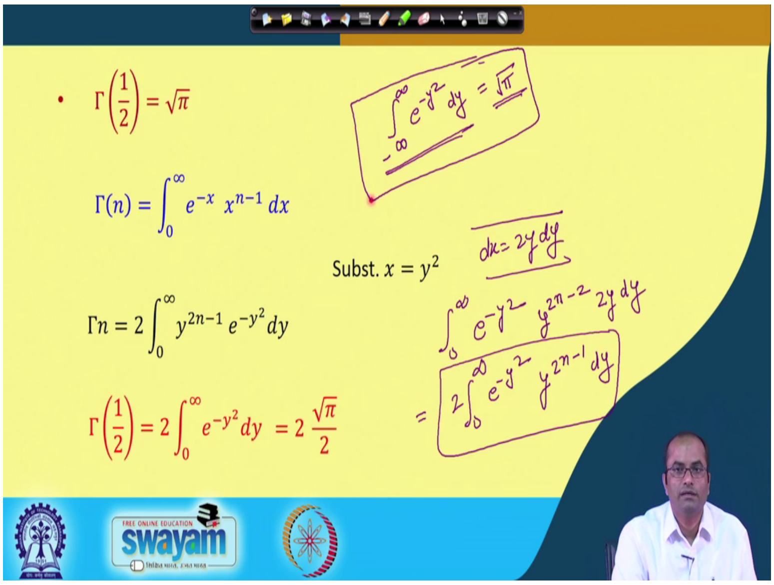

Now, moving next we have another nice of relation not only for the interior, but also for this half at least gamma half which is going to be again useful. So, if we have that reference relation gamma n plus 1 is equal to gamma n gamma n. So, for some non-integer number as well we can compute using that reference relation and at the end if we end up with this gamma half then certainly we can get the value. So, this will be again helpful. So, gamma half is square root pi which we can observe by the definitions of gamma n is 0 to infinity e power minus x and x power n minus 1 dx. So, if we substitute this x is equal to y square in this integral if we substitute x is equal to y square this will have another form of this gamma integral.

So, what will happen for the substitution the dx will become the 2y dy the limits will remain the same. So, we will have 0 to infinity e power minus x square and this x will be y square. So, y 2n and then minus 2, but dx is again 2y dy; so, this y. So, this 2 time 0 to infinity e power minus y square and we have y power 2n minus 1 and dy. So, this is another form of this gamma integral which we will use now.

So, this is 2 times 0 to infinity power minus y square y power 2 n minus 1 dy and now, you will substitute for this n as n is equal to half. So, having this we have 2 time 0 to infinity this y power when n is half so, this becomes 0 so, y power 0 1. So, we have e power minus y square dy and this is well known integral which we will also evaluate in when we will be talking about the double integral later on, but this one the value of this integral 0 to infinity e power minus y square and this dy. In fact, we will see later that value of this minus infinity to plus infinity power minus y square comes to be square root pi.

So, at present we will assume this, but it is very useful integral and better to also remember this value square root of pi and having this one now we can evaluate this 2 times and the value of this interval is coming to b square root pi by 2 because this is in the half range not from minus into infinity, but it is 0 to infinity. So, this is 2 times and the square root pi by 2 and this to get cancel and then we have the value square root pi. So, gamma half is square root pi what provided we have just without evaluation we have use this value, but after few lectures we will evaluate this when discussing the double integrals. So, this for later and.

