Engineering Mathematics - I
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Lecture – 25
Beta & Gamma Functions

So, welcome back to the lectures on Engineering Mathematics-1, and this is lecture number 25. And today we will be talking about some special functions beta and gamma which fall into the class of these improper integrals. So, mainly we will be discussing again the convergence of such a special functions which are improper integrals.

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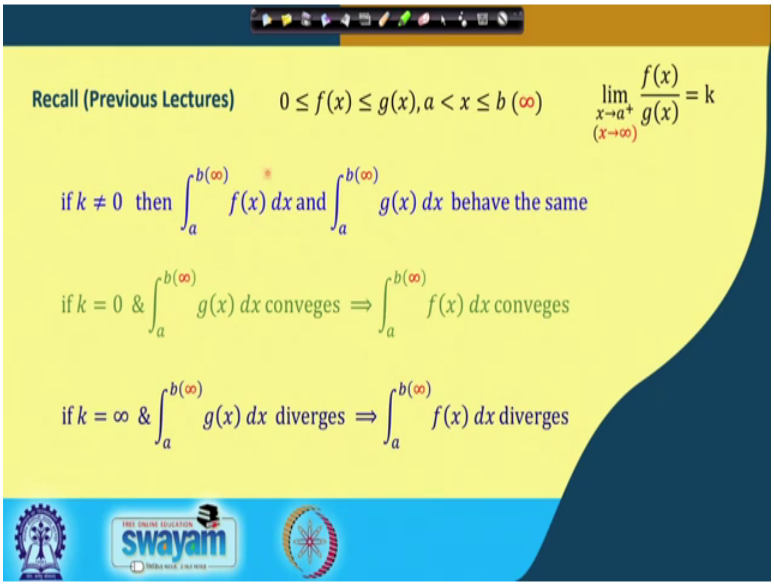

So, in the previous lecture we have seen very useful comparison test. So, for example here we have the function f and g which satisfy this inequality that f x is a non-negative. So, all the values of f at any point here x between a to b or a to infinity in the case of improper integral of type-1. So, we have this inequality that f x is taking in non-negative values. And also this g x is taking even larger values then f x for all the given range. And in that case, we compute this ratio of his f and g and based on the limit of this ratio you will conclude, whether the f converges for given the convergence of the integral of g or other way round.

So, here we are considering both the cases the integral of type-1 as well as integral of type-2. So, for integral of type-1 we have the problem because of the infinity where the integrand is bounded, in type-2 integral we have a unbounded function somewhere between this a and b. So, we considering both together here. So, in case of the improper integral of type-1 you will take this limit as x approaches to infinity, in case of integral of type 2 we will consider this limit as x approaches to a plus.

So, based on this limit here if this k comes to be non-zero number in that case the both the integrals, the integral a to b or a to infinity this f x d x; and the other one a to b or a to infinity now g x d x, they both will behave the same this was the conclusion. And there was another inclusion form there that if k comes to be 0, and this g x the integral over this g x in either case if this converges, because when k is coming to b 0 this g x growing faster as we are approaching to infinity what to a.

So, in that case if that integral converges the other one will also converge and we had also the result that if this k is infinity, in that case if the other one the g integral this diverges in that case we can also conclude that this f will also diverge. So, based on this these resources some on convergence we will prove the convergence of two special functions which are the beta and gamma functions. So, first I will define what are those functions.

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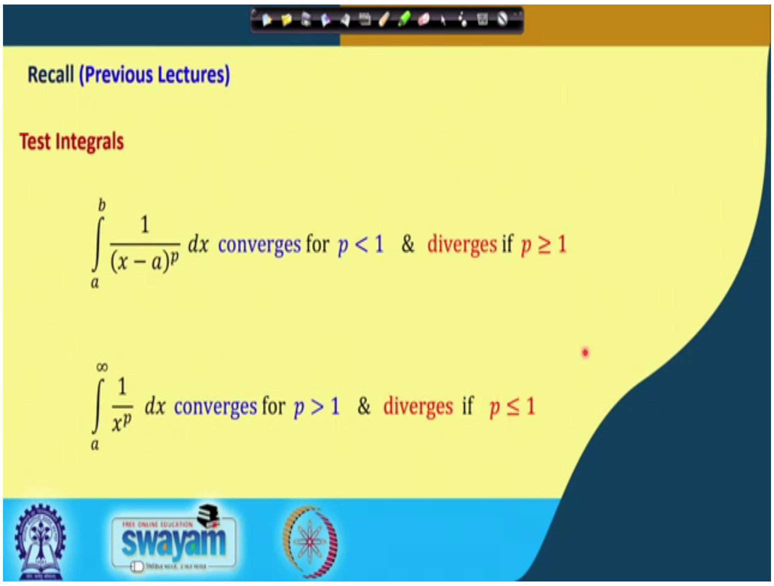

So, before that we also need these test integrals again from the previous lecture. And there we have seen these two types of integrals; this was used when we were talking about type-2 integrals. So, here the convergence this integral converges for p less than 1 and diverges when this p is greater than or equal to 1.

In this case, when we were discussing the integral of improper integrals of type-1; in that case this integral 1 over x power p converges when we have p greater than 1, and this integral diverges for p less than equal to 1. So, these two integrals will be used again for the comparison test of those special functions.

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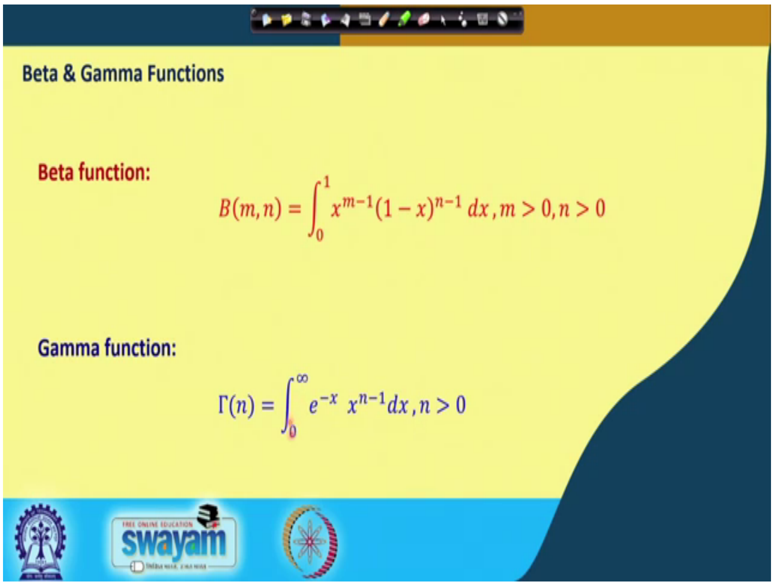

So, coming to these functions we have beta and gamma functions. And this beta function is defined by this integral the notation we used here B m n, and this B m n is defined this integral as 0 to 1, x power m minus 1 and 1 minus x power n minus 1 d x where this m n are positive.

So, we will see that this integral converges only for these values when m and n both are strictly positive. And there is another function it is called gamma function. So, gamma n is defined by this improper integral. So, 0 to infinity and e power minus x and x power n minus 1 d x. And we will also observed that this integral as well converges when n is strictly positive.

So, coming to this again this first integral here on B m n. So, this integral is integral of type-2 improper integral of type-2, because this integrand integrand may become unbounded when x approaches to 0 or x approaches to 1. In case this x power m minus 1, so the power here is negative, then this will become unbounded when x approaches to 0 or when x approaches to 1 in case this n minus 1 is negative. So, this will be improper integral of type-2.

In the second case, you have 0 to infinity. So, this infinity appears in the limit. So, this is a improper in this is an improper integral of type-1 as well as it can be in proper integral of type-2, because this x power n minus 1 and if this n minus 1 is negative in that case when x approaches to 0 the integrand will become unbounded. So, this is a mixed kind integral or the type 3 integrals as per our notations used in previous lectures. So, we will be discussing today the convergence of these integrals based on the knowledge we have received from earlier lectures.

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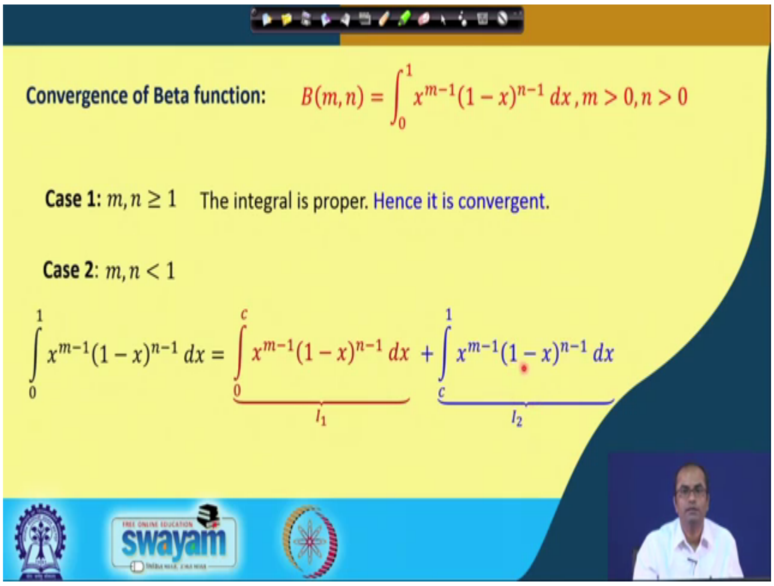

So, here the convergence of the beta function and we will consider the first case when m and n both are greater than or equal to 1. So, in this case what is happening here the m and n? So, this power here for x is positive and also for 1 minus x the power here, because n is also greater than equal to 1 that power is also positive.

So, in both the cases this when this m and n are strictly greater than 1, we see that the integrand is not unbounded function. So, this integral is indeed a proper integral and we do not have to discuss about the convergence of improper integrals. So, for this case this is special case and this integral is proper and hence it is convergent.

So, the interesting case here is when m and n both are less than 1. So, in this case when the both m and n are less than 1. So, here this is m minus 1 and here n minus 1; so these powers will become negative, and in that case we have the improper integral of type-2, because the integrand is becoming unbounded. So, we will consider now this one. So, we will break into two, because we have the problem at both the ends when x approaching to 0 as well as when x is approaching to 1 because of this term.

So, we will break this integral into two parts, 1 0 to sum number c, c is between 0 and 1 and this integral and plus we will take the c to 1 and c to 1 again the same integral. So, let us call this I 1 and I 2, because you will discuss now separately the convergence of these two integrals.

And in this first case because of this x power m minus 1 term, this integral is becoming improper. And in the second case because of this 1 minus x as x approaches to 1, this integral is becoming improper.

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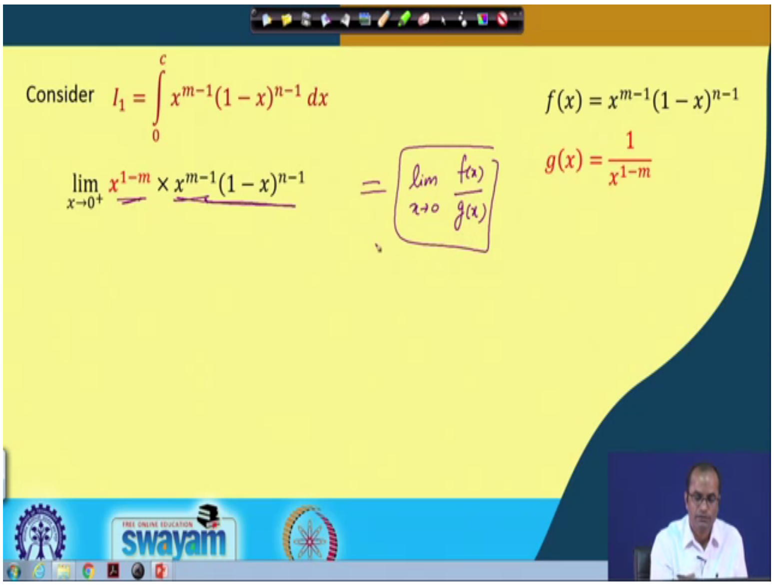

So, let us discuss one by one. So, in the first case we will take this I 1 as 0 to 1 x power m minus 1 1 minus x power n minus 1 d x. And we take this f x, the integrand of this improper integral as x power m minus 1 1 minus x power n minus 1. And now for the comparison we have to take another function.

So, we consider here g x, because the problem is clear we have in this integrand as x approaching to 0. So, the problem here is when x approaching to 0. This is not creating any unboundedness as x approaching to 0. So, this is the function here the part of the function, which is actually making this integrand unbounded. So, the behavior of this function as x approaches to 0 will be taken from this x power m minus 1.

And that is the reason we have chosen here this g x as this 1 over x power 1 minus m or the same function here x power m minus, so this is nothing but x power m minus 1. So, we have taken this g, and now you will take the ratio, so that this which is making it unbounded will cancel out, and you will get some finite limit in that case.

So, if we take that limit f x is equal to so we are taking now the limit as x approaching to 0, and this f x over g x, so taking this limit here. This is equal to this x power 1 minus m and x power m minus 1, this is the f x function here. This is our 1 over this g x, so it will be x power 1 minus m will be multiplied there.

And when taking the limits, so here this is cancel out and when taking the limit x approaching to 0, so we will get this as 1. So, this limit here the ratio of this f x and g x is getting 1. And then from the comparison test, we have just reviewed in previous slides, we know that the behavior of this integral this improper integral will be the same as the behavior of the integral over this g x function.

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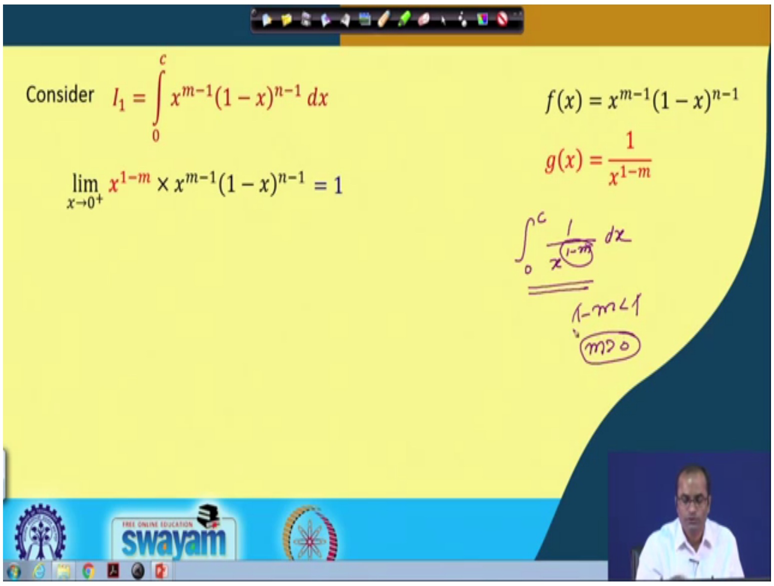

So, what is our integral of g x function, this is a 0 to c and 1 over x power 1 minus m and d x, this is the test integral we have discussed. And this integral converges, when this 1 minus m is less than 1, so that means the m is greater than 0. So, this is this integral converges, and m greater than 0.

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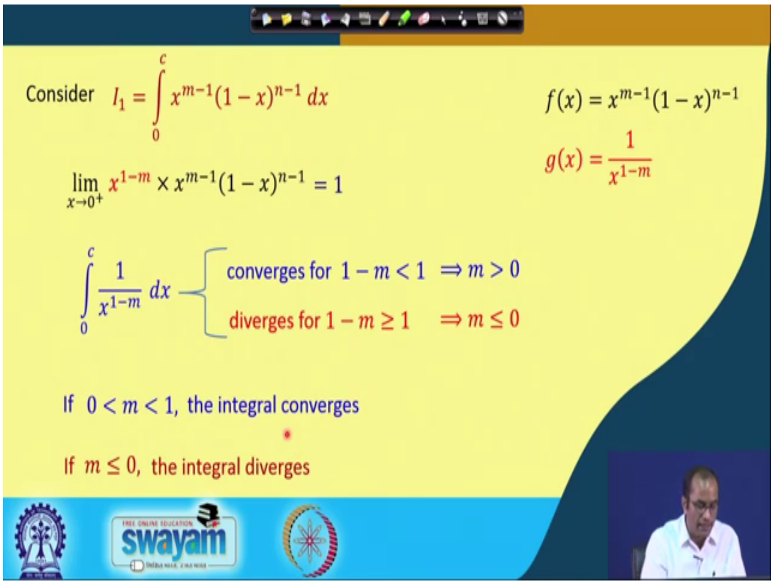

And here we will conclude that again the test integral converges, when this 1 minus m this power here less than 1 meaning this m greater than 0, and also we know about the divergence as well. So, this integral diverges, when this 1 minus m greater than equal to 1 meaning this m is less than equal to 1. So, we know the convergence and the divergence behavior of this test integral here. So, this was the test integral. And behavior of this test integral, and the given integral is the same, because this ratio is coming to be constant.

So, as per the comparison test our integral I 1 will also converge, when m is greater than 0; and it will diverge, when m is less than or equal to 0. So, what is the conclusion now here about the integral I 1. So, when we have this m greater than 0, and we are considering a case when m is less than 1. So, if this m lies between 0 and 1, the integral test integral convergence; and when m is less than equal to 0, this integral diverges.

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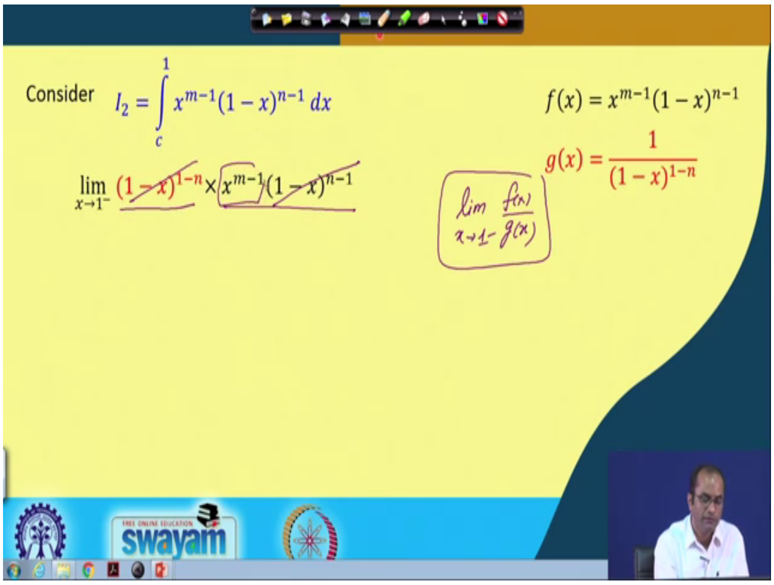

Now, coming to the second integral, because we have break the original integral into two parts. So, this was the second integral c to 1 and x power m minus 1 1 minus x power n minus 1 dx. And in this case again, we take this integrand as this f x function.

And now in this case remember; now the behavior is governed by this 1 minus x as x approaches to 1, because this n minus 1 is negative. And this part of this function is making this integral divergent this unbounded. So, this is 1 minus x power n minus 1, when x is approaching to 1. So, this is the term here in this function, so accordingly we will choose now again our g x.

So, our g x will be the same here 1 minus x power n minus 1 or 1 over 1 minus x power n minus 1. So, having this g, now we will compute again this limit as before, so will compute this limit x x approaches to 1, and then this f x over g x. So, this limit here f x over g x is precisely this one, so 1 minus x power n minus 1 will go to the numerator, and this is the f x here. So, again this term will cancel out, and this we have x power m minus 1, so as x approaches to 1. So, again we are getting the similar result, which was earlier one.

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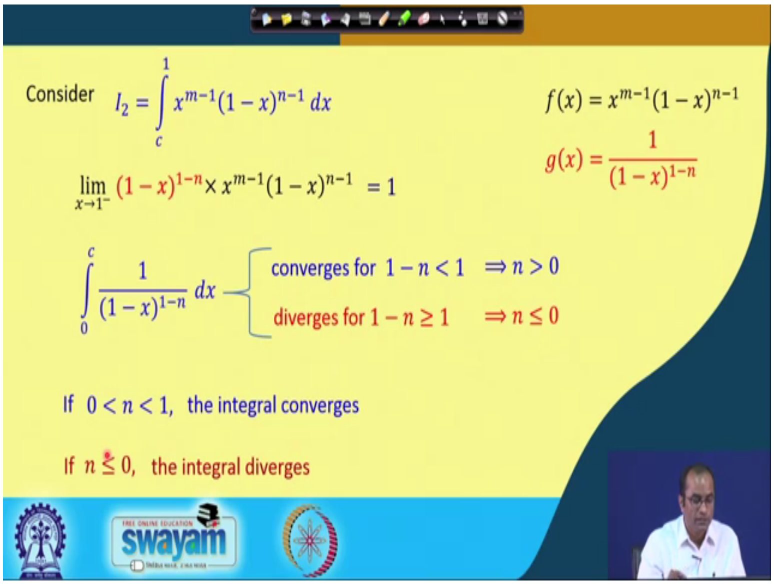

So, this limit is again 1, and now the behavior of this integral as well. We can conclude from the behavior of this improper integral this test integral. So, this was our test integral here 1 over 1 minus x power 1 minus n dx. And this converges, when 1 minus n this power is less than 1 that means, n is greater than 1 greater than 0.

And this diverges, when 1 minus n is greater than 1 here that means, the n is smaller less than equal to 0. So, this part of the integral also have this behavior that when n is between 0 and 1, the integral converges; and when n is less than equal to 1, this integral diverges. We had the similar results for the other one, but that was the condition was on m.

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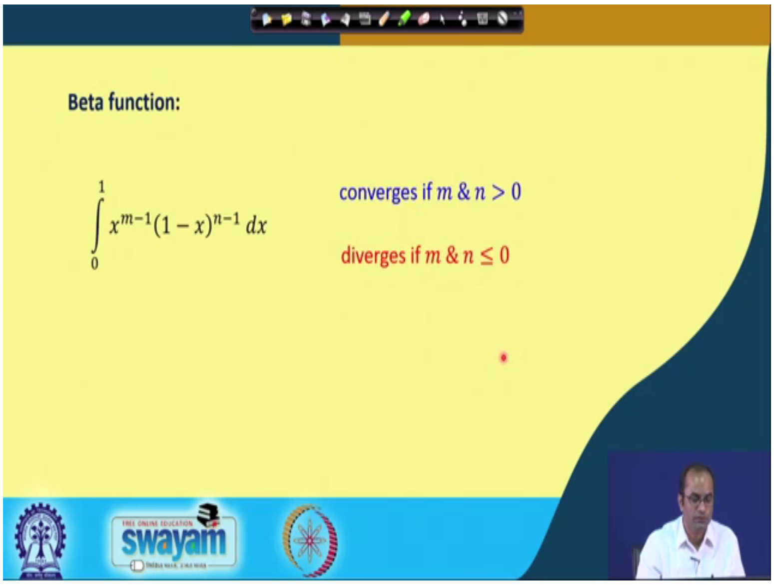

So, combining these two what we get that our beta function, which was 0 to 1 x power m minus 1 and 1 minus x power n minus 1. And this was taken into two parts, and we have discussed each separately for the convergence. So, this integral will converge for m and n, they are positive greater than 0. And any value they can take, because we have already discussed the convergence.

And for the divergence, if m and n are both are less than equal to 0 in that case this integral diverges. So, whenever we will be taking these beta this beta function, we will assume this m and n greater than 0, because the integral diverges when these 2 m and n are less than equal to 0.

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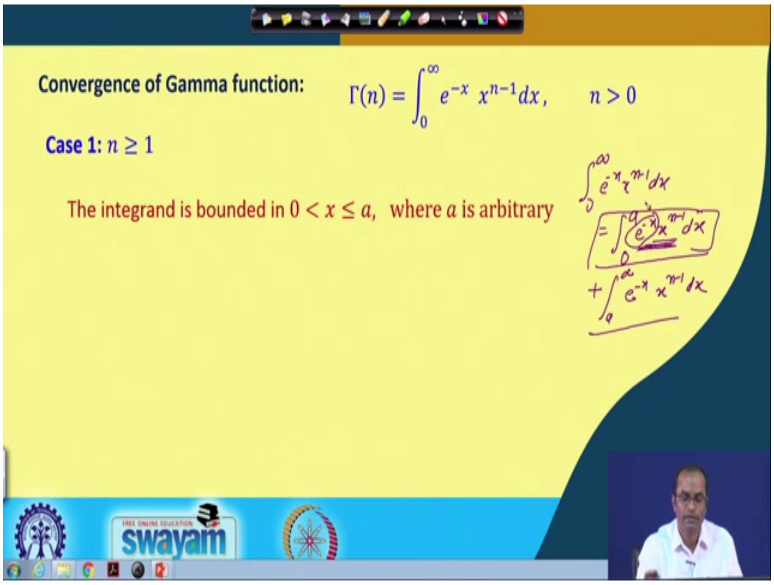

Now, we will discuss the convergence of gamma function, which is 0 to infinity now. So, this is the mixed type of integral e power minus x and x power n minus 1 dx, and here we will conclude that this exist when n is strictly greater than 0.

So, we consider the case that n is greater than equal to 1. And in this case, the integrand is bounded. So, we will take into two parts again this integral one is 0 to a, and then the other one is a to infinity. So, when we take 0 to a, the first integral so meaning this 0 to infinity here e power minus x and x power n minus 1 dx. We will again take like 0 to a and e power minus x, x power n minus 1 dx, and this plus here a to infinity e minus x and x n minus 1 and dx.

So, this part here, when n is greater than equal to 0. So, n is greater than equal to 0 means this is there is no singularity, there is no unboundedness coming here of due to this part. And this is nice here e power minus x, so that means, this integrand is bounded now in 0 to a whatever a we take choose here.

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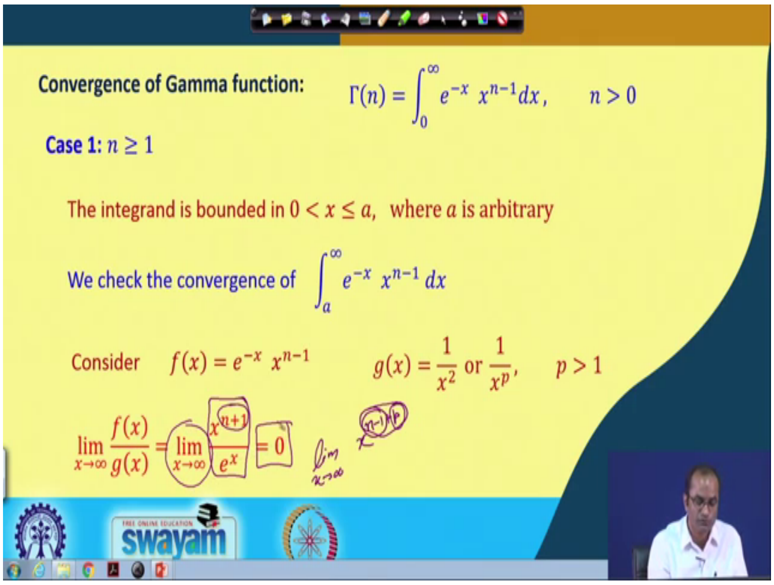

And the other parts, so we will check the convergence of the second integral, which is a to infinity e power minus x x power n minus 1, because here we have now this infinity problem even in this case, when n is greater than 1. So, we are considering the case n greater than 1 and because of this infinity now, we have this improper integral of type-1, and we will go through the convergence.

So, here this is have a integrand f x e power minus x x power n minus 1. And based on this behavior of this function, which is as x approaches to infinity because of this x here, we will consider this g x or let us considered this g x 1 over x square or anything 1 over x power p, where p is greater than 0. Anything we can take either 1 over x square or 1 over x power p, when p is greater than 1 what is the reason taking this.

So, if we take this limit here f x over g x as x approaches to infinity, so what we will get here, we have the x power suppose we have taken this x square. So, this x square will be multiplied here with x power n minus 1, so we have x power n plus 1, and we have e power x. The reason whether we take here x power p, so in that case this will be n minus 1 plus p does not matter, and n is greater than equal to 1. So, here we have in that case also some x power positive there.

So, now this limit we know already the behavior of these exponential function is this is a is a fast growing function, then x x power whatever positive power we have there. So, this limit here, because this e power x will go will grow faster to x to infinity than this one, so this limit is going to be 0, this limit is going to be 0 whatever positive power we have.

So, for example if you take x power p, so this limit will become the limit x power x infinity and x power n minus 1, so n minus 1 and then plus p will come. So, this is again, so n is greater than 1 so n is greater than 1, so we have here or greater than equal to 1. So, this is greater than equal to 0. And here we have the p greater than 1.

So, in this case also we have this power greater than 1, so this power x here is greater than 1 in any case, and the same scenario will happen that when we take the limit x approaches to infinity, the limit will be 0. And now we have the comparison test, when this x then the limit is going to 0 that means this g x is growing faster than this f x function, and this is our g x here 1 over x square.

So, if this g x integral g x converges, then we can conclude that the f will also converge, because from this ratio if we are getting this 0 and that was one case in the comparison test. If this g is the meaning here is that g is growing faster or getting unbounded faster here or to when x approaching to infinity, now going growing faster than the f function. And if the integral of this g converges, then naturally the integral of this f will also converge.

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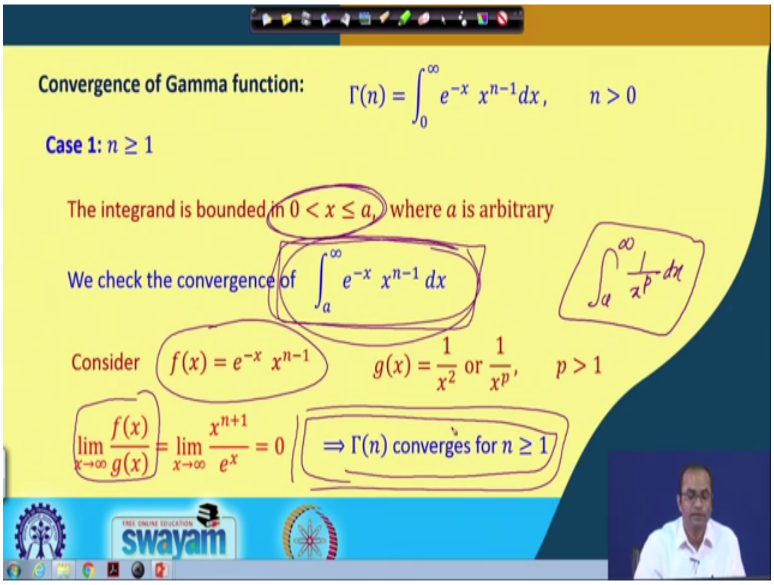

So, here the gamma n converges for n greater than equal to 1, because this integral the g the integral over the g the integral a to infinity of this x power p or x square this converges, when p is greater than 1. So, we have seen by taking this g that this ratio here is 0.

