Engineering Mathematics – I

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Lecture – 02
Mean Value Theorem

Hi. So, welcome to the second lecture on Engineering Mathematics – I and today, we will discuss Mean Value Theorems.

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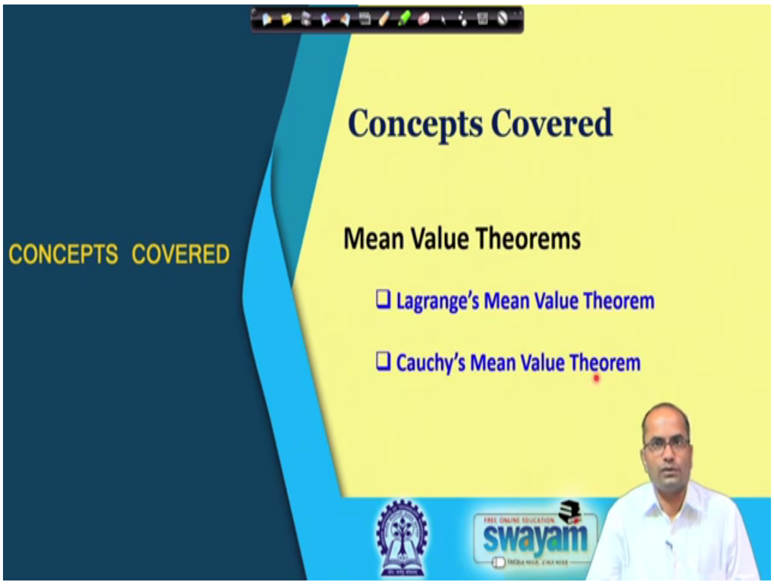

So, let us go through the concepts covered. So, we will discuss the Lagrange mean value theorem; a very important concept which is the extension of the previous lecture where we have a studied Rolle's theorem and there is another generalized a mean value theorem or the Cauchy mean value theorem which will be also discussed in today's lecture.

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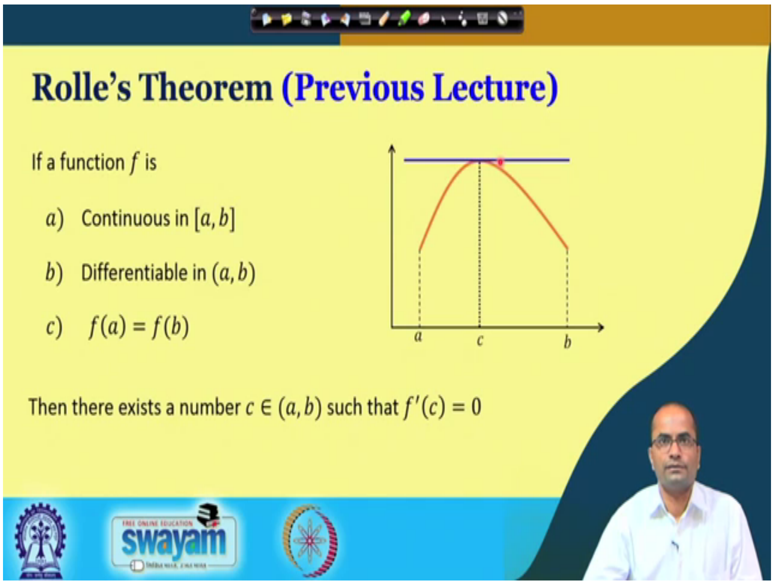

So, let me just recall from the previous lecture. So, if a function f is continuous in a closed interval a b and differentiable in open interval a, b and the function value at the point a and the point b. So, the end points of the interval is equal, then there exist a number c in the open interval a, b such that the derivative vanishes at this point.

The geometrical interpretation is as clear from this figure. So, we have a function f which is continuous and differentiable and the function value at a and b both are equal. Then, there exist a point c here where the tangent is parallel to the x axis.

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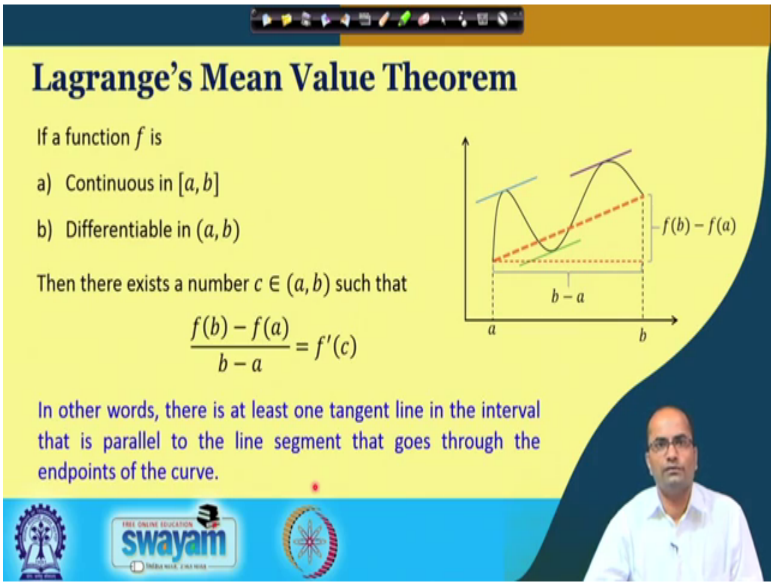

So, now coming to the Lagrange mean value theorem we have the function f which is continuous similar to the previous conditions of the Rolle's theorem and differentiable in the open interval a, b.

The third condition where the function was equal at the two end points is not required here. So, it is more general and less restrictive and in that case again there exist at least one number c in the open interval a, b such that this quotient here f b minus f a over b minus a is equal to the derivative at a point c. So, let us first discuss the geometrical interpretation of this Lagrange mean value theorem.

So, if we have a function which is continuous and differentiable in some interval a, b and then let us take a look what is this quotient here. So, if you join these two points f at a and f at b then we get this line segment. So, what is the slope of this line segment? Let us compute.

So, in this case if I draw this triangle here the height here will be f b because the distance from here to this point is f b and the distance from this point to this point here is f a. So, f b minus f a is this height of this triangle and the base here is b minus a, because up to this point is b and up to this point here the co ordinate of this point is a f a.

So, here this distance is b minus a and this one is f b minus f a and. So, this quotient here f b minus f a were b minus a this perpendicular divided by the space will be the tangent of this angle. So, basically this expression here f b minus f a were b minus a is the slope of this line segment which we have drawn by meeting these two end points of the a curve and now, what this theorem says that this will be equal to f prime c. So, the slope at some point in the domain a to b.

So, the geometrical meaning is that they will be at least one tangent; in this particular case we can see these three tangents which are parallel to this line segment. So, this Lagrange mean value theorem says that there will be at least one point where the tangent will be parallel to this line segment joining these two points, the end points of the curve. So, as I have written here in other words there is at least one tangent in this interval that is parallel to the line segment that goes through the end points of the curve.

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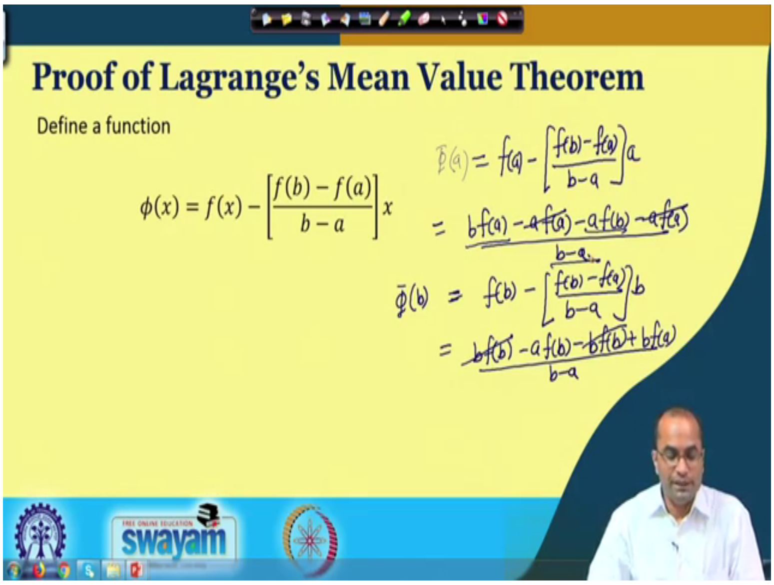

The proof is very simple if we consider this function phi x is equal to f x minus f b minus f a over b minus a and x. So, if you take a close look at this function it is a basically difference of two functions f x and minus some constant times x. So, if f is continuous in the closed interval a, b and differentiable in the open interval a, b and x is also a function which is continuous and differentiable in those intervals, then this difference will be also continuous in closed interval a, b and differentiable in the open interval a, b.

So, for the setting of this function is done because if you compute here for example, the phi at the point a and phi at the point b then we will realize that these two values are also equal. So, phi at a is nothing, but f a minus the f b minus f a and divided by this b minus a and at point a so, here would be a.

So, this if I simplify then and this will become b f a minus a f a and then minus a f b and minus a f a and divided by this b minus a. So, this a f a will get cancelled and then we will get b f a minus a f b over b minus a and now, if I compute here f b. So, here you have then f b minus this course in f b minus f a and divided by this b minus a and then here b.

So, if I simplify now this so, b f b minus a f b minus b f b and plus b f a divided by this b minus a. So, in this case this b f b gets cancel and we get b f a minus a f b over b minus a. In the earliest case also we got b f a minus a f b over b minus a. So, the function is taking same value at a and b and if we recall again the condition was for Rolle's theorem other than the continuity and differentiability that the function should be having the same value at the two end points. So, in this case this function f satisfies all the conditions of the Rolle's theorem.

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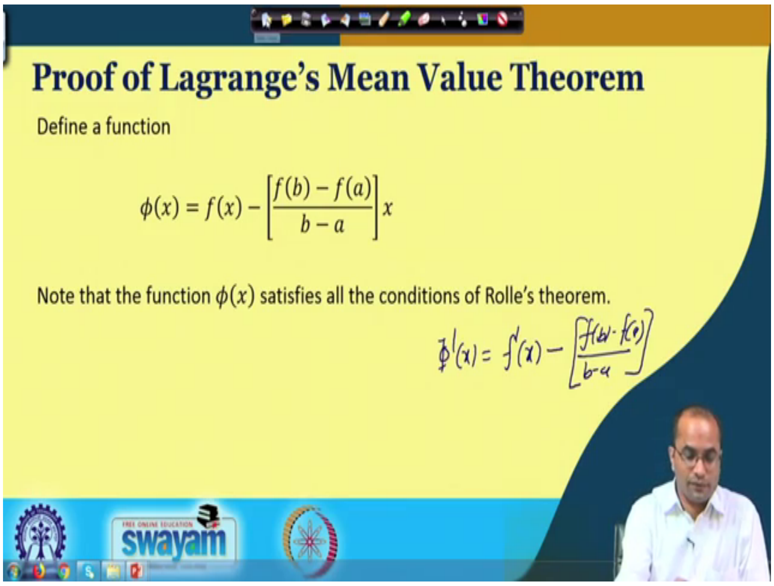

And therefore, we can apply the Rolle's theorem to this function phi x. So, what will now give us if we take the derivative here the phi prime x is equal to the derivative of x minus this is a constant. So, here f b minus f a and divided by this b minus a and the derivative of x will be 1. And the Rolle's theorem says that the there will be a point where the function will be the derivative of the function will be 0.

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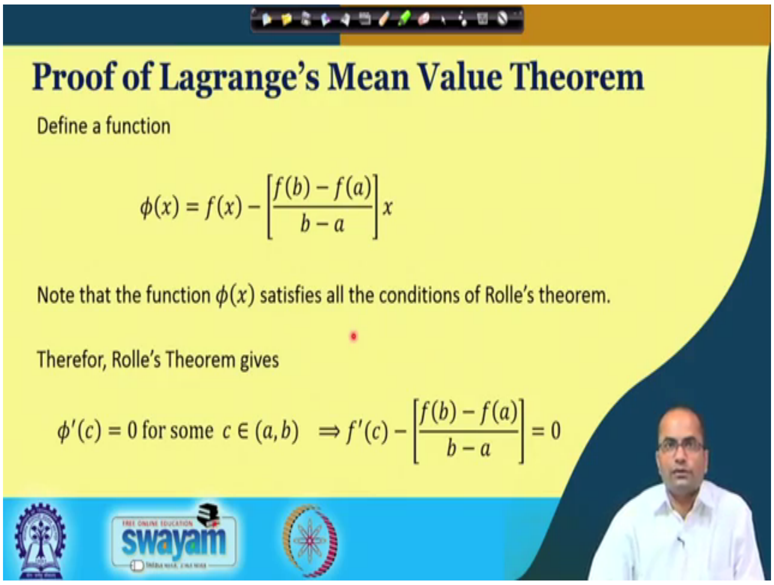

So, in this case now if we apply the Rolle's theorem then phi prime c will be 0 and for some c in the interval in the open interval a, b and which implies precisely that this is 0 and that is the Lagrange mean value theorem that f prime c will be equal to f b minus f a over b minus a is equal to 0.

So, the construction of this function here was important to prove the Lagrange mean value theorem and this phi here satisfy all the properