Engineering Mathematics – I
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Lecture – 20
Constrained Maxima & Minima

Welcome back to the lectures on Engineering Mathematics-I and this is lecture number 20. So, today we will discuss the Constrained Maxima and Minima.

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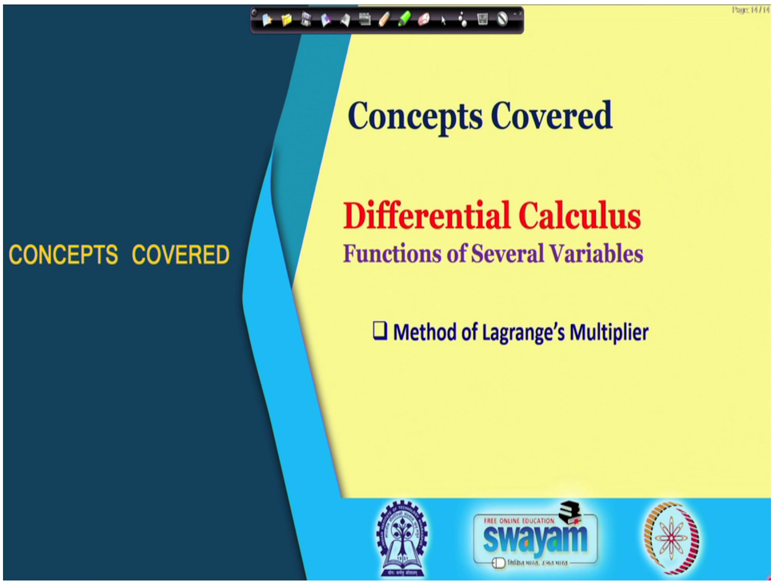

So, in particular we will be talking about the method of a Lagrange's multiplier which is used to solve such problems, where we have along with the function some constraints.

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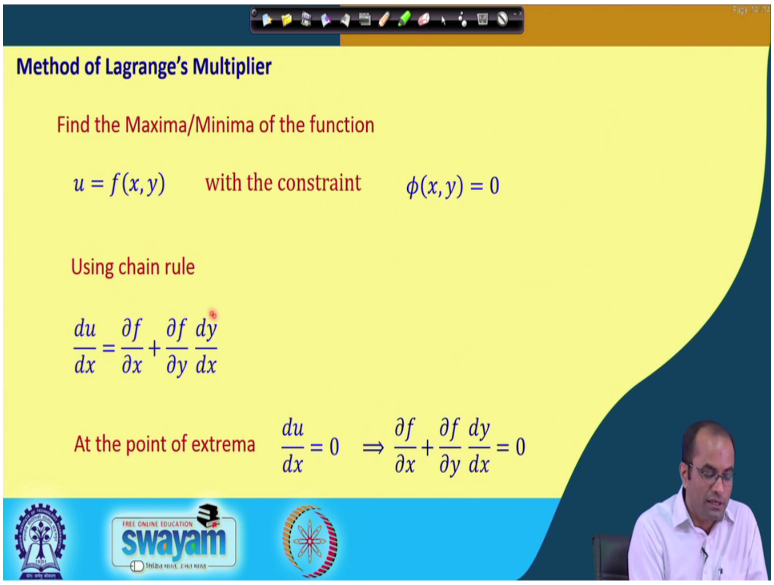

So, what is the method of Lagrange multiplier? So, let me just discuss the problem. So, if you want to find the maxima and minima of the function, let us say u is equal to f x y where we have some other constraint that phi x y is equal to 0. So, we have some relations some conditions on x y is given. So, basically this is similar to the problem we have discussed in the previous lecture where, we had taken the boundaries into the consideration.

So, those boundaries were actually the additional constraint on the function that x y satisfies either x is equal to 0 or y is equal to 0 or there was a line there y is equal to 9 minus x. So, there was an additional condition along with the function given. So, there we have substituted the value of the y into this f x y and then the function was converted into a function of 1 variable problem which we have discussed for along each of these conditions.

So, today we will have this method of Lagrange multiplier where we do not have to substitute the value of y in this one and then getting a function of 1 variable and proceeding further for extrema. In this case we have some direct method which without substituting we can actually get all these critical points where, the function may take maximum minimum or it may be a saddle point. So, let me just explain that what is the method of Lagrange multiplier. So, using this chain rule here u is equal to f x y we can get the du over dx directly because, here the function y is a function of x out of this we have the relation here between x and y.

So, by removing this y from here we have basically this function of x. So, this du over dx make sense here, but with the rule with the idea of this chain rule we can get that derivative in terms of the partial derivatives here. So, the partial derivative of f with respect to x and then dx over dx that is a 1; so, plus this partial derivative with respect to y and then dy over dx. So, with this du over dx and at the point of extrema because we know that the du over dx must be 0, it is a 1 variable problem where we have to get this derivative of u with respect to x and that has to be 0 at the point of extrema. So, we at this point here we get that this derivative must be 0. That means, this del f over del x plus del f over del y into dy over dx must be 0.

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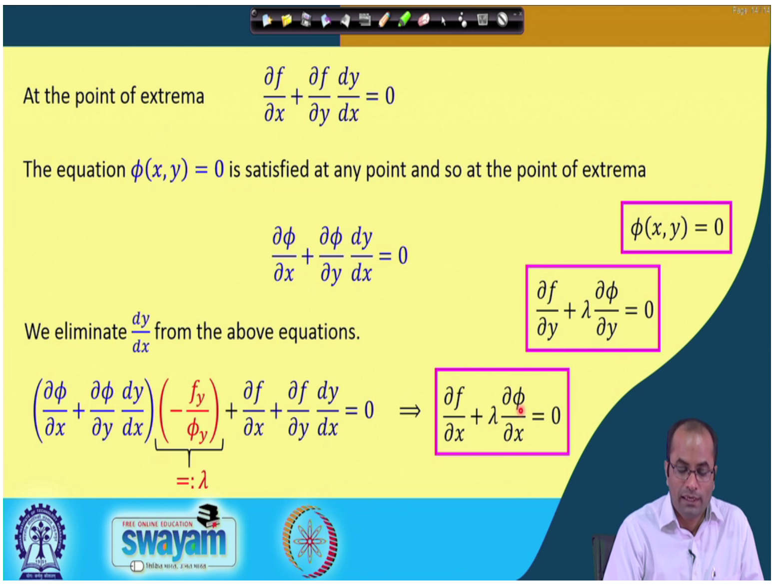

And at the point of this extrema we have this df over dx plus this del f over del y dy over dx is equal to 0. So, that is a one condition 1 equation we are getting which has to be satisfied at the point of extrema, and then this equation which is the constraint that a phi x y must be 0. So, whatever point is the point of extrema this condition has to be satisfied because, that is given in the problem that the relation x and y must be satisfied with this relation.

So, in that case whatever point we have the point of extrema this equation must be satisfied at any point; so, naturally at the point of extrema as well. So, we have won this equation out of this equation if we get the derivative with respect to x, if we differentiate this one. So, we will get a partial derivative with respect to x and the chain rule plus this del f over del y and then dy over dx is equal to 0. So, we got this another equation which is satisfied at the point of a extrema. And now we will try to eliminate out of these 2 equations this dy over dx term and we will have everything in terms of partial derivatives.

So, here we have this del phi over del x plus del phi over del y this second equation the left hand side of this equation here and we have multiplied by minus fy over phi y. The idea is because this phi y and this phi y will get canceled and we have this with minus sign dy over dx and this del f over del y and here we have del f over del y and dy over dx. So, these two will get cancelled and we have the term free from this dy over dx. So, in this case when we added here; so, these terms will get canceled and we will get simply assuming that this here is lambda just for simplicity now.

So, here we get del phi over del x into this lambda and this del f over del x is equal to 0. So, this is the equation we got that that has to be satisfied at the point of extrema that is a one equation, another one since lambda has appeared here. So, we have this relation from here that del f over del y is equal to this phi y into lambda which is the relation here. So, we have another relation which has to be satisfied, and we have this equation phi x y is equal to 0. So, we have these 3 equations which has to be satisfied at the point of extrema there.

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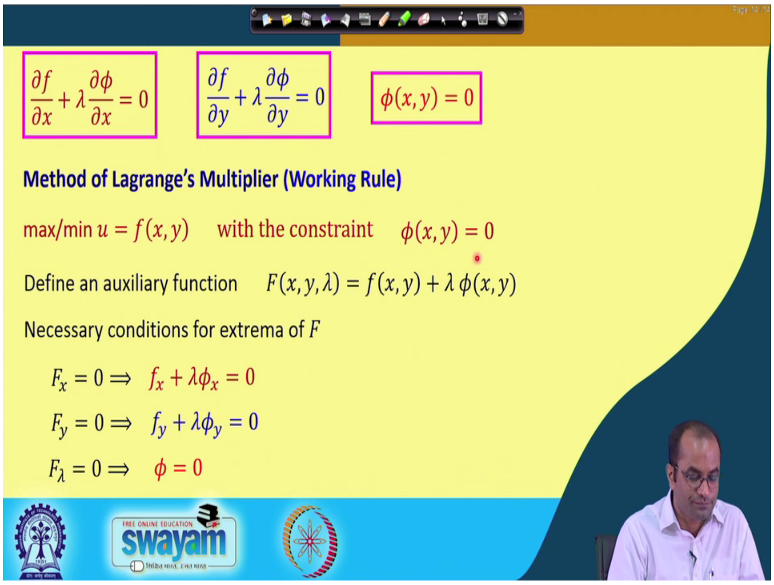

Now, moving further so, we have considered we have discussed that these are the equations which has to be satisfied at the point of a extrema and we have here 3 parameters. So, 1 will go 3 unknowns basically the x y will be the points we are looking for and also this lambda we have introduced. So, there is a x y and lambda. There are 3 points we have to we have to get by solving these 3 equations and that those points will be the points of extrema in terms of the x y.

So, what is the method of the Lagrange multiplier which we have just discussed we can there is a way to remember easily how to set up these equations. One way which we have just seen bit along bit long derivation, but now we will here write down in a more precise form which is very easy to remember and how to get these equations directly. So, we have a problem that we want to minimize or maximize this u is equal to f x y under this condition that x y satisfies this relation.

And we just define in an auxiliary function. So, we define that F x y and lambda the function of x y and lambda by introducing this lambda here. So, we take this function f x y which we want to minimize or maximize and plus this lambda and this constraint this phi x y. So, we need to define such an auxiliary function. In fact, this idea can be extended for when we have many constraints like phi 1 phi 2 and so on.

So, again we have to introduce the lambda 1 phi 1 then plus lambda 2 phi 2. So, we have to introduce more lambdas. So, we are discussing just for under one constraint. So, this is the auxiliary function we define and then we find the necessary conditions on for the extrema of this F. That means, the derivative of this F x a with respect to x will be set to 0. Derivative of this F with respect to y will be set to 0, the partial derivative of F with respect to this lambda has to be set to 0 to find the a critical points of this F here.

So, by doing this we will precisely get those equations which we have derived that at the point of extrema these equations must be satisfied. So, when we derive when we get the derivative with respect to x here we will get this partial derivative of f with respect to x lambda and the partial derivative of this phi with respect to x. That means, this F x is equal to 0 is nothing but the f x plus lambda phi x which was the first equation there; the F y the partial derivative with respect to y. So, here again this f y lambda phi y will be 0 which is the second equation listed there. And, the third one is the partial derivative of F with respect to lambda and that will be just the phi x y is equal to 0.

So, we have these 3 equations which can be obtained just by defining this auxiliary function here F, do as that you we substitute here this f x y as it is, introduce one lambda and we have just one constraints here. So, lambda phi x y, but we can extend this idea for many constraints as well. So, if we have for example, 2 constraint phi 1 x y is equal to 0 phi 2 x 2 y is equal to 0 then we will just introduce more lambdas there lambda 1 phi 1 then plus lambda 2 phi 2. And again then we have to discuss these necessary conditions. So, there will be 2 lambdas there so, 2 one more equation will be coming with respect to lambda 1 here or with respect to lambda 2 there. So, at another constraint will appear here and then some more terms there.

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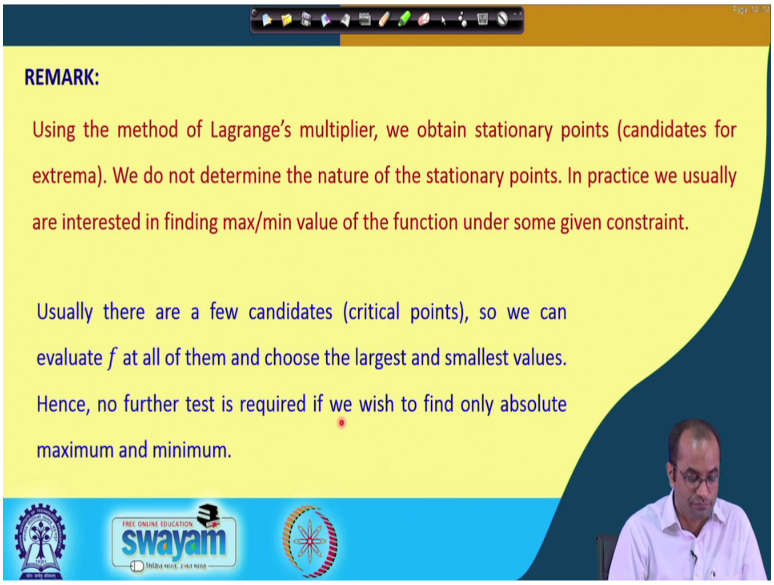

So, anyway let us discuss for this one very one constraint and the remark here that using this method of Lagrange multiplier, we will obtain the stationary point or rather we call the candidates for the extrema. This will be the candidates for the extrema; we will not compute the behavior of this point whether it is a point of local minimum local maximum.

We will we do not determine the nature of the stationary point in practice we in many problems we are usually interested finding the maximum and the minimum; so, the global maximum and minimum value of the function under given constraint. So, here in this lecture today or by this Lagrange multiplier we basically find all the candidates. So, called the critical points of the problem where the maximum or the minimum may take place. So, we will find all these critical points and find the values of the function at all these points.

And then, we can identify that which one is the point of maximum the global maximum or which one is the point of a global minimum. So, usually there are a few candidates as to the critical points. So, we can evaluate f at all of them and choose the largest and the smallest values. And, hence no further test is required if we wish to find only absolute maximum and minimum. So, our aim is now to find only the absolute maximum and minimum. And therefore, we do not need any other test to compute whether this point is a point of a local maximum minimum or a saddle point. But rather we will look for just the global maximum minimum of the problem under that given constraint.

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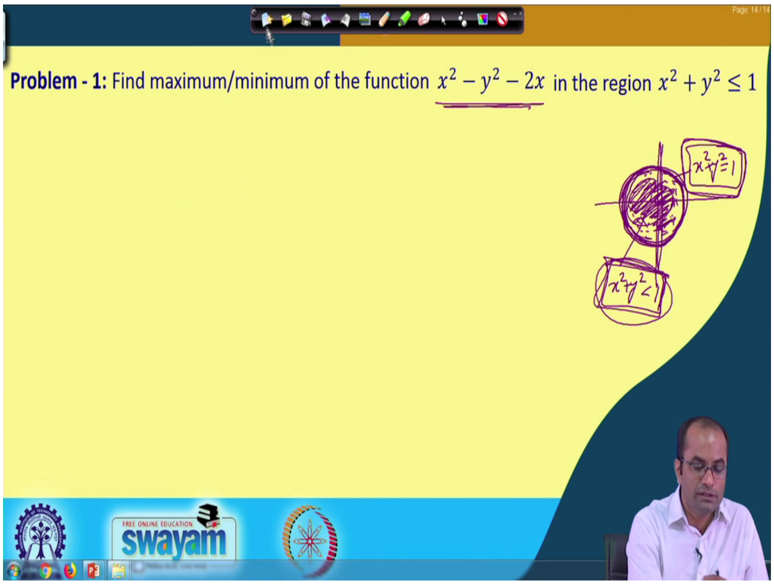

So, let us move to the example here. So, we have the maximum minimum of this function x square minus y square minus 2 x in the region here, x square plus y square less than equal to 1. So, this is the constraint is given that we want to find the maximum minimum of this function x square minus y square minus 2 x under this constraint that the x y satisfies only x square plus y square less than 1.

So, we are talking about this disc including the boundaries including this circle here. So, x y's are restricted only within this a domain here. So, there is a restriction on x y. So, this is a problem of a constraint and now we will deal in two ways. So, in all these problems when we have such a reason is given; so, this boundary is the boundary is here x square plus y square plus is equal to 1. So, we will break into two problems. So, we will find the maximum or the critical points of the problem here x square minus y square minus 2 x inside this domain.

And then on the boundaries; that means, with the restriction x square plus y square is equal to 1 exactly and that will be the problem which we have discussed that using the Lagrange multiplier we will get; when we are talking inside this domain that means, this x square plus y square is strictly less than 1. So, our domain is open here and we will apply the idea which is discussed already in the previous lecture that we will find directly, the critical point and then we will see that if some of the critical points fall in this region.

So, we will take them for further evaluation of f at those points, if the critical points does not fall in the domain they are outside the domain then we can leave them because that is not of our interest. So, this problem is taken into two parts: one is finding the critical points or the candidates for local extrema minima inside the domain. So, x square plus y square less than 1 and the one problem will be that finding these maxima minima the candidates for maxima minima on the boundary. So, let us move to inside the domain first.

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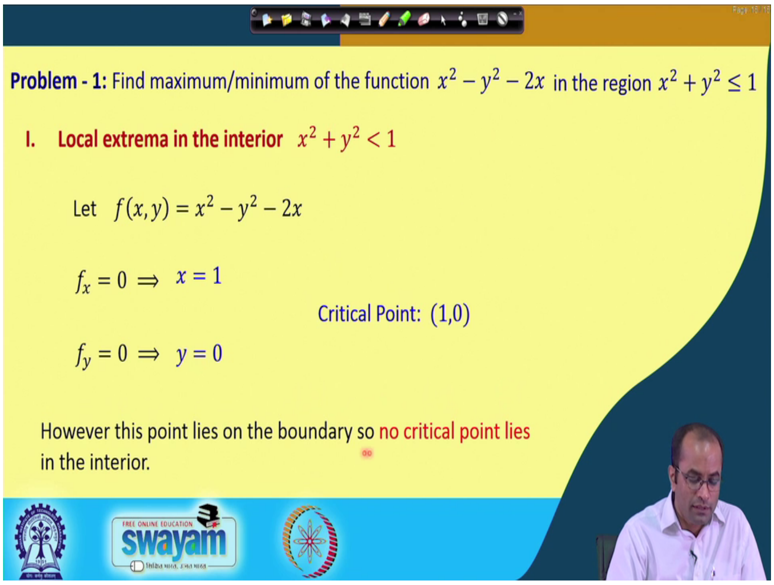

So, for the interior; that means, x square plus y square less than 1. So, here we do not have to use any other idea then the discussed in already in the previous lecture. So, we will have the f x y is equal to x square minus y square and minus 2 x. So, we will get f x here which will give us the 2 x and minus 2 is equal to 0. So, x is equal to 1 and we will set this f y to 0 so, we have 2 y is equal to 0. So, it means y is equal to 0.

So, the only critical point which we are finding for this f x y is equal to 0 is 1 and 0, but this 1 and 0 point does not fall in our interior. Here indeed this is the point on the boundary, but that will be automatically taken care when we will consider the boundary in the next slide. So, here there is no critical point inside this interior here. This is outside the interior or in this case rather it is sitting exactly on the boundary, this point may be absolutely outside the domain which we are considering.

So, in any case you will not consider this point now in this sub problem where we have we are looking for this extrema in the interior of the domain. So, this is a point certainly because falling on the boundary. But, this will automatically come in the problem and we discuss when we find the critical points on the boundaries.

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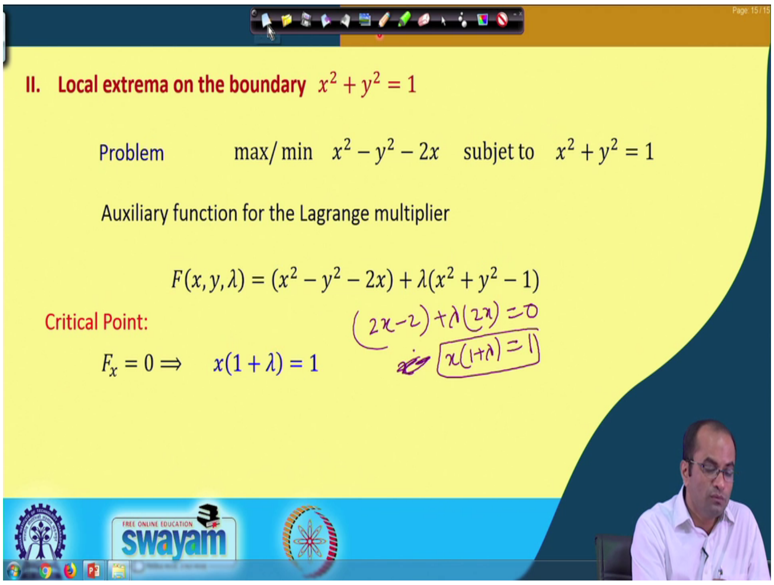

So, the next problem will be that we will look now for the local extrema on the boundary the x square plus y square is equal to 1. So, the problem is to find the maxima minima subject to this condition and that is the problem which we will solve using the Lagrange multiplier. Because here, we have the exactly this constraint that x square plus y square is equal to 1 and for that we have to define the auxiliary function, so that will be this function itself x square minus y square and minus 2 x plus this lambda times this constraint x square plus y square minus 1.

So, you will find the critical points f x is equal to 0 which will be 2 x and minus 2 plus this 2 x. So, then we can simplify; so, x and 1 plus lambda is equal to 1. So, just to look again here so, what we get with respect to x we got 2 x and we got 2 here plus this lambda and we got 2 x here is equal to 0 we want to set this. So, this 2 will be cancelled from everywhere we have x and plus this lambda x. So, x we can take common here. So, we have 1 plus lambda and then this 1 this minus 1 the other side will go plus 1. So, this is x 1 plus lambda is equal to 1.

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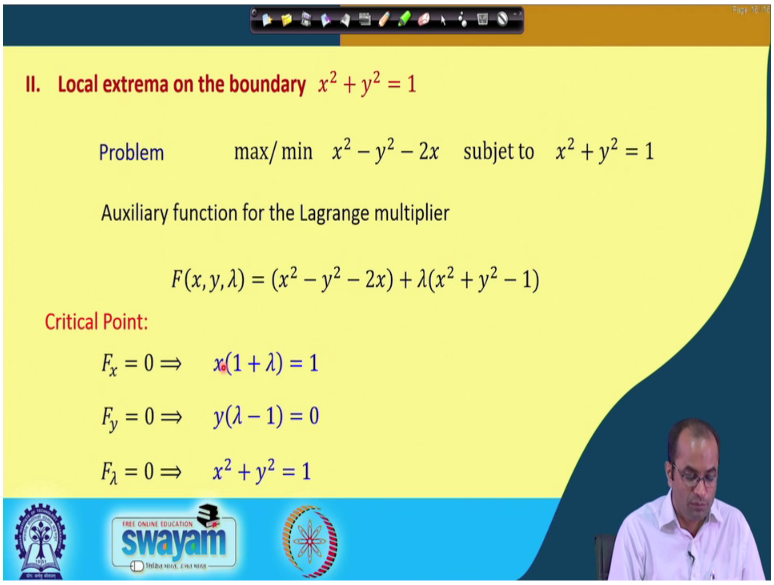

And then F y is equal to 0. So, we have to differentiate now this with respect to y treating lambda and x as constant. So, we will get y and lambda minus 1 is equal to 0 another point. So, with respect to lambda when we differentiate we will get exactly our constraint. So, it is x square plus y square is equal to 0. So now, we have to solve these 3 equations to get the points to get the critical points and those points will be the candidates for the local for the maximum or the minimum of the problem.

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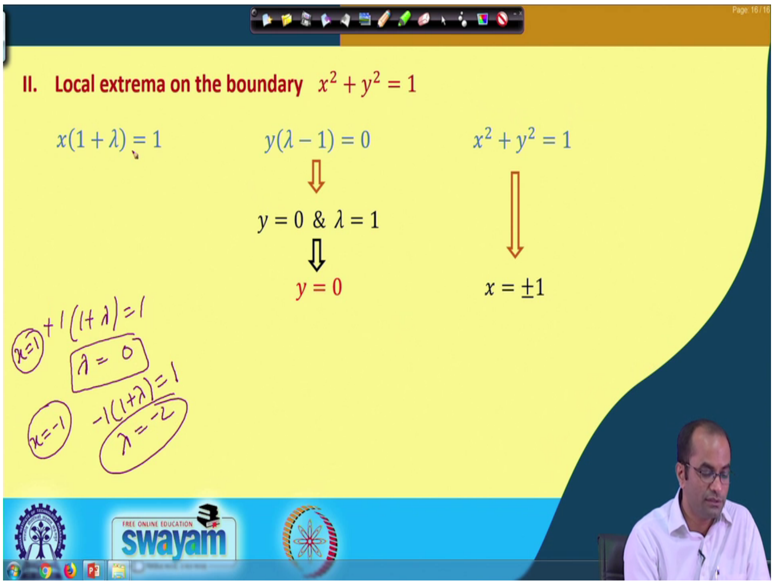

So, we have these 3 conditions. So, these 3 equations which we want to solve now which let us start with this middle one. So, this is satisfied when lambda is equal to 1 or this y is equal to 0 or both. So, here if we consider the y is equal to 0 first. So, let us consider that y is equal to 0 so, this equation is satisfied. Now, for this y is equal to 0 from this third equation when we set y is equal to 0 here we have x square is equal to 1; that means, x is equal to plus minus 1 this equation is also satisfied.

So, if we have y is equal to 0 and x is equal to plus minus 1 these two equations are satisfied, now from the left one having this x is equal to plus minus 1. So, if x is x is plus 1 and we have 1 plus this lambda is equal to 1 so, this lambda will be 0 in that case and when we have x is equal to. So, this was x is equal to 1 and then if we have x is equal to minus 1 in that case we have minus and the 1 plus lambda is equal to 1. So, this lambda will be minus 2 in this case. So, we have lambda 0 and lambda minus 2 from this first equation.