And then if this g converges, then the integral over the f will also converge that means, the integral or the gamma n will converge. So, here the gamma n converges for n greater than equal to 1, because this gamma n we have the two integral. One was 0 to a which converges for any arbitrary a, and the second one we have just tested that this integral will also converge, whenever we have this n greater than equal to 1. So, you need a case we got this convergence for n greater than equal to case.

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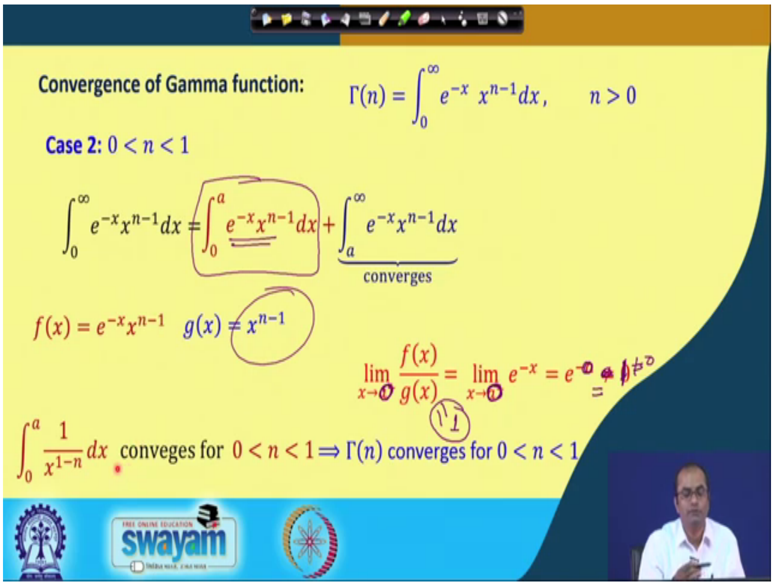

Now, we will consider the second case, which is 0 to 1, when n is lying between 0 and 1. So, in this case we have this integral. So, again we have taken 0 to a, and a to infinity. So, based on the earlier study again this converges the same reason, which we have use the same argument which we have used here, we will choose this function and again here when n is between 0 and 1, the same limit you will get 0. And since this integral g converges, the other one will also converge.

So, in that case with the second integral converges. And now we will test for the second integer, because when n is between 0 and 1, we have this x power negative, so because of this when x approaches into 0, we are getting the unbounded function meaning this is a type-2 improper integral.

So, consider the convergence of this we take this f x the integrand again the same procedure, and the g because this function is getting unbounded because of this x power n minus 1. So, we will take this g exactly x power n minus 1 and if we take this limit as x approaching to a, when x approaching to 0.

In fact, not to a because the problem here is when is approaching to 0. So, e power 0, so when x approaching to 0. So, we will take x approaching to 0, because the integrand is getting unbounded, when x approaching to 0. So, when we take x approaching to 0, so e power 0 and this will become 1, which is not equal to 0. So, this is this is 1 here. So, this limit is coming to be 1, and which is not equal to 0. So, the behavior of this function here will be the same as the behaviour of this function.

So, now we know already that this function here 1 over 1 minus n. So, the integral over this g 0 to 1 and this converges whenever this n is between 0 and 1, this integral converges we know this test integral. And based on this we can conclude that our gamma integral will also converge, because they will behave the same this integral. If this integral of converges, the other one will also converge; if this diverges, other one will also diverge because of this limit is one here. So, in this case this integral converges again, when this is n is between 0 and 1.

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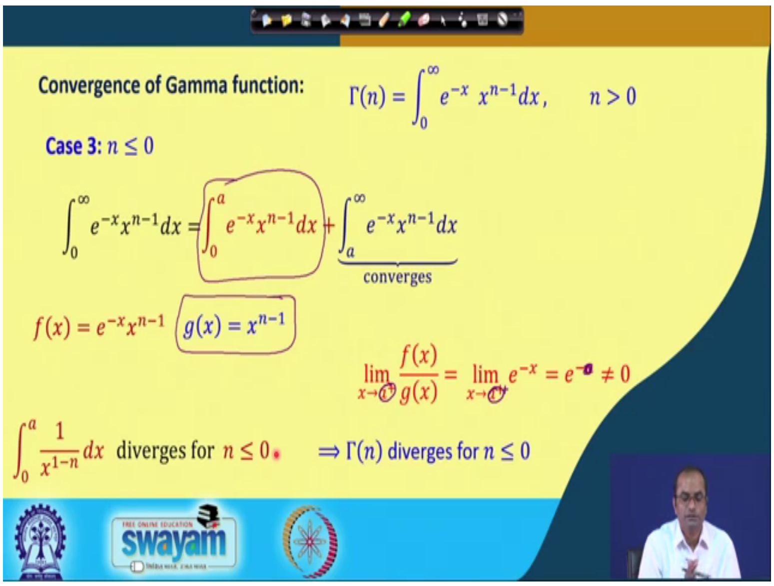

So, now the 3rd case, we will discuss when n is less than equal to 1. So, so n is less than equal to 0, so that is the case left case-3. So, in this case, we again take the 0 to infinity e power minus x x power n minus 1 dx. And we have the 0 to a, and this a to infinity and we will separately discuss the convergence.

