Engineering Mathematics - I
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Lecture - 23
Improper Integrals (Contd.)

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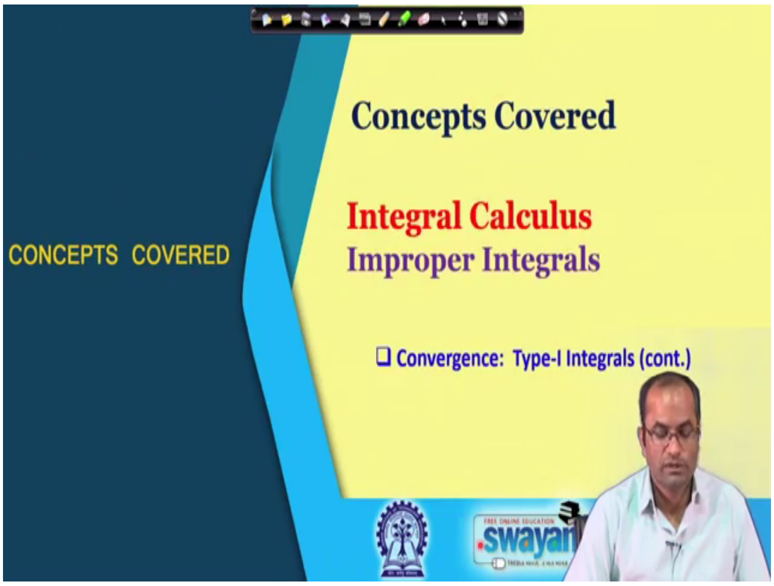

So, welcome to the lectures on Engineering Mathematics-1, and this is lecture number-23. We are talking about the Integral Calculus in particular Improper Integrals. So, we will continue on the discussion on the convergence of type-1 integrals. Now, we have one more test, which was not discussed in the previous lecture that is called the Dirichlet's test. And this is again a very important test for testing the convergence of improper integrals of type-1.

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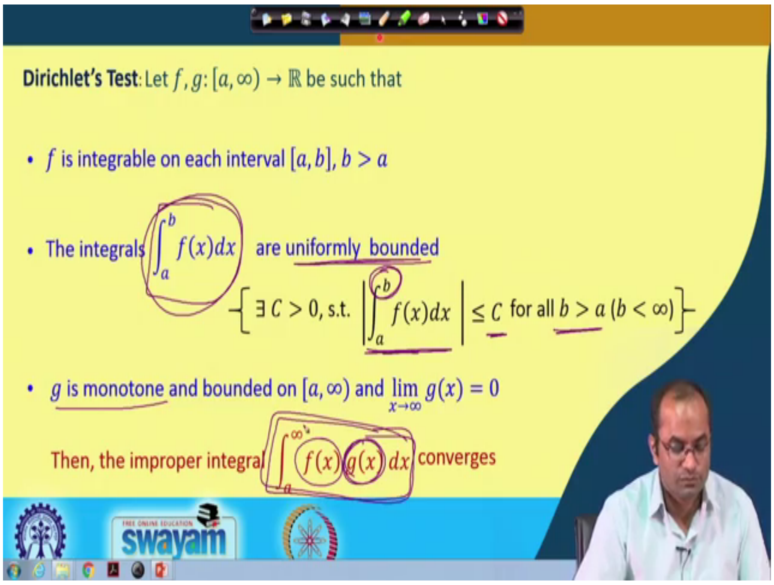

So, here we suppose that this f and g, these are the two functions defined from this a to infinity, and they are taking the real value. And they are such that, that f is integrable on each interval, so a, b and b can take any value greater than a. So, this f is integrable on each integral a to b.

And the other one the integral a to b this f x, this is the second condition, these integrals are uniformly bounded. So, integrals means that this b can take any value, so we are not restricting on b. So, these integrals here are uniformly bounded. So, what does that mean that there exists number C here, a constant C greater than 0 such that the value of this integral, the absolute value of this integral a to b f x d x is less than equal to that constant and for all b greater than a.

Naturally, b is some finite number we are talking about. So, here for any value b here, which is greater than a, if this integral is bounded by a constant C, it is this C is not depending on the number b here. So, in that case we call that this is uniformly bounded. So, this bound is independent of this b. Then we call that this integral is uniformly, these integrals are uniformly bounded. So, these are the two conditions on f, f is integrable. And this is uniform, these are these integrals are uniformly bounded.

And the function g is monotone and bounded. So, again the g is either monotonically increasing or monotonically decreasing, and the bounded so the values of these functions are also finite the bounded And then we have this limit as x approaches to infinity this g x is 0. So, this is a g is a monotonic function which is approaching to 0 as x approaching to infinity, so that is another condition.

And in that case this improper integral a to infinity f x g x dx. So, we have now the product here in the integrant f x g x, and this converges that is the result. So, we will not go through the proof of this result. So, what is important here the important is to check that these integrals here, they are uniformly bounded. So, meaning we have to compute this integral, and see whether we can bound that integral by some constant C, which is independent of this number b here for any finite number b if we can do that, so we have this condition uniformly boundedness.