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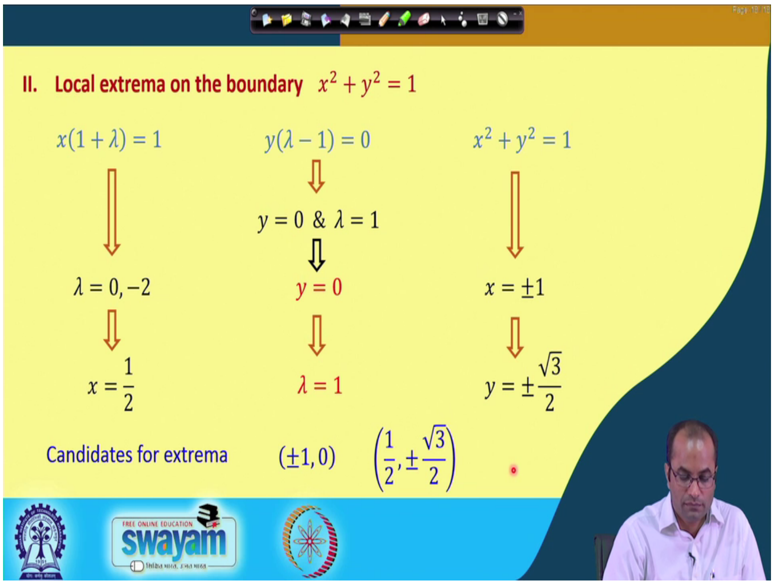

Lambda is equal to 0, lambda is equal to 2. So, what are our points here? Our points are like x is equal to 1 y is equal to 0 and lambda is equal to 0. This is the point which is which satisfies all these 3 equations. The second point we have here minus 1, we have 0 there and we have minus 2 there. This is another point which satisfies all these equations.

Now, from here we have another possibility is that we get lambda is equal to 1. So, if we take lambda is equal to 1 from here; when lambda is equal to 1 we can get x from this first equation when lambda is equal to 1 we can get x has to be a one half to satisfy this equation. So, we have x one half lambda 1 and since x is one half so, from here we can get y. So, y square is 1 minus 1 by 4. So, we get y plus minus 3 by 2. So, we got another points here. So, our points are now, so x is half and then y let us take the plus 1 plus 3 by 2 lambda is 1, this is 1 point which satisfies all these 3 equations.

And then we have x 1 by 2 we have minus 3 by 2 and then we have 1 again here, this is another point which satisfies all these 3 equations and these are all possibilities; we do not have any more possibilities which can satisfy all these equations. So, the candidates for this extrema I am just writing here in term the x y points because, we will compute the function value at the x y point lambda is not interesting for us there. So, we have the candidates now the plus and the minus 1 with the 0. So, the first two points there and these two points so, plus this half and plus minus 3 by 2. So, these are the candidates now for the extrema.

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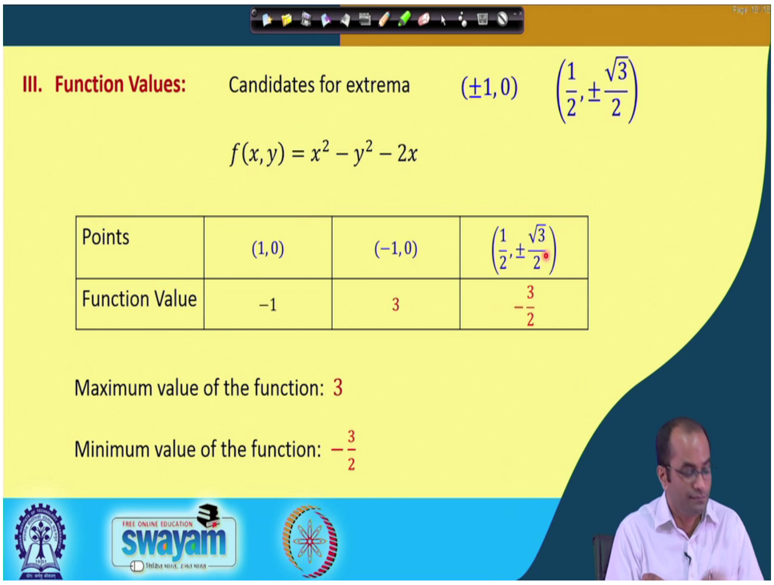

So, now the function values we will compute at this plus minus 0 and this 1 by 2 and plus minus 3 by 2, this was our function now. And, these are the points 1 0 minus 1 0 and then here the y is taken plus minus 3 by 2, because it is a y square there. So, does not matter if we take plus sign there or minus sign the value will be the same. So, the function value at these points so, when 1 0 so, 1 here the 0 and then 1 so, minus 2 plus 1 minus 1.

So, the function value at this 1 0 point is minus 1, here we take minus 1 0 so, minus 1 for x. So, we have 1 there and then this is 0 and then here we have 2. So, we have 3 and at this point similarly, if we compute we will get minus 3 by 2. So, here we see that the maximum value is achieved at this point minus 1 0 which is the value of the function is 3 at this point.

So, the maximum value of the function is 3 and the minimum value is at these 2 points here or the value of the function is minus 3 by 2. So, in this problem again to conclude that we have computed all the critical points of the problem. And, then we have evaluated the function value at all these critical points and then we have selected well this is 3 is the maximum value which is attained at minus 1 0 minus 3 by 2 is the minimum value of the function which is attained at this point half plus minus 3 by 2.

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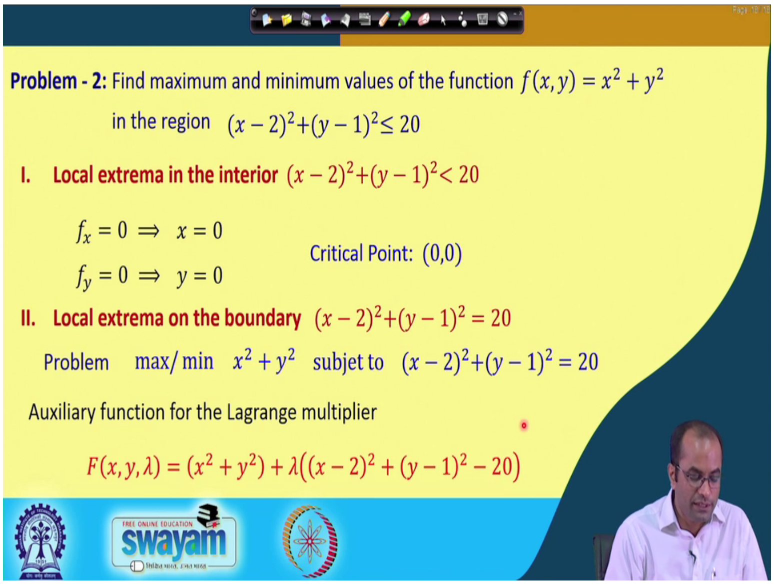

Next problem so, we want to find the maximum and the minimum value of this function x square plus y square and in the region x minus 2 whole square plus y minus 1 whole square minus a 20. So, again we have the similar problem, we have the region given here and this function x square plus y square. So, we will deal exactly in a similar fashion. We will first look for the extrema in the interior point; that means, when strictly less than 20. So, we have this open domain no boundaries and we will compute the f x is equal to 0 and which is just 2 x is equal to 0.

So, x is equal to 0 we will compute f y is equal to 0 so, we see here y is equal to 0. So, our critical point of the problem here at 0 the only point which falls inside the domain. So, we have this point 0 0 in the domain itself. So, we will consider, now for this we will find the value of the function and then that will tell us whether the function has the local minimum or the rather minimum or the global minimum at this point.

So, the local extremum on the boundary so now we will consider the boundary points that x square x minus 2 whole square y minus 1 whole square is equal to 20. So, this with equality and now we have to use the method of Lagrange multiplier. So, the problem is that we want to minimize maximize this function subject to this constraint. So, we will define this auxiliary function as usual by putting the lambda in front of this constraint there and find all the critical points by differentiating F with respect to x y and the lambda partially.

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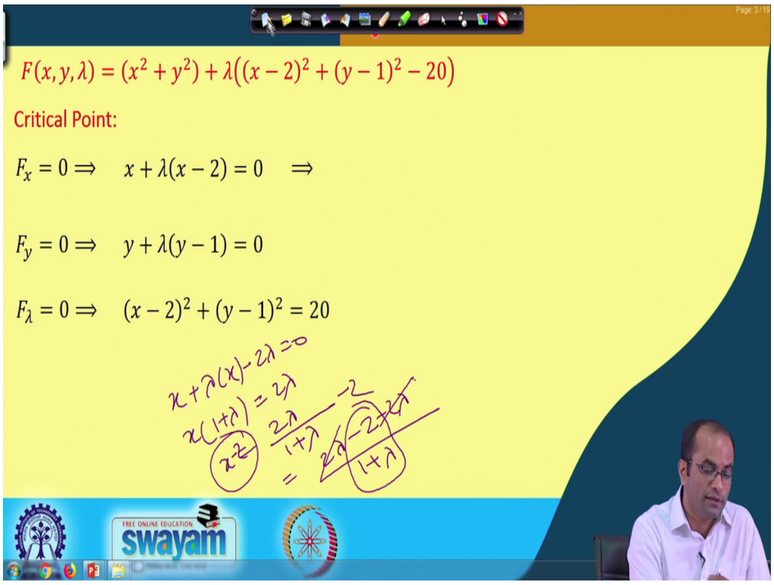

So, the critical points by differentiating F x is equal to 0 we will get this equation F y is equal to 0, we will get this equation and F y z is equal to 0 we will get this equation. So, out of these 3 equations we have to find all the critical points, from this equation first we will write down this x minus 2 in terms of lambda. From the second equation also we write y minus 1 in terms of lambda we will substitute here. So, we will get possible values of lambda. So, in this way we will solve this let us from the let us look at the first problem here. We have x and we have the lambda x and then we have minus 2 lambda is equal to 0.

So, x 1 plus lambda and then is equal to 2 lambda. So, the x is 2 lambda over 1 plus lambda. So, if we want to get x minus 2 so, x minus 2 and then minus 2 there. So, we will get 2 lambda minus 2 minus 2 lambda over a 1 plus lambda. So, in this case this is minus 2 over 1 plus lambda is the value of x minus 2 from this equation number 1.