So, this integral converges which we have already seen before, when we have a to infinity, because we take this now for this one f x e power minus x x power n, we take this function f x. And g we take this x power n minus 1, so when n is negative. So, this is getting unbounded. And again we have to take this limit as x approaching to 0, so when we take this limit x approaches to x approaches to 0.

So, in this case again let me point out here, this is x approaching to 0 from the right side. And in this case also this e power 0 will be 1 which is not equal to 0. So, we have again the same situation that if this integral if this integral over g, and this integral here, they will behave exactly the same. And that can be seen here, so we have 1 over this 1 minus n. And this integral diverges, when n is less than equal to 0. Because, when n is less than equal to 0, we have this integral 1 plus something. So, this integral will diverge, because this integral converges when 1 minus n is less than 1.

But, in this case this power here, when n is negative this will be a greater than equal to 1. And in that case this integral will diverge, and therefore the integral 0 to a will also diverge. So, we have this gamma n that it diverges, when n is less than equal to 0.

So, we have considered all the possibilities for n, and less than equal to 0 and between 0 and 1 and greater than equal to 1. So, in those cases other than this one that integral converges, but when n is less than equal to 0, this integral diverges because of this reason, which we have explained here. And now that is the conclusion now, so both the integrals this gamma n the beta, they have the convergence issues and based on this power here. So, in case of this gamma n whenever n is positive, this converges. And the other one also for m n positive, it converges.

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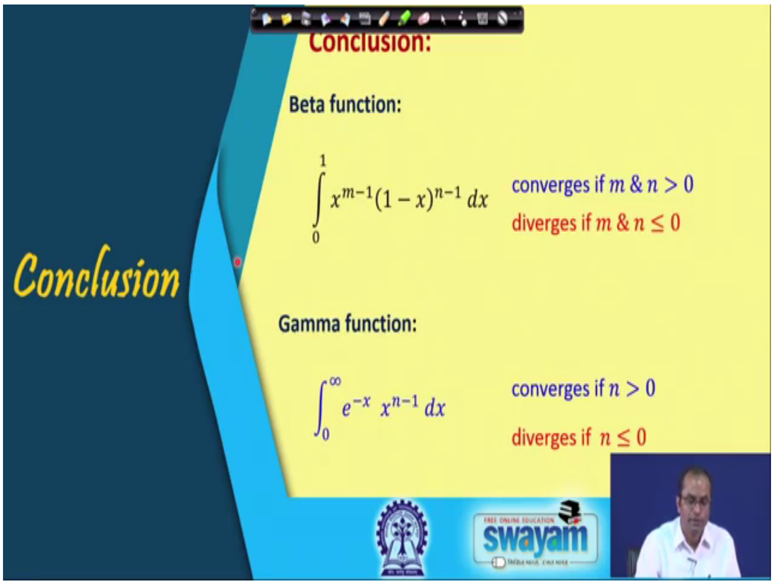

So, coming to the conclusion we have this beta function, which converges if m and n so both are strictly positive and it diverges when m and n are less than equal to 0. And this gamma function, so we have this 0 to infinity e power minus x x power n minus 1. And we have the same situation here that it converges, when n is greater than 0 or diverges when n is less than equal to 0. So, in the next lecture, we will also see some nice properties of this beta and gamma function also the evaluation of these such special functions.

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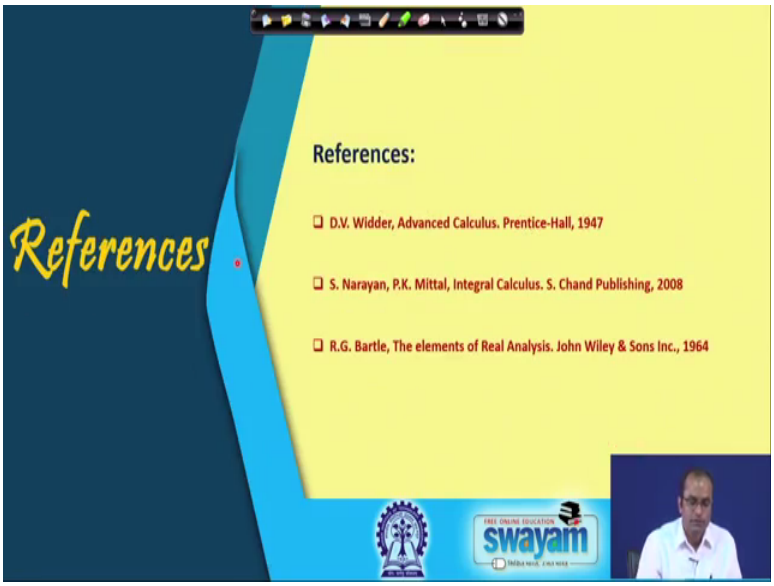

So, these are the references we have used to prepare these lectures.

And thank you very much.