Engineering Mathematics - I
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Lecture – 22
Improper Integrals (Contd.)

So, welcome back to the lectures on the Engineering Mathematics-1, and this is lecture number-22. And we will continue our discussion on integral calculus, so in particular improper integrals of type-1. And today we will be talking about the convergence of type-1 integrals. In the last lecture we have seen how to evaluate those integrals or the integrals of a type-1. And with the basis of the evaluation indeed we can conclude whether the integral converges or it diverges. But today we will be talking about, some test based on which we can conclude that the integral converges or diverges without evaluating the integral.

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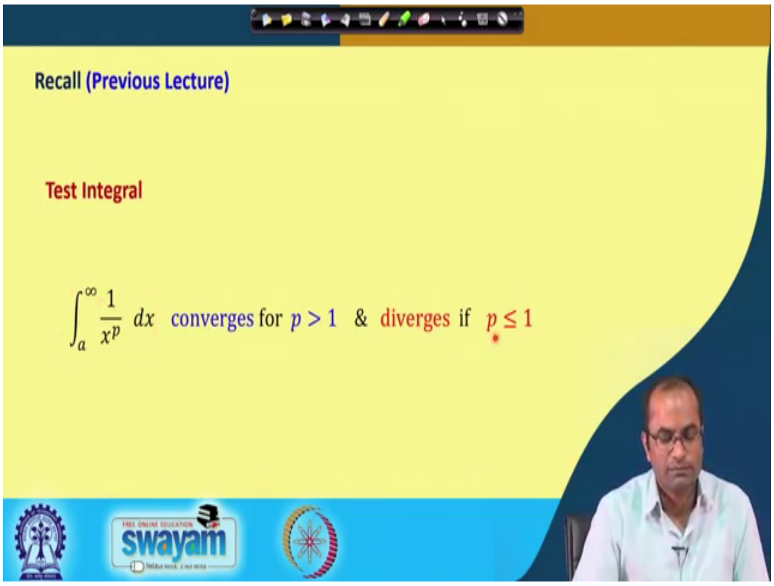

So, just to recall in the previous lecture we have a discuss this test case, which was a to infinity 1 over x power p d x, and it convergence for p greater than 1 and divergence for p less than equal to 1. So, this test this test integral will be very useful to compare the integrals with other improper integrals and to conclude the convergence of the underline integral. So, here again this convergence of this depends on p. So, for all values of p greater than 1, this integral convergence; and p less than or equal to 1, this integral diverges.

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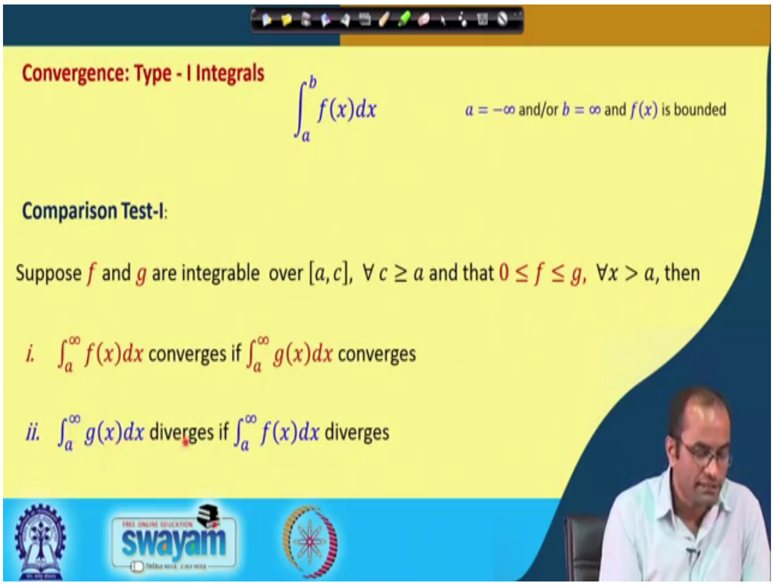

So, here we have we will be talking about type-1 integrals again to recall, we have this type-1 integral when our f x is bounded for any value of in this range a to b, and one of the is a and a and b are infinity. So, both are so like minus infinity to plus infinity that is also type-1 integral. So, we will be talking about the convergence of such integrals where infinity appears in the in the limit.

So, this is a very useful test comparison test it is called comparison test-1. So, here we suppose that f and g are integrable over a range here a to c, where the c is greater than or equal to a. And there is another relation given that f is taking non-negative values, and g is taking larger values then the values of f for any value of x greater than a. So, in that case we have this result that if this a to infinity f x d x converges, then this will converge if the integral a to infinity g x d x converges.

So, here what is this integral, integral g so g is taking larger values then f. So, in that case if this integral, which is the bigger integral in that sense, because g is taking larger values. So, we have indeed this relation when we have this g bigger that this integral this g will be having the larger value then this integral f d x d x, whatever range we are talking about here like a to infinity and a to infinity if these integral exist naturally in that case, we have such a result here.

So, if this integral converges if this integral converges, because and this is the larger value, because g is taking a larger value at any point x greater than a so, in that case naturally that this integral also converges, because if this value exists and this is a smaller than that so naturally this also exist.

And there is another conclusion which we can conclude out of it that a to infinity g x diverges if this integral diverges. So, if a to infinity f x the smaller integral this diverges. So, naturally we can conclude that the integral over a to infinity of this g will also diverge, because this f is taking the smaller values at any point x here greater than a; and if this integral exist, so naturally if it does not exist this f with a smaller values, then naturally this will also not exist this will also diverge.

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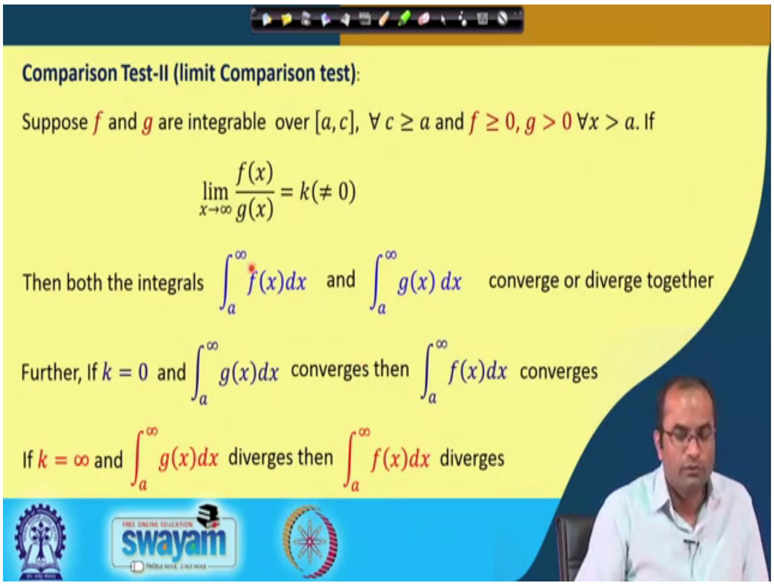

So, here we have another test which is called the limit comparison test or the comparison test-2, so this is again a useful test here. So, we have this f and g they are integrable over the range a to c for any value of c greater than a. And we have again this that f is taking non-negative values and g is strictly positive for all values of x greater than a.

And then we will be talking about, when we take this limit x approaches to infinity. So, if we take this limit x approaches to infinity f x over g x fx over g x, we have two functions we are actually getting this behavior that when x approaches to infinity of this ratio of the two functions, and if it comes to be just k a non-zero number; in that case we can conclude from here that the behaviour of this f and g is similar when x approaches to infinity, because we are getting some constant there.

So, regarding to this integrals where we have a to infinity f x d x the type-1 integral. And this table can converge or diverge together because of this reason here; we have the problem when x approaches to infinity. And what we have observed that this f x and g x, they behave a similarly as x goes to infinity and in that case we can easily conclude that so this integral a to infinity f x d x or this integral a to infinity g x d x; either both will converge or both will diverge when we get this limit as k non-zero number.

So, we are not proving all these results, but by intuition we can see all these results that they hold. Here for example, so again as I said here this behavior of both the functions is same as x approaches to infinity. And in that case, we can conclude easily that if this converges the other one will also converge and if this converge this was also converge. And this is going to be very useful, because for a given integrals suppose this f x is given we will construct a g whose a behaviour is known or that the integral over this g is known whether it converges or diverges. And based on the behaviour of this integral, we can easily conclude the convergence of the other integral.

Further, what will happen if this k is 0 for example; k is 0 means that this is converging faster to infinity than this one than this function, then only this will dominate here the g will dominate and then x goes to infinity this will go to 0. So, in this case here g is a dominating function, therefore we will get this k 0. And in this case if this g integral converges, so we have this function which is a dominating as x approaches to infinity than this f x.

And if this converges then naturally the other integral will also converge, so that is again a simple so, because this limit is coming to be 0 as x approaches to infinity. So, here the g is a growing faster than f as x approaching to infinity, so this is approaching to infinity faster. Then this one, and therefore we are getting this 0 there. So, this is a and that is a reason here if this g converges, so the f will also converge.

And again in the other way around if this k is infinity here. So, if this is approaching to infinity in the limiting case this ratio, so in that case this f is dominating and then if we can conclude that if g diverges. So, if we have the result that this g diverges, so we can tell something about the integral f which will also diverge, because this g the f is dominating function. So, this is taking its growing much faster than this one and that is the reason we are getting infinity there. So, in that case this integral f will also diverge. So, these are the simple comparison test which we have discussed and now we will go for some problems sample problems.

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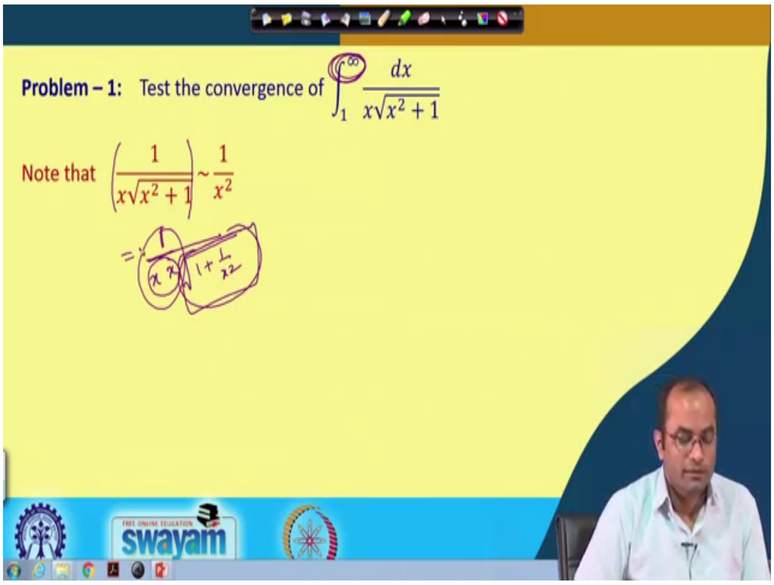

So, for example this we will test now the convergence of this integral 1 to infinity d x over x and square root x square plus 1. So, we have to here now actually find another function which we can relate or which is comparable to a similar to this as x approaches to infinity, again we have the problem at infinity only that is what we are talking about these improper integrals. So, here we will see that how this function is behaving as x approaches to infinity.

So, basically what usually we do here, so we take x and here also we will take x common. So, we have then the 1 plus 1 over x square. So, this term will just approach to 1 as x approaches to infinity, so we will not create any trouble. And now we have here x square, so 1 over x square. So, this function here the given function the integrand is behaving as 1 over x square behaves as x goes to infinity and that will help us to set the functions for comparison.

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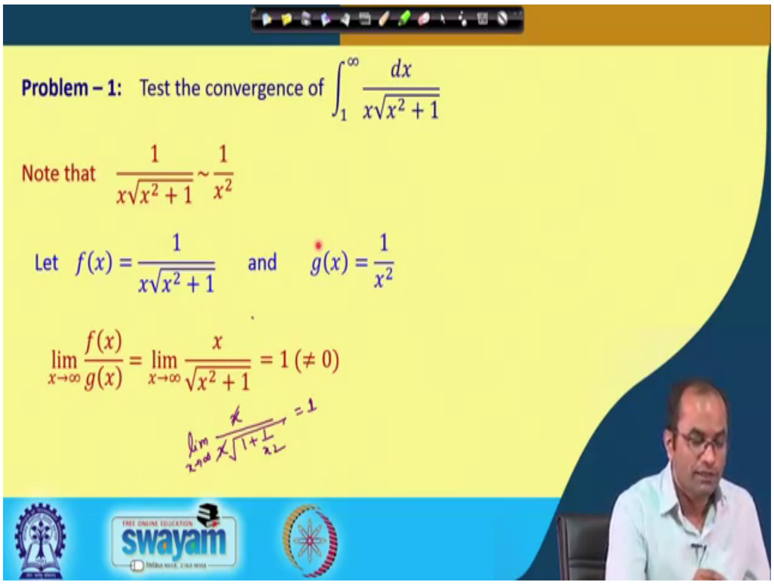

So, we have this f which is the integrand 1 over x square root x square plus 1. And we will take g as 1 over x square, because the behaviour of this function is like 1 over x square. So, we have taken the second function g 1 over x square, and which is well which is well known for us about this integral properties of this function.