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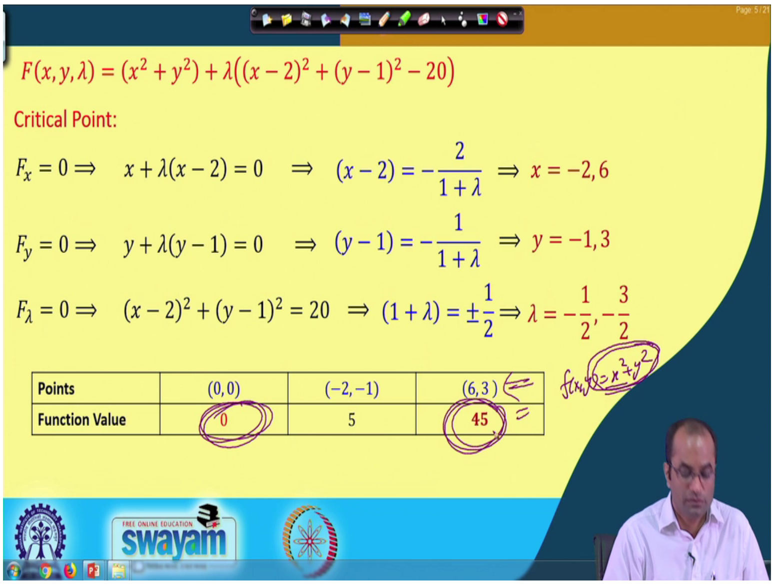

So, here we have x minus 2 minus 2 over 1 plus lambda. Similarly, we can get out of this equation y minus 1 which will come as minus 1 over 1 plus lambda. And, from the third equation now we can substitute this x minus 2 and y minus 1 here and we will get a equation in lambda which can be easily solved. So, we have 1 plus lambda is equal to a plus minus half. So, from here we got basically lambda; once we have the lambda here we can get x and then corresponding y from this first and the second equations.

So, from here we get lambda minus half and minus 3 by 2, these are the possibilities which will finally satisfy this equation. From this equation when this lambda is minus half we will get minus 1, when lambdas minus 3 by 2 we will get 3 and from the first equation we will get x minus 2 and 6. So, we have the solution of this problem. So, minus 2 minus 1 and minus 1 by 2 is one point another one is 6 3 and minus 3 by 2.

So, naturally we will take just the x y points, because we want to compute the value of the function at the x y point. So, we have 3 points 1 was in the interior that is a 0 0 point and 2 points we got here minus 2 minus 1 and 6 comma 3. These are the 3 points we will evaluate the function value. So, at the 0 0 the function value is 0 minus 2 minus 1 it is 5 and the 6 3 if we evaluate that x square plus y square. So, the function was the function was f x y is equal to x square plus y square.

So, here it is 5 and then here 6 square 36 plus 9 45. So, we have these functions values and we realize now here this is the minimum value of the function takes which is naturally true and directly we can get from the function itself where, the function is x square plus y square. So, the minimum value will be a 0, because it has to be greater than equal to 0. So, the minimum at 0 0 will be 0 and the maximum value here is attained at the point 6 3 and the value of the function is 6 45. So, we have the minimum value 0 and the maximum value of this problem is 35.

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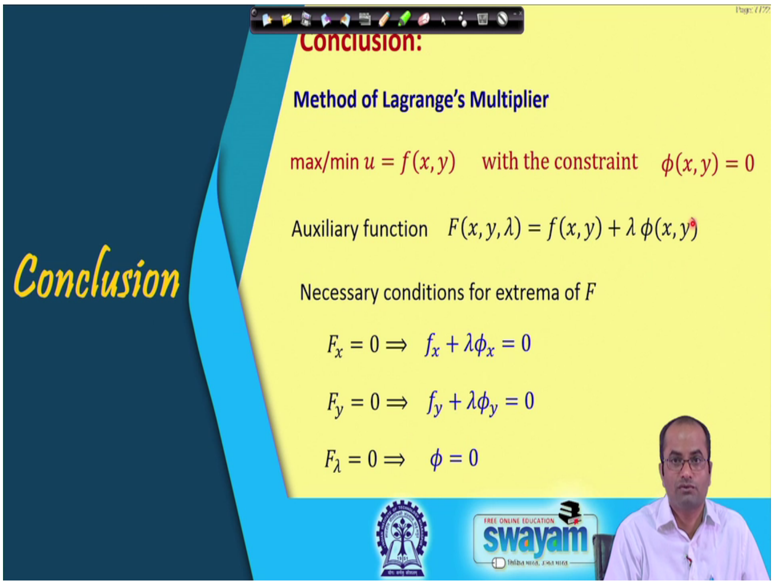

Conclusion now: so, the method of Lagrange multiplier is that we want to find the maximum or minimum of a function here f x y with the sum given constraint. The idea is that to formulate an auxiliary function here by introducing this lambda in front of this phi. And, then the necessary condition for extrema of this F will be the setting F x is equal to 0 and then which gives us this equation F y is equal to 0 we get another equation and F lambda is equal to z we get the phi x y is equal to 0.

So, out of these 3 equations we have to find all the points meaning this x y and lambda. So, all possible values of x y lambda we have to see which satisfy all these 3 equations and then at all those points we have to compute the function value f x y and see where the function is attaining its maximum and its minimum. So, in this way we can find the global maximum and minimum of the problem.

But, we have to be careful that the problem when we have that close and the boundary domain we have to look inside the domain. Because, there may be some candidates where the function may take maximum or minimum value and there may be the points on the boundaries where the function may take its extrema.

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So, these are the references we have used for preparing these lectures.

Thank you very much.