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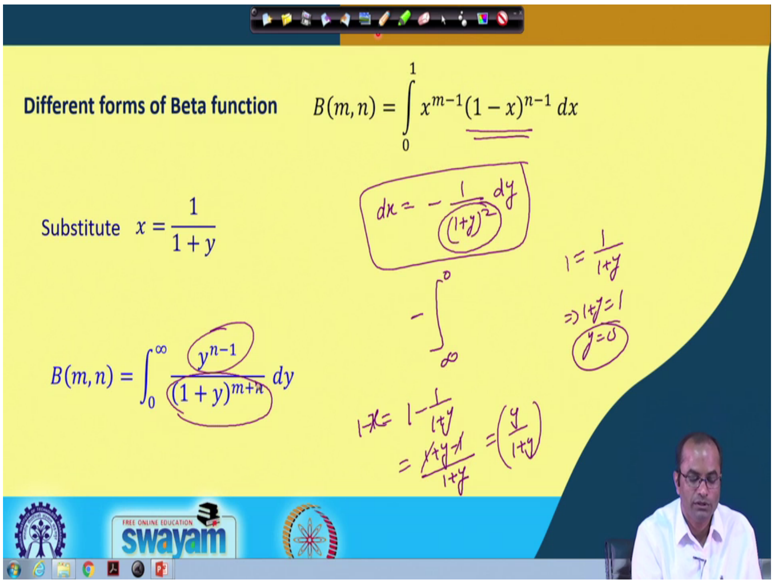

Now, we have also several different forms like we have seen just for the gamma function by some substitution we got another form, same thing happened for beta function. So, let us just go through some forms that could be useful to evaluate the integrals or to recognize integrals in a different format. So, we have this standard definition of the beta m, n and if you substitute for example, here x is equal to 1 over 1 plus y; so, in that case your dx will become minus 1 over 1 plus y whole square and then dy. So, this will be the relation and then this beta m, n will become. So, let us talk about the limits. So, x, when x was approaching to 0 here, the y is approaching to infinity. So, this we will get this integral as this minus sign and this will be approaching to infinity when x is approaching to 1 when x is approaching to 1 the y will approach to.

So, 1 is equal to 1 over 1 plus y; that means, the 1 plus y is equal to 1. So, y is 0. So, y will approach to 0 and then we have this minus sign so, which will revert the limits. So, we have 0 to infinity there and so, 0 to infinity and then we have this x is replaced by 1 over 1 plus y. So, we will have here 1 over 1 plus y power m minus 1 1 minus x. So, here from here we can also get this 1 minus x. So, what will be 1 minus x 1 minus x. So, 1 minus x will be 1 minus 1 over 1 plus y and this is 1 plus y minus 1 over 1 plus y. So, this will be y over 1 plus y. So, here y power n minus 1 which is exactly here and this 1 plus y power n minus 1.

So, combining those powers now so, here we have this m minus 1 power them here we have n minus 1 power. So, we will have a m plus n n minus 2, but this 1 plus y power 2 here will get cancelled with those 2 here. So, we will get this power m plus n. So, this is another form of the beta function given here y power n minus 1 and 1 plus y power m plus 1 and we should note that with this substitution the integral limits change completely.

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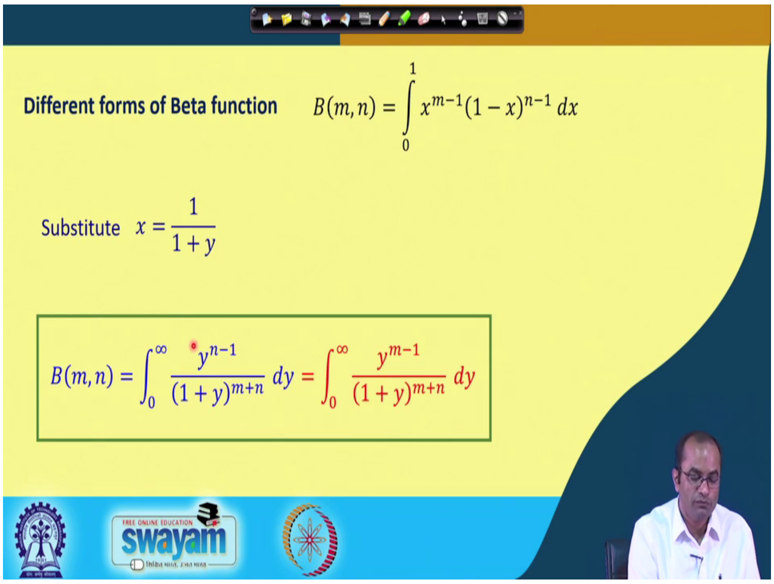

So, in the original integrate was 0 to 1, but this is also the same function, but having this 0 to infinity now, into the picture or we can because this is a symmetric function. So, m, n we can replace m by n and n by n the value will not differ you have y power here m minus 1 and 1 plus x power m plus n. So, this integral or this integral they are the same having the same value as beta m, n.

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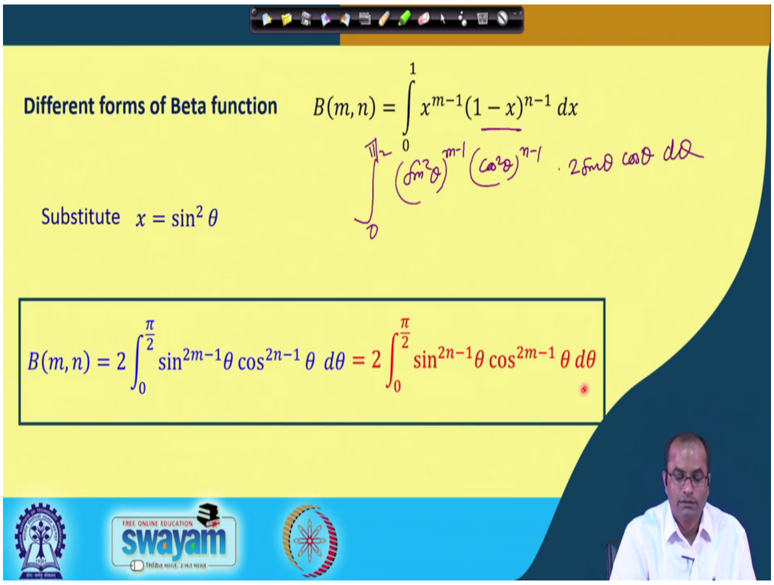

We can get another form when we substitute this x is equal to sin square theta. So, having this x is equal to sin squared theta so, again this is this will become sin squared 2 m minus 1 this will be 1 minus sin square theta. So, that will be cos square theta. So, here we will have this cos squared theta power n minus 1, here we will have sin square theta power m minus 1 and then the dx will become 2 sin theta and sin theta will become cos theta and d theta and then the limit.

So, limits when x is 0, so, that will be also 0 when 1, so, this will be pi by 2. So, we are getting here now this is 2 times 2 times and then sin power. So, 2m minus 1, so, 2 this was a square. So, 2m minus 2, but 1 sin is there. So, we have 2m minus 1 and then same thing for cos cos power 2n minus 1 d theta. So, this is the one another form of this beta m n in terms of the trigonometric functions. So, beta m, n is coming to be 2n 0 2 pi sin 2m minus 1 cos 2m minus 1 d theta or by just changing this all of this m and n we can also get this sin 2 n minus 1 theta and cos 2m minus 1 theta. And, now we can also look for some different forms of the gamma function which is e power minus x x power n minus 1.

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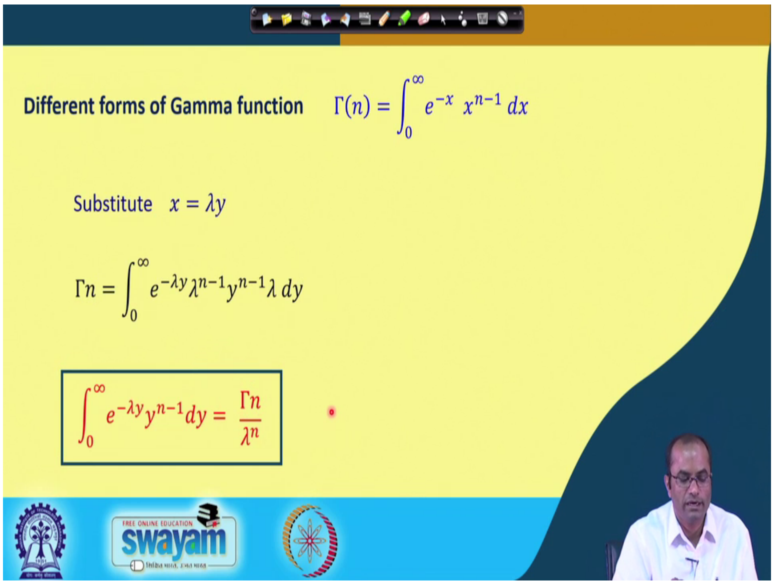

And, in this case if you substitute this x is equal to lambda y. So, x is equal to lambda y the limits will remain the same and we have a slightly different form. So, this lambda n minus 1 will come out from this term and basically we have this integral, 0 to infinity power minus lambda by y power n minus 1 dy is gamma n over lambda n.

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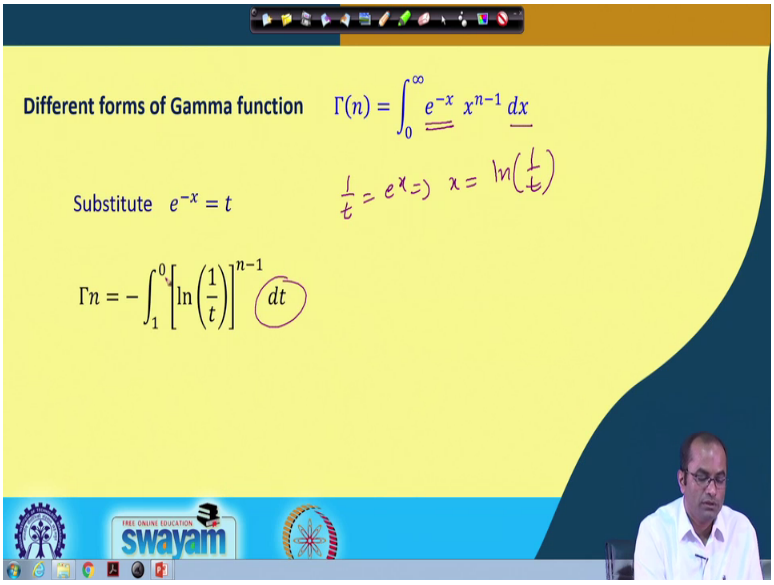

Also if we substitute this e power minus x is equal to t so, we will have again very difficult form because from here now we get 1 over t is e power x so, we get x as ln 1 over t e power minus x e replaced by t and then x will be replaced by ln 1 over t power n minus 1. So, this e power minus x with this dx will become this dt. So, there will be no extra term will be coming here and then we have gamma n as minus 1 to 0 which will be again replaced by 0 to 1.

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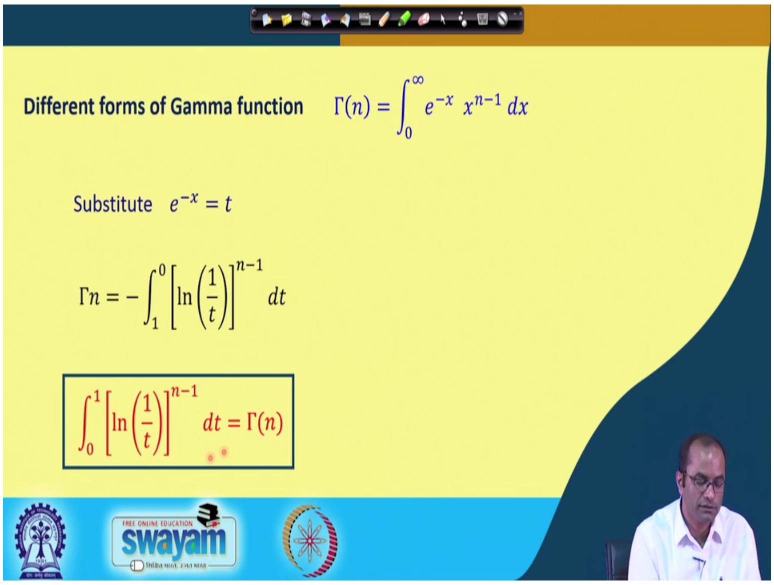

So, this is another form of this gamma function and in terms of this integral that such a integral can be just writ10 as a gamma n.

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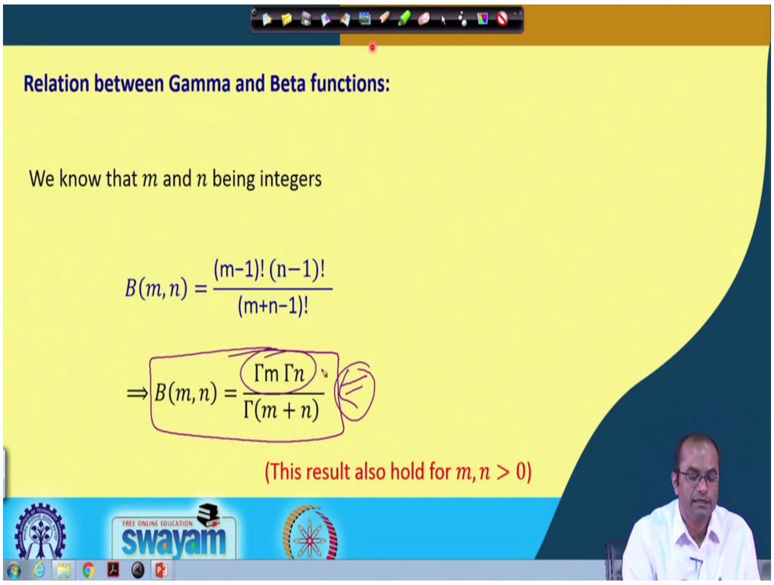

So, the relation between this gamma and beta function we know that when m n both are integers we have seen this relation that beta m, n is can be written as the factorial m minus 1 factorial n minus 1 and m plus n minus 1 factorial. In terms of the gamma function because we have seen that relation that this gamma m is nothing, but the factorial m minus 1 and gamma n is nothing, but the factorial n minus 1 gamma m plus n will be factorial m plus n minus 1.

And, what is interesting which we are not showing here that this result with which at the end here this result also holds when this m and n are not integer, so, they are just the positive number in that case also this results holds because this gamma is defined for non-integer values as well naturally. So, this beta m, n is defined as gamma m gamma n over gamma m plus n.

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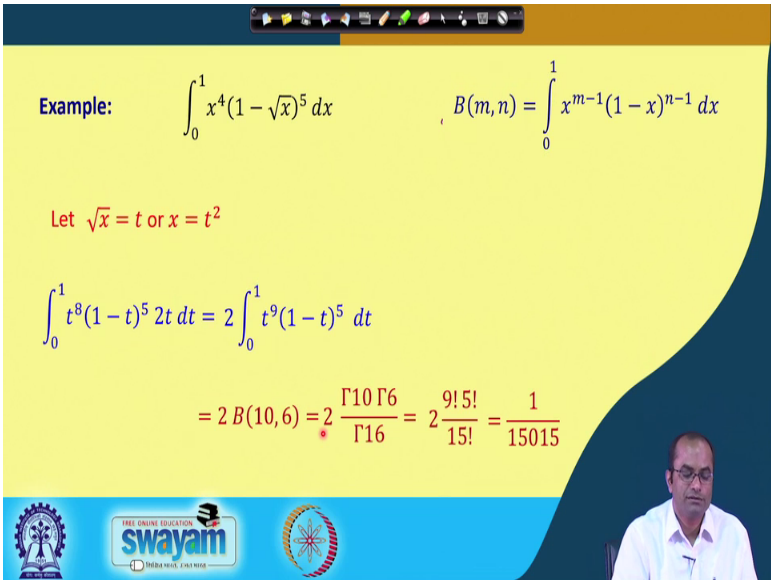

Just an example to so, if you have this integral which we want to evaluate 0 to 1 x power 4 1 minus x up of 5 dx we know the definition of the beta function. So, sometimes difficult to recognize, but in this case it is not so difficult we by some substitution we can easily convert into this form and then we can evaluate.

So, for instance in this case if we take this square root x as a t so, if we have taken here square root x as t. So, this will be 1 minus t which we want to have to see this beta form. So, that means, this x is equal t square we can write. So, this would become t power 8 and this is 1 minus t power 5 and dx will become 2t dt. So, this t power 9 and 1 minus t power 5 dt. So, this is exactly the integral we are talking about in this beta. So, we can write down this as the beta.

So, here we have m minus 1 that 9; that means, we have 10, so, here 6. So, we this is beta 10, 6. So, this is 2 times and beta function, the arguments 10 and the 6 which we can evaluate now. These are the integers here. So, gamma 10 gamma 6 and gamma 16 the sum of this by the formula just shown before and this gammas we can evaluate. So, this is factorial 9, this is factorial 5 and this is factorial 15 and again we can simplify this and this will be the number coming 1 over 15015. So, we can you evaluate several such integral with the help of this beta function.

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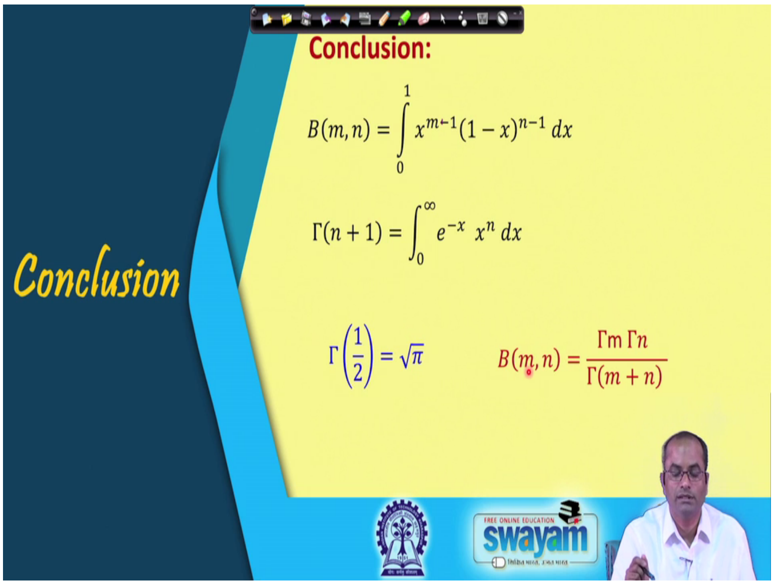

So, the conclusion is that we have seen these two definitions of the beta function and how to evaluate these functions and mainly the interesting was that the gamma half we have a computer that this square root pi and also this relation is very useful between beta and the gamma functions. So, beta m, n is gamma m gamma n over gamma m plus n.

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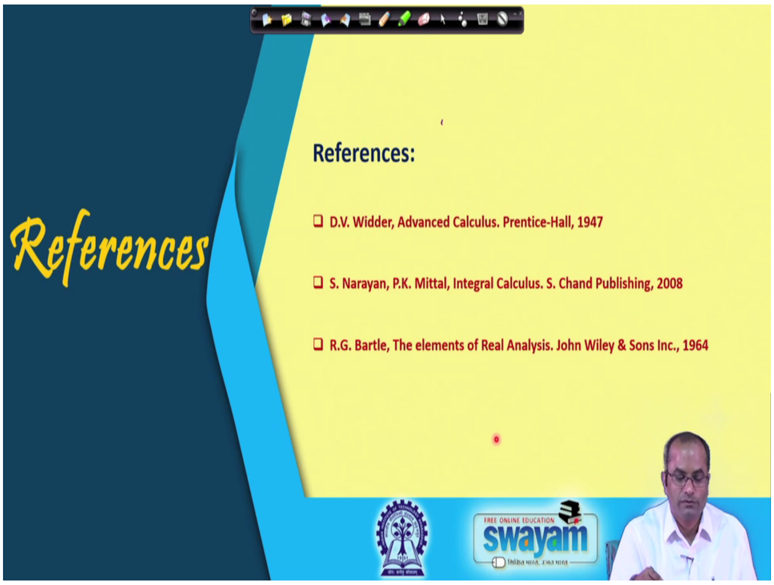

So, these are the references used and thank you very much.