And the second one then g is monotone, and so monotonically it is going to 0 as x approaching to infinity. So, in that case so these are the two main conditions here. And that those conditions we have that this integral the product here converges. So, this is a very useful integral, because if we know that one of them, so for example g is here this bounded and monotone going to 0 here. And then these integral over f converges, so then we can conclude about this product as well that what will happen to to this integral. And it will converge, if these conditions are satisfied.

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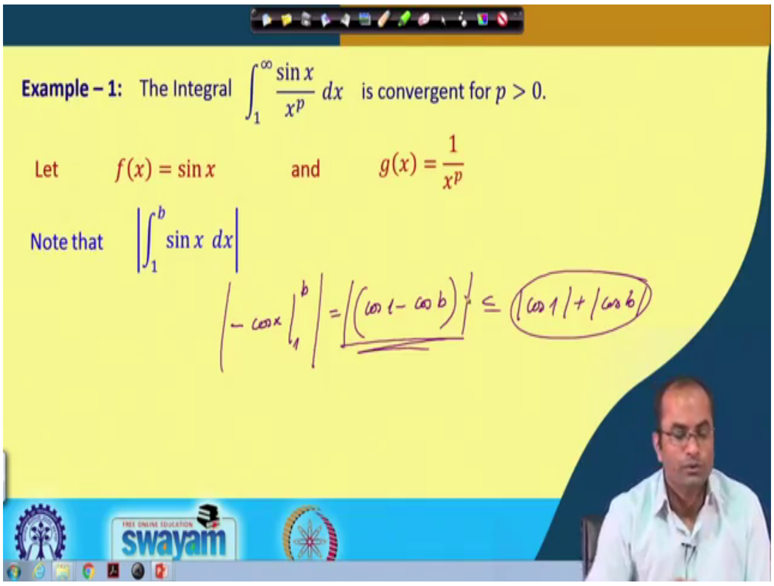

So, here let us go through this example, where we can use this result easily to prove such integrals 1 to infinity sine x over x power p is convergent for p greater than equal to 0.

So, for any p here greater than 0. This integral 1 to infinity sin[e] x over x power p is convergent, we will show with the help of this is Dirichlet test.

So, in this case we let that f x is equal to sine x, so we have this function here f x as sin[e] x. And the other one g x, we will take 1 over x power p. So, how this is useful now, because we know about this g that this is monotonically decreasing to 0 as x approaches to infinity for this p positive. And this integral the sine x, which we can easily show that 1 to b this sine x dx.

So, this integral will be the minus this cos x, and then we have this limit a to b, so which will b this cos 1 minus the cos b this integral value, and then the absolute value of of of these we have to consider. So, this cos is bounded by 1 always, and we can use this inequality here that this is less than cos 1 plus cos b, and in that case this will b bounded by 2.

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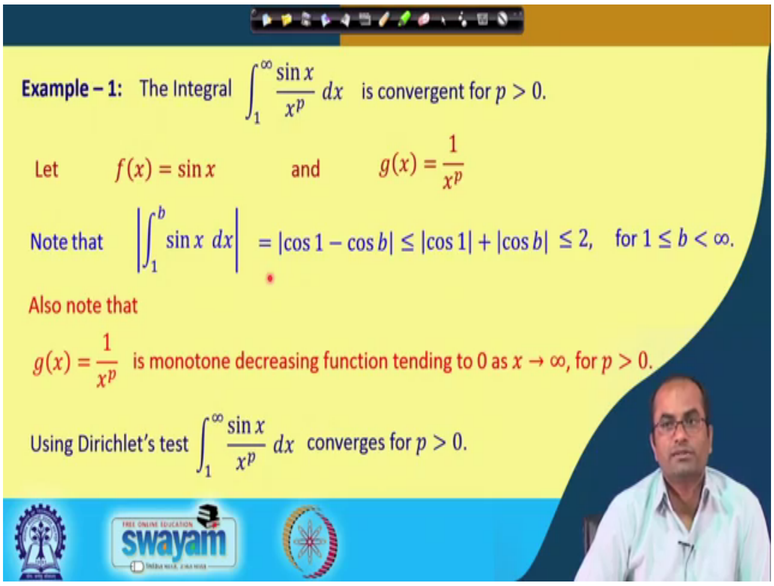

So, we can easily show that this integral here is bounded by by 2 for any value of b we take, which is greater than 1 less than infinity. So, what we have seen that there is a uniform bound here. So, whatever values of b we take, the value of this integral will be bounded by 2. And this is what we we want for uniform boundedness of this integral now.

The other function g, which is monotonically decreasing function and naturally this tends to 0 as x tends infinity for any value of p positive. So, using that Dirichlet test now, so we can talk about this product the integral of this product sin x and product with 1 over x power p that means, this integral sin x over x power p dx from 1 to infinity this converges for any value of p greater than 0.

So, this Dirichlet test is very useful now that we have just consider as a ratio of the two function again one was f x, and another was g x. And this function f x has this nice property above this uniform integrability. And the second one was monotonically decreasing to 0, therefore we could conclude with the Dirichlet test that this converges.