So, here if we take this limit now f x over g x. So, if we compute this limit, so this is a because this g x is 1 over x square. So, if we have 1 over x square here, so this x will cancel out there for x square, so we will get x over square root x square plus 1. And this limit will be a so again we can cancel this x also. So, we will take common from here we have 1 plus 1 over x square. So, this will get cancelled and when we take the limit as x goes to infinity. So, here it will be 1.

So, we have this limit as 1 which is not equal to 0. So, if this is not equal to 0, our comparison test-2 tells that this integral given integral will be behave as the integral of this g similarly. So, we have already the result for this integral g and remember this 1 over x square over this 1 to infinity.

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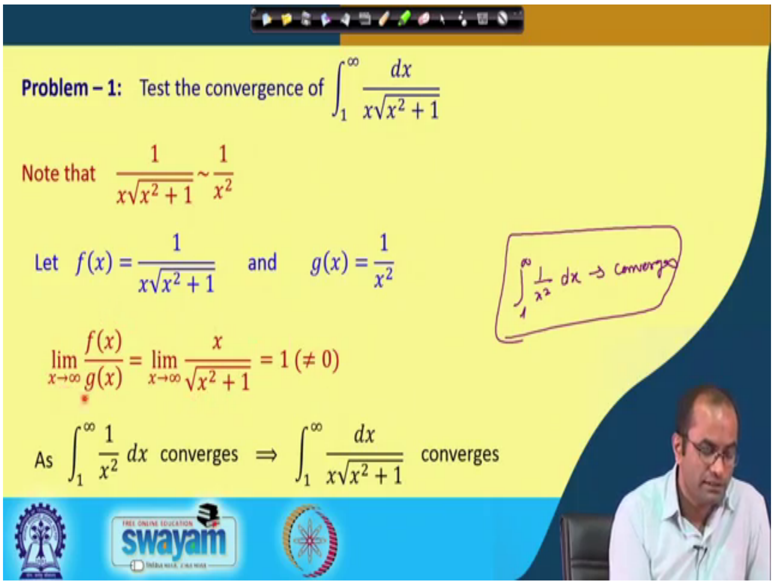

So, this integral here 1 over, so 1 to infinity and we have taken here 1 over x square and d x, this integral converges that we know from the test integral. So, if this converges, and this ratio is coming to be 1 as constant. So, in that case the other integral will also converge. So, as this converges this will imply that the given integral g x over x is square root x square plus 1 will also converge.

So, here the main point is that we have to look for the behavior of this integrand as x approaches to infinity, how this behaves in terms of the simple function 1 over x power p type function. And then we can easily get this limit, in this case it is like non-zero, so 1 and then we can conclude that this converges. So, this will also converge.

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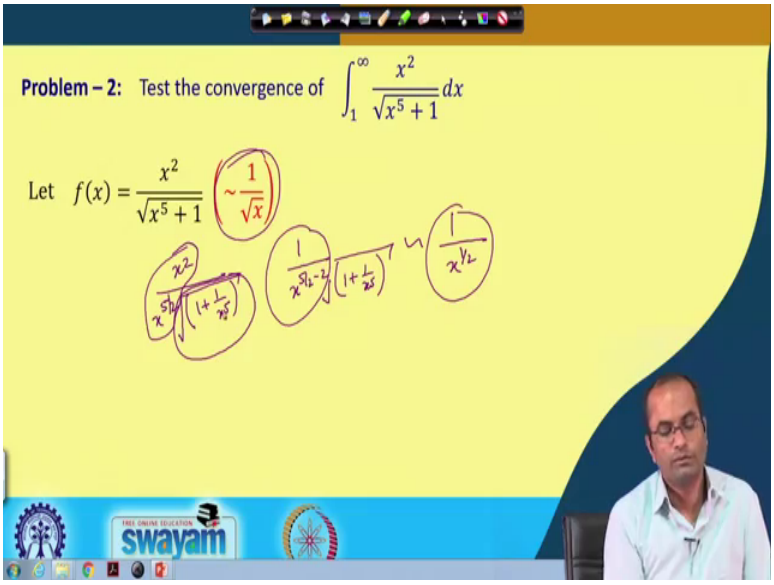

Well. So, the next problem we will test the convergence of this integral 1 to infinity and we have x square divided by x power 5 plus 1 d x. So, in this case also you will also approach same as before. So, we have this f x x square whole square root x 5 plus 1, and then the behavior of this function we will test here. So, we have x square in the numerator and here if you take x power 5 by 2. So, x power 5 and there is square root there are so 1 plus 1 over x power 5 and then there is a square root here.

So, this term will not create any problem well as x approaches to infinity, because this will be just one here. So, the behavior will be influenced by these two terms. So, we have 1 over here x power 5 by 2 and then minus 2, and this term square root which is not creating any trouble as x approaches to infinity, and this one here. So, this is behaving like 1 over x power 1 by it is a 4 5 minus 4 by 2, so 1 by 2. So, square root x, so 1 over square root x. So, this is behaving like 1 over square root x as x approaches to infinity. So, based on this now we can select the function, we can choose the function g.

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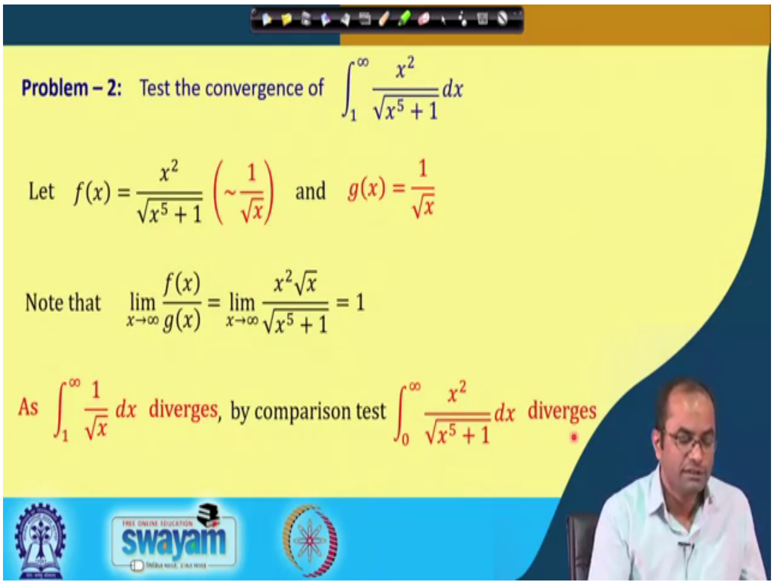

So, g we have taken 1 over square root x. And now in this case, we will take again this limit f x over g x. So, this limit here f x over g x will be x square and then the square root x will go there, and we have here square root x power 5 plus 1. And we will take this x power 5 common from here. So, we will have x power 5 by 2, and we have also there x power 5 by 2. So, this limit will be coming as 1.

So, again we are getting a non-zero limit here as the ratio of these two functions. So, both the integrals will behave the same. So, in this case we know about this g x integral that this 1 over a square root x. So, here the power of x power is half which is less than 1. So, this basically diverges this integral as this was a test integral and we have the divergence whenever this power of this x is less than or equal to 1.

So, this integral diverges and since both the integrals will behave the same. So, the integral given in the problem will also diverge by the comparison test, so this will also diverge. So, we have proved the divergence of this integral again using these comparison test ok.

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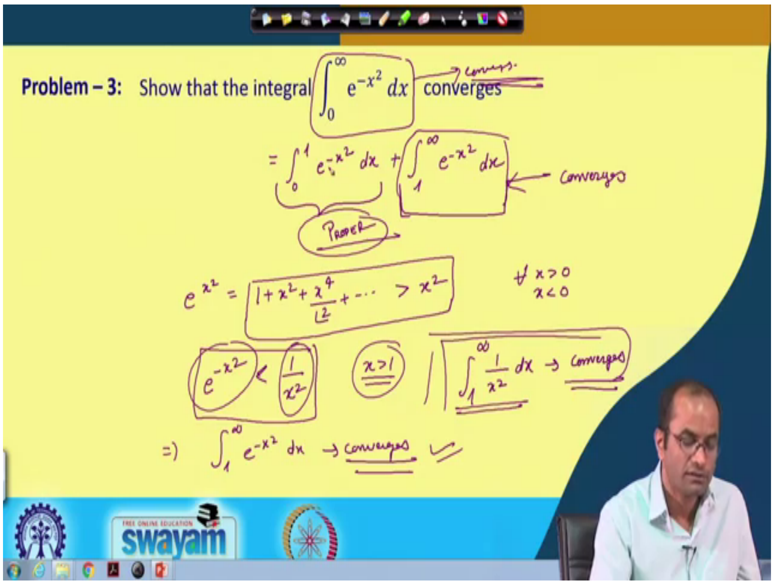

So, this is again another problem where we will show that this integral 0 to infinity e power minus x square d x converges. So, in this case we will break this integral as 0 to 1 and e power minus x square d x and plus this 1 to infinity and e power minus x square d x. So, this is the trick very often we used to remove this part here 0 to 1, because this is a proper integral. So, this is not bothering as now, so this is proper integral that means we have the value of this finite value of this integral.

Now, we will discuss only this integral here. So, if it is a convergent integral, then everything will converge; if it diverges naturally, the given integral will diverge. So, here now to apply the comparison test what we observe, now that this e power x square if we expand this one. So, we have 1 plus x square plus x power 4 over factorial 2 and so on, this is the expansion. And this will be greater than always greater than x square term.

So, we have all these positive term whether x is strictly positive. So, for all x greater than 0 or x less than 0 this term is going to be larger than this x square term, because x square term already sets here. In fact, x is equal to 0 also, so for all x this is this is true now. So, 1 plus x square this will be larger than this one, because this is exactly the same term we have taken here and all other are positive terms.

So, in any case this for all x this equality this inequality will hold. And now what we will do here. So, this we have the result now that e power x square is greater than x square. So, what we can invert it, so we have e power minus x square and then inequality will be changed here and 1 over x square. So, we are not taking here x is equal to 0 indeed in our problem now x is larger than 1. So, this is naturally true for x larger than 1.

So, we have this result that e power minus x square our integrand here is less than 1 over x square term. And we also know the test integral that this test integral 1 to infinity 1 over this x square d x that this converges. So, this converges and then by this comparison test, because this function here 1 over x square is taking larger values then e power minus x square, and this integral which is dominating integral this converges. So, naturally this one will also so this will imply that this integral 1 to infinity e power minus x square d x also converges.

And now this integral converges and also this is proper integral which converges, so the given integral here it will converge, so this converges which we want to show. So, this was easy by handling in this way that we have taken this two integrals into consideration the idea was because if x is greater than 1, we can easily in fact this not equal to 0; we can just divert it here 1 over x square and by breaking into two integrals that help, because the first one was the proper one.

And now we have x greater than 1 anyway. So, it convergence and then both the sum converges and the original the given integral converges. So, now you will take one more example. So, so that this integral 0 to infinity sin square x square a d x converges.

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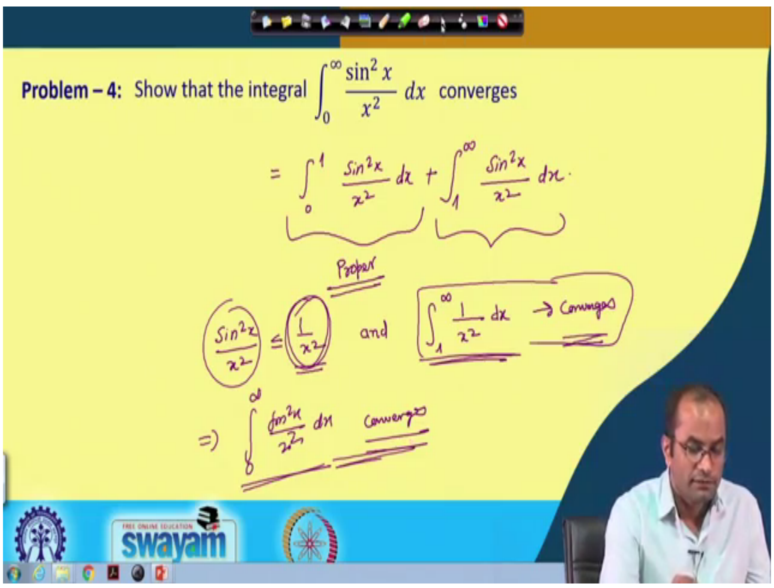

So, in this case we will also deal this because at x is equal to 0, we have this problem here with x. So, let us just break again this 0 to 1 and then sin i square x over square d x and plus then here 1 to infinity the rest sin square x over x square d x. And this the first integral here is a proper integral, we have discussed already in the last lecture, because this limit as x approaches to 0 is finite; so this is a proper integral where that was the only problematic point as x approaching to 0, but this limit is finite. And in that case, we do not have any problem in the integrand even when x approaches to 0, so this is a proper integral.

The second one, now we will discuss that this also convergence and the reason is clear, because the integrand which is given here sin square x over this x square, this is always less than equal to 1 over x square. And we know that the integral 1 to infinity that test integral 1 over x square d x that also converges. So, here this is a dominating integral, because this 1 over x square is taking larger values then the sin square x over x square.

And this integral converges, so by comparison test we know that this integral sin a square x over this x square d x 0 to infinity will also converge. So, this will also converge based on this test integral which was 1 over x square 1 to infinity d x this converges. So, here this was a dominating integral. So, naturally this will also converge. So, we will take another example now.

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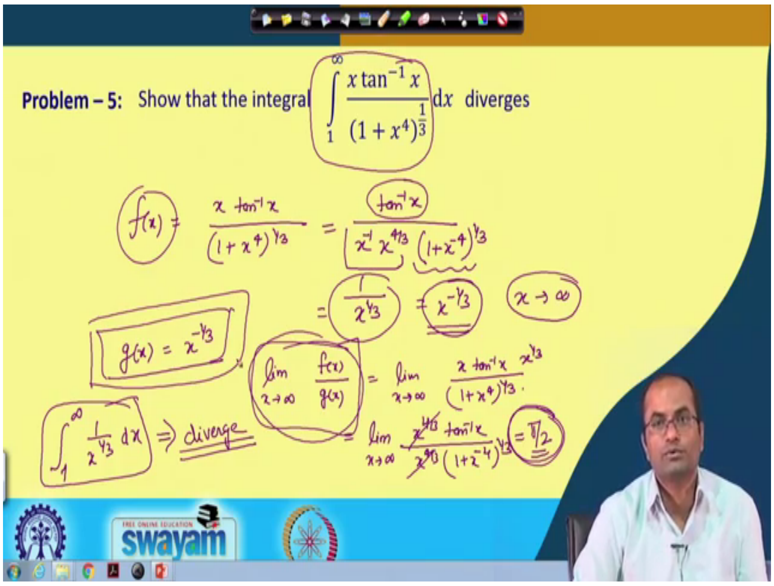

Here we will show that this integral x tan inverse x over 1 plus x 4 1 plus x 4 power 1 by 3 d x diverges. So, now we will show here that this integral diverges. So, let us take this f x the integrand term, so let us assume that this is f x. So, over f x is x and tan inverse x over 1 plus x 4 and we have here 1 by 3. So, this x we can bring to the denominators. So, tan inverse x and we have x here and this x 4 also we will take out of this, because we want to see the behavior as x approaches to infinity. So, in that case we take this x power 4 by 3 here out, this is x power minus 1 this x.

And then we have again here x power minus 4 plus 1. So, 1 plus x power minus 4 and this is 1 by 3. So, as x approaches to infinity this will be 1 and here was x approaches to infinity, this will be also a constant term pi by 2. So, mainly the behaviour of this we can see by this term here which is a x power, so 1 over x power 1 by 3 or this is x power minus 1 by 3 as x approaches to infinity.

So, this function here the integrand behaves as x power 1 minus 1 by 3 when x approaches to infinity. And now we got this function g x. So, if we take this function as g x is equal to 1 over x cube x power minus 1 by 3, then we can gather limit. So, the limit as x approaches to infinity of this f x over g x function. So, the limit x approaches to infinity the f x was x tan inverse x, and then we have here 1 plus x 4, and then 1 by 3. And this is g x which is x power minus 1 by 3. So, I can take to the numerator x power 1 by 3.

So, now what do we have here. So, this is limit x goes to infinity and there we have x power 4 by 3, and this tan inverse x and here also we will take this x power 4 by 3, and then we have the same x power minus 4, and this is power 1 by 3. So, as x approaches to infinity this is pi by 2 tan inverse infinity this will be pi by 2, and this will go to 1, so we got this pi by 2 limit.

So, now since this limit is same this limit is a non-zero number. So, now the conclusion is that both integrals integral over f and integral over g, they will behave the same. So, in this case we know that this integral here 0 to so 1 to infinity and this 1 over x power 1 by 3 d x. So, here the p the power is less than 1. So, this integral will diverge so diverge. So, this integral will diverge and so the integral over f will also diverge, which is given here in the problem.

So, this integral diverges because we have a compared this with 1 to infinity 1 over x power 1 by 3 d x integral, and they both has two different has to be behave the same because of the reason that they behave their behaviour is same as x approaches to infinity, well.

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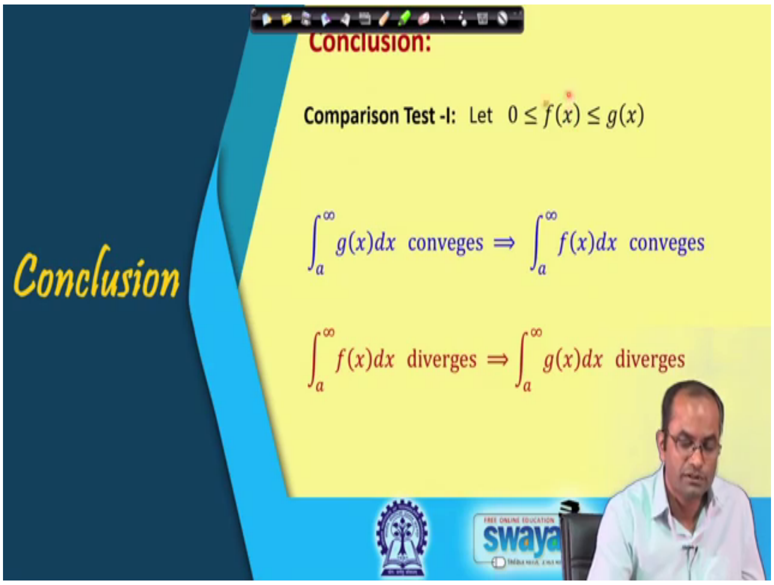