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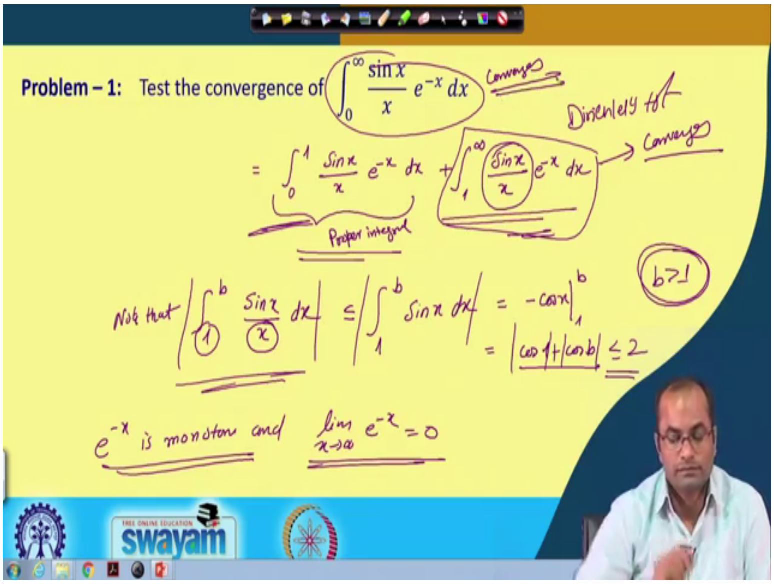

Another of similar kind we can test the convergence of this 0 to infinity sin x over x e power minus x this integral here.

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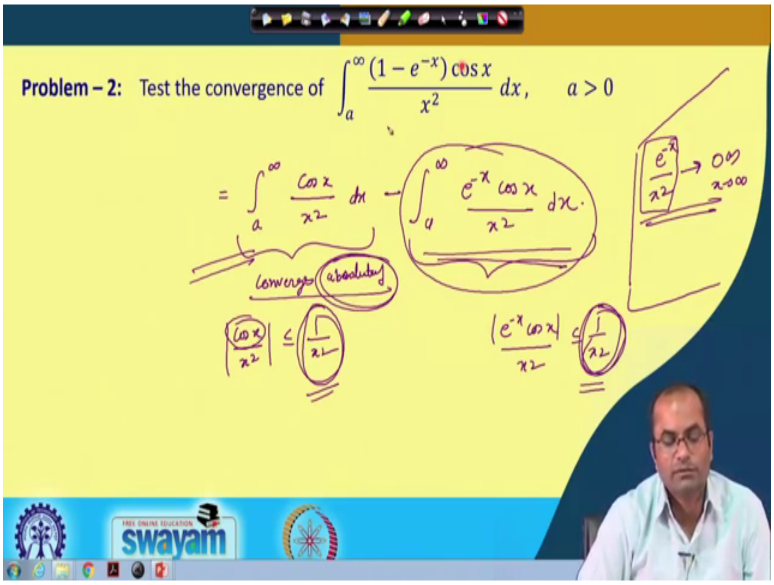

So, for any value of this a positive. So, what we will do we can prove this again easily, so this is e power minus x cos x wait so this was the first problem e power so sin x over x e power minus x. So, we have already done before that was sin x over x power p, when 1 to infinity. And now we will do with sin x over multiplied by e power minus x.

So, in this case we will again this use the idea that we can break this easily 0 to 1, and then we have sin x over x e power minus x dx plus this 1 to infinity and we have sin x over this x and e power minus x dx this function. So, the first integral is a proper integral can proper integral the reason is clear, because the sin x over x has the value 1 as x approaches to infinity and this is one. So, the limit here of the integrant exist, when x approaches to 0 and at all other point there is no singularity is there is no unboundedness. So, this is a proper integral the first one. The second one now because of this x infinity, we have to test that what type of result we get from here.

So, now we take this integral here a to b, so note that the integral one and then above one we can take this b is sin x over x dx. So, if we take this integral here sin x over x dx, this x is going from 1 to any number this b. So, here the x we can replace, because this is the minimum value if you replace this one, so that the integral will be the larger one.

So, we have 1 to b and this x is replaced by 1 the lowest possible value, so that the integrant is the larger one. So, we have this integral bigger than this one as sin x dx. And this we know now that we can integrate this with minus cos x and then 1 to b, so this value will be cos 1 and minus b cos b, and we can take that absolute value there. So, this absolute value of these functions will be again the absolute value here, which is bounded by 2.

So, we can prove that that this integral here sin x over x the sin x over x 1 to b for any value of b greater than 1 any finite value. This is uniformly bounded, and this value is 2. So, this is uniformly this integral is uniformly bounded by 2. And this exponential minus x function, so exponential minus x function is is monotone, and it approaches to to 0 if we take the limit as x approaches to infinity this e power minus x that approaches to 0. So, we have that property also satisfy.

So, then we can use Dirichlet test there, because the one integral with the sin x over x we have seen that there is monotone that is uniformly bounded, and the other one here the g x, which is e power minus x, it is going to 0 as x approaches to infinity. So, in this case we can apply now Dirichlet results Dirichlet test, which says that this integral converges. So, this integral converges and the other one is proper integral. So, we have the original integral here 0 to infinity sin x over x that converges well.

So, we do one more problem here that is problem number-2. So, here we will it is a similar kind of problem, we have 1 minus e power minus x cos x over x square in this case. And the idea again we can break into two parts, so I will take here a to infinity, and then this we have 1 over x square cos x as 1 integral. So, we have cos x over x square dx, and the other one is a to infinity, and we have e power minus x, and then cos x over x square dx.

So, this first integral this is converges absolutely this converges converges absolutely. And the reason is clear, because by the comparison test we can easily conclude, because this cos x over over x square this is less than this 1 over x square indeed this absolutely a concept we will explain again.

So, here the absolutely means that the if we take even the absolute value of the integrant this is this converges, so this cos x over x square is bound is less than equal to 1 over x square, because this cos x the values are bounded by 1. So, we have 1 over x square and we know, this integral as a to infinity 1 over x square that converges. And in that case this integral will also converge, because this is dominating function 1 over x square and the whose integral converges. So, naturally that integral will converge.

The second one here we can again use the same argument, because again this e power minus x cos x over this x square. So, this value here will be bounded by by 1, because e power this is minus x is also decreasing function. So, it will take the highest value, and we have then x will approach to this a. So, in any case this will b will also bounded by 1, so this second integrand is also bounded by 1 over x square. And by the comparison test, we can again conclude that this also converges absolutely.

We can also use here the Dirichlet test, because we can take this e power minus x. So, if you want to use this Dirichlet test here, we can take this function e power minus x over x square. And this function approaches to to 0 as x approaches to infinity as x approaches to infinity is this is monotone function e power minus x is decreasing, and 1 over x square is also decreasing. So, both the functions, so here we have the decreasing function for whatever values of x. And this cos x the integral of the a to infinity cos x, we can again show that this is bounded by 2.

So, we can use that Dirichlet test like we have done in the earlier problems, so this is similar to the problem number-1. So, we can here also show that this converges, so this also converges. And the given integral the 1 minus e power minus x cos x over x square dx converges.

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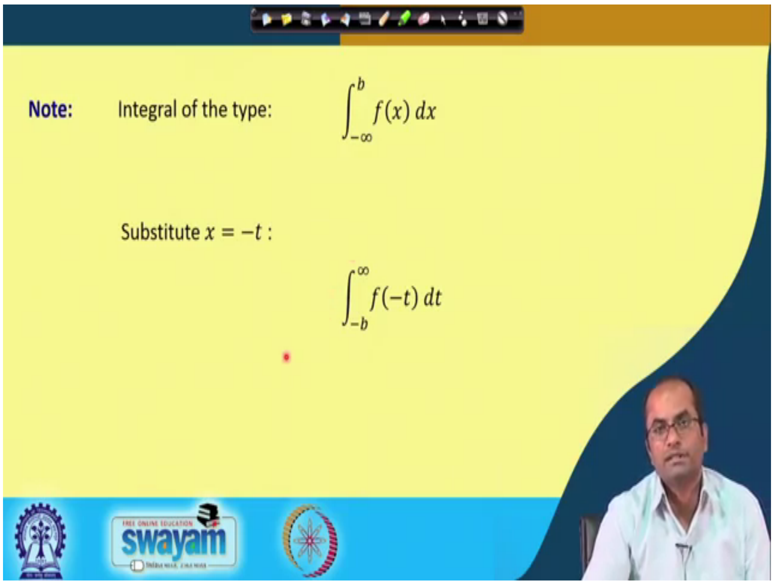

So, now just in note here that this integral of this type minus infinity to b f x dx, we are not you normally discussing here, we are just discussing the integrals where we have infinity in place of this b. So, this a to infinity type of integrals we are discussing, but this also we can easily b, because we can substitute for example here x is equal to minus t.

And then what will happen that this integral will be just minus b to infinity. So, again we have a similar type integral, which we are discussing for the convergence where the infinity appears above. So, if we have this type of integrals, we can just by simple substitution, we can converge into this type of integral, and then discuss the convergence the way we have discussed earlier.

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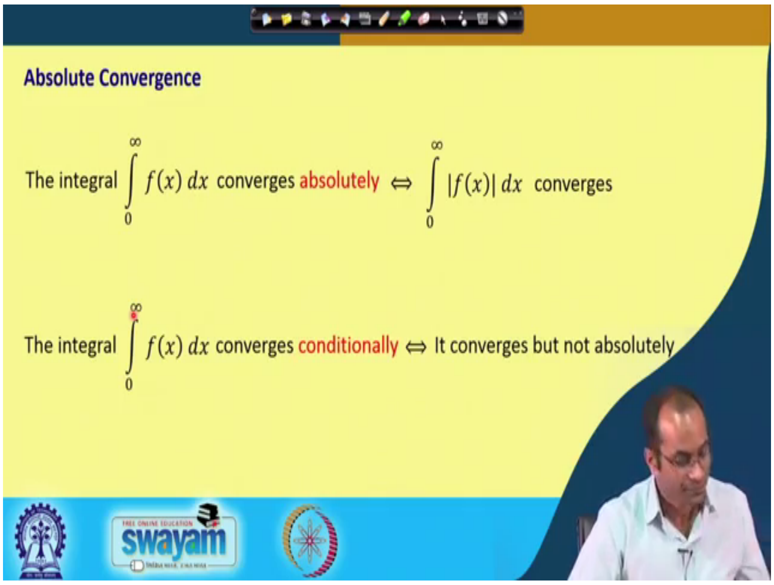

So, this is the point here, we were talking about the absolute convergence, another important factor here that what is the absolute convergence. So, as I said before, the absolute convergence means that this integral converges absolutely meaning that if with the absolute value over the integrant, if this converges. So, if this integral converges, we call that this integral converges absolutely, because we have taken the positive values of this integrant for the range here 0 to infinity. So, if this converges, then we call that this integral converges absolutely.

And there is a one more term, which is used here that this integral converges conditionally meaning that this integral converges, but does not converge absolutely. So, here meaning is that this converges, but not absolutely. In that case, we call usually that this integral converges conditionally.

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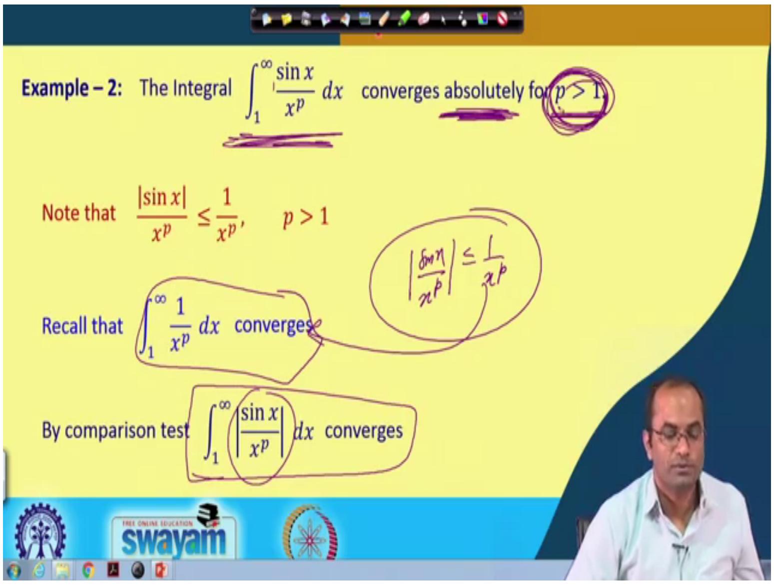

So, here like coming to this example here 1 to infinity is sin x over x power p dx, and we can show now that this integral converges absolutely for p greater than 1. For any value of p greater than 1, this integral converges absolutely. The idea is clear which was also mentioned in previous example a bit. So, here we have the inte; if you take the absolute value of this integrant that means, the sin x and x power p, so here x taking positive values, and p is greater than 1.

So, here we have the sin x absolute value over x power p, because this is positive. So, we do not have to use the absolute value here. And this is bounded by 1 over x power p, because the sin x will never take value greater than 1. So, we can bound this by 1 over x power p, and we know now by the comparison test that the integral 1 to infinity 1 over x power p.

This integral converges. So, what we have shown here that this integral also converges, because this we have by the comparison test. So, we have taken the integrant here that the the sin x over x power p, which was less than equal to 1 over x power p. And we have we know the result that this convergence the integral of 1 over x power p 1 to infinity, this converges.

And therefore, by the comparison test we have shown that this converges meaning that integral in other words converges absolutely yeah. So, we can take (Refer Time: 18:47) of because at present this is taking positive, negative values in the range. So, if we take indeed all the positive values, in that case also this integral converges for this is important that we have shown here for p greater than 1. This integral converges absolutely, when p is greater than 1. We will see later on that when p is equal to 1, this integral does not converge absolutely indeed it converges, but not absolutely.

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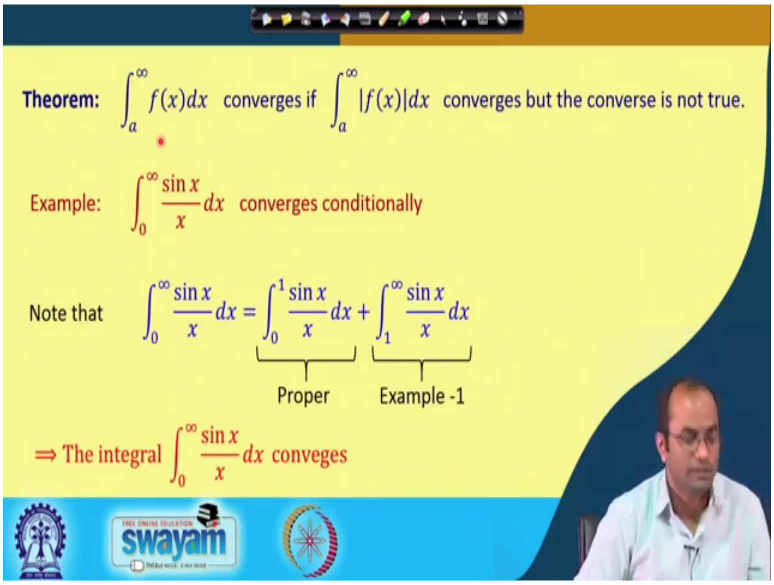

So, going to the next problem. So, this is there is a result also that this is such integral converges if this converges meaning with the absolute value, because this is going to be a larger value than this value of the integral. So, if this converges naturally this converges, because here we have positive negative and many this cancelation will come and this value will be less than the value here with while taking the absolute value of the integrant.

So, this converge so if this converges, but the converse is not true meaning that so this will converge if this converges, but if this converges this may not converge. So, if the integral converges, the integral may not converge absolutely. And that is the way standard example, which we will discuss now here that 0 to infinity, the sin x over x dx this integral converges conditionally meaning that this integral converges, but when we take the absolute value here over the integrant, then this integral does not converge.

So, first we will show that this integral converges, which is a trivial task now. So, 0 to infinity sin x over dx, we have written as the sum of these 2 integrals 0 to 1 sin x over x dx and 1 to infinity sin x over x dx. And note that the first integral is is a proper integral, so naturally it converges.

And the second one, we have this infinity this we have to to discuss. So, this is a proper integral. And here we have seen in this example-1 that here given x power p and p is greater than 1, so that integral converges. So, naturally this integral also converges sin x over x, we have seen already in example-1, so that means, that this integral converges. So, the integral 0 to infinity sin x over x dx converges.

And next, we will show that when we take the absolute value over the integrant that integrant does not converge. So, we have this example, which says that the converse is not true, because here we have the convergence of the integral, but the integral does not converge absolutely. The other way around this is obvious, because if we taking all the positive value is still we can get this integral, so naturally when we take the when the function takes positive and negative values, it will certainly converge.

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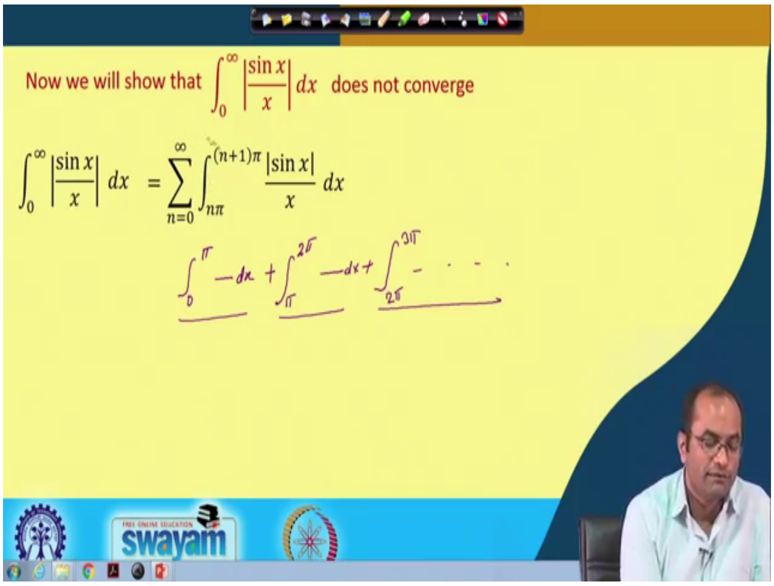

So, here we will show now that this integral does not converge, and this is a bit involved now. So, we take this absolute value sin x over x dx. And we write down this as a sum of n goes some of over this n goes from 0 to infinity, n pi n plus 1 pi and sin x absolute value over x dx.

So, here this integral is is exactly this integral. So, we have broken here like n is equal to 0, so we are going from 0 to pi. And then this integrant and the value, then we are going from n is equal to 1, so that means pi to 2 pi, and this integrant and so on. So, we have taken the 0 to pi, pi to 2 pi, and then 2 pi to 3 pi, and so on. So, this integral 0 to infinity, we have broken into several this is small segments over the range. So, 0 to pi, pi to 2 pi, 2 pi to 3 pi, and so on, which we have written in this summation form.

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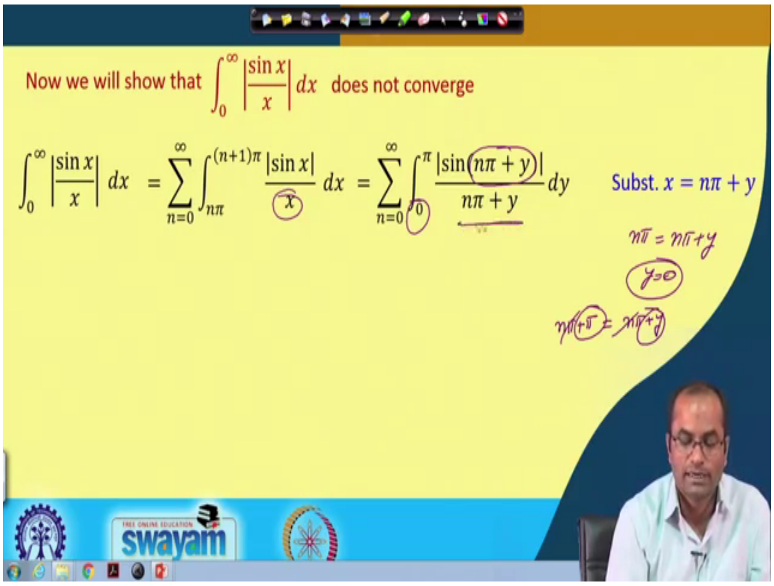

So, this is the integral. And now, we will make a substitution here that x is equal to n pi plus y. So, we make the substitution x is equal to n pi plus y in this integral, so by substituting this, so dx will become this d y. And then this n pi will be replaced accordingly and n plus 1 pi as well. So, when we substitute this in this integral, so when the x was taking n pi values, the y will take 0 values.

So, again to see here, because this relation was x is equal to n pi plus 1, so x was taking the lower value as n pi, so n pi plus y. So, this y will take as the value 0. So, this goes from 0 now. And when x was taking the value n pi plus this pi, so in that case we have this n pi plus y, so the y will take the value pi. So, our integral is now 0 to pi. And this absolute value sin this x is n pi plus y, so we have n pi plus y. And here also we have substituted for x that is n pi plus y.

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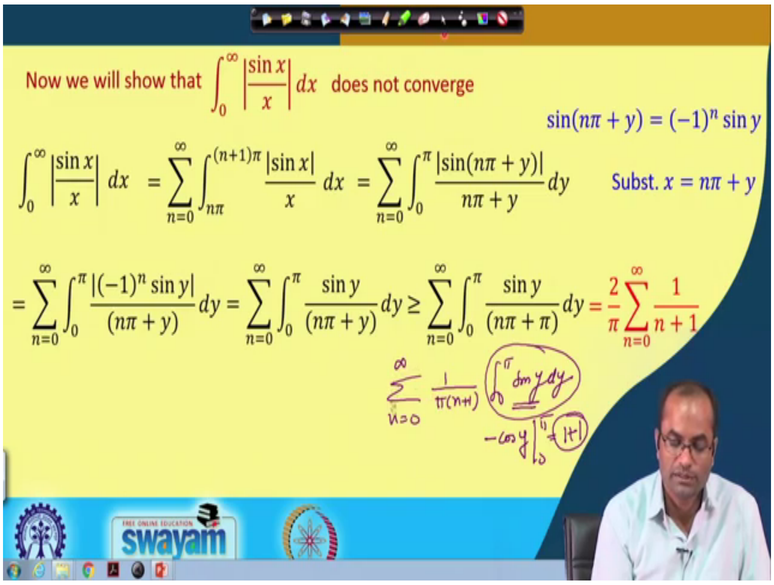

So, we will make another use of this inequality a of this equality that [sin/sine] sin n n pi plus y sin n pi plus y is equal to minus 1 power n n sin y. For example, n is 1. So, we have sin pi plus y, which is minus sin y, and similarly for other n. So, we have this equality the technomatric equality, which we will use here now in this numerator, because we have sin n pi plus y.

So, y using this we have now in the numerator minus 1 power n, and then we have this sin y over n pi y dy the same integral, which was here. So, this sin n pi plus y is replaced by minus 1 power n sin y. So, now this summation as it is we have 0 to pi, and this minus 1 power n, because we are talking about the absolute value, so that will be as it is and this sin y again.

So, here we should note that we have we are not using now the absolute value, because the this sin y is will take positive values from 0 to pi takes value 0 there, and then at pi by 2 1, and then gradually goes to 0. So, this never takes negative values in 0 to pi, therefore we have remove this absolute value here. So, divided by n pi plus y and d y.

So, in this case now this denominator if you take a look here, so y in this case for this integral as taking 0 value at 2 the pi value. So, if we want to make this integral larger, then some other integral here, so we can replace this y by the larger value or this denominator will be the taking the largest value here at pi here. So, we have n pi plus pi. So, this y is replaced by this pi, so that this is this integral here becomes smaller integral for this integral, and therefore we have this inequality that this integral is larger, because this is taking now the integrant is taking lower value, because this denominator was replaced by the lowest value the highest value here which is pi there. So, n pi plus pi, so this is a smaller integral.

And now here this is nothing but because there is no y here. So, we can take this out, so we have n sorry this pi we can take out. So, we have pi there, and the integral also we can evaluate. So, we have this summation n goes 0 to infinity and 1 over n so will remain as it is so we have also this pi here. So, 1 over pi and this n plus 1, so 1 over pi and n plus 1 or again let me write down.

So, we have this summation n goes from 0 to infinity, and this is 1 over pi and n plus 1, and integral 0 to pi and we have this sin y dy. So, this will be minus cos y and then 0 to pi y. So, when we substitute pi, we will have again minus 1 there. So, minus minus will be plus, and then minus minus will be plus again, so say 1 we have to value 2 here. So, this is replaced by 2. So, we have the value 2 over pi and the summation over 1 over n plus 1.

So, this integral is equal to this 1, and this is a well known series here, which is the divergent series, so that means, the sum is infinity here. So, what we have seen that this integral, which we have started with the absolute value. This is greater than this sum of the series, which is infinity, so that means this integral is a divergent integral, because the value of this interval will b larger than the value of the sum, which is coming already to infinity.

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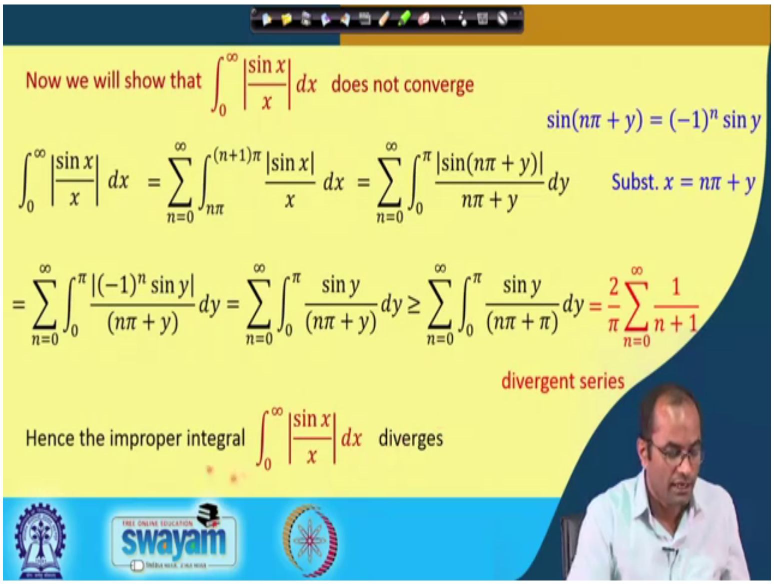

So, in that case we have that this improper integral, the given integral this diverges.

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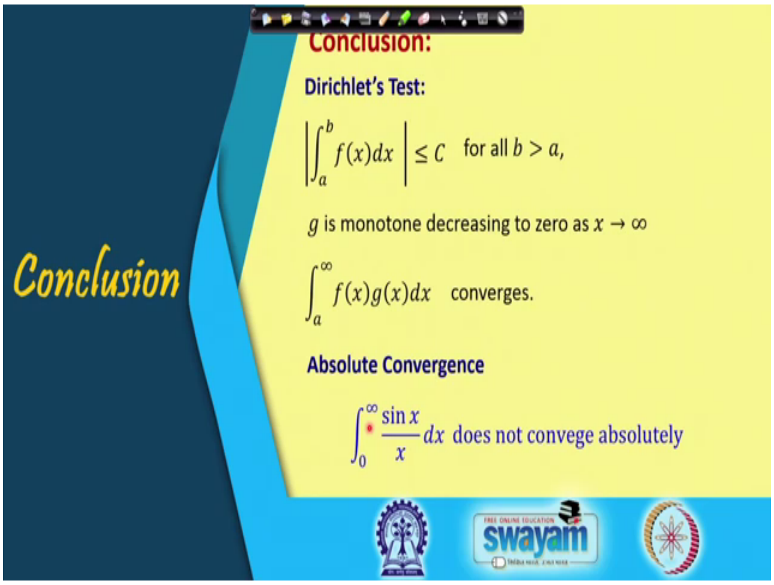

So, the conclusion we have the Dirichlet test, which says that if we can show this uniform boundedness of this integral for all b greater than 1. And g is a monotone decreasing to 0 as x approaches to infinity. Then we have this product in the integrant a to infinity this dx converges. And we have also discussed about the absolute convergence there. And we have seen this nice example which does not converge absolutely, but it converges, if we do not take this absolute value for the integrant.

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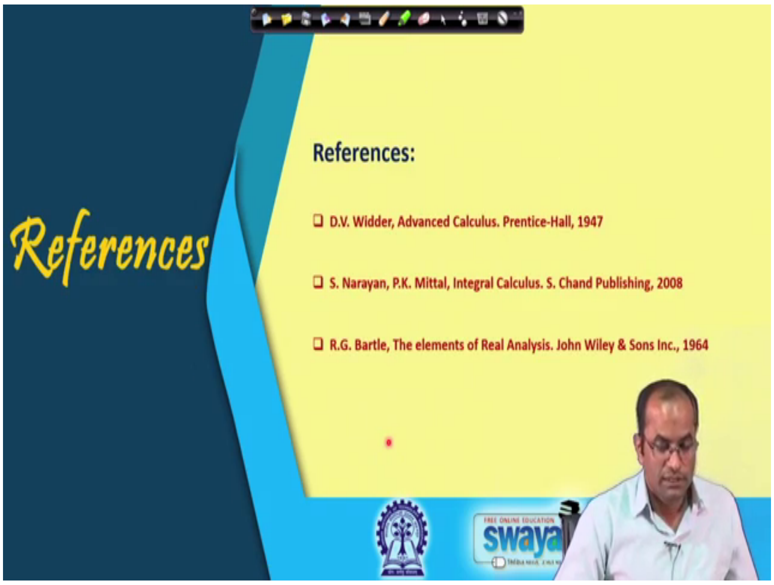

So, these are the references used for preparing these lecturers.

And thank you very much.