So, having this now we go to the conclusion now. So, we have seen two comparison tests test number 1 was when we have a this relation in f that f is greater than equal to 0, and g is greater than equal to f. And in that case if we know that this g converges, this g converges the larger integral which is taking the function taking large values. If this converges, the other one will also converge; and if that diverge the smaller one if that diverges, the other one will also diverge.

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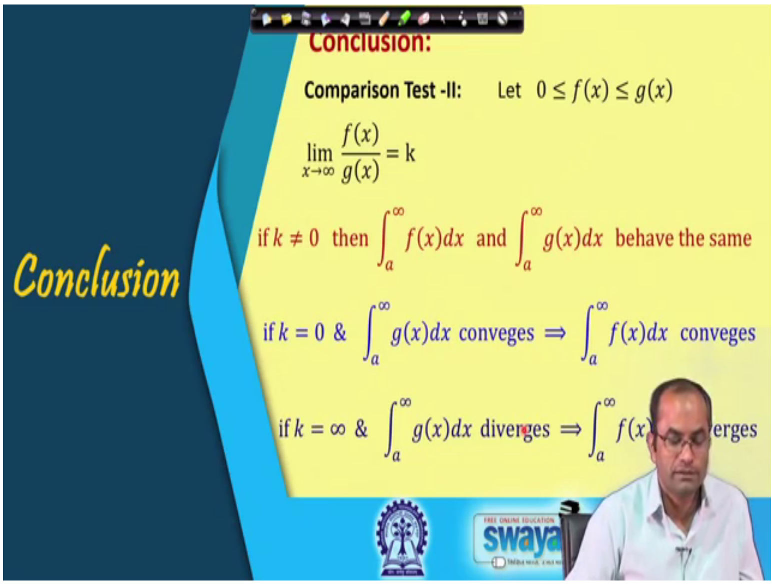

So, the comparison test-2 was again we have these relations. So, it can be strictly equal also because that x approaches to infinity at least this g x should not be 0. So, we have if this value is k, and in that case if k is not equal to 0, they both will behave the same and if k comes to be equal to 0 that means, this is growing much faster; in that case if this converges, the other one will also converge.

And we have also the result if this k is come infinity, and this diverges the g integral which is the smaller one now this f dominates. So, in that case if the smaller one diverges naturally, the bigger one will also diverge so that was the conclusion of this lecture these comparison test-1 and the comparison test-2.

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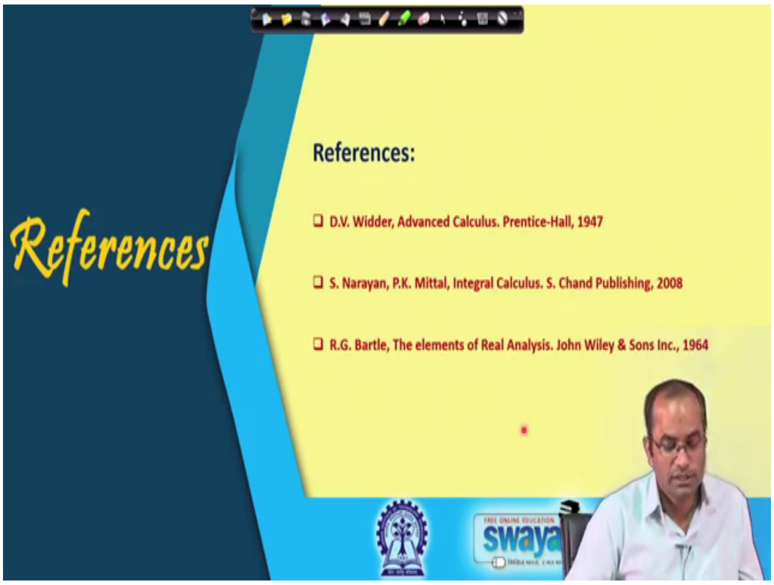

So, these are the references we have used to prepare these lectures.

And, thank you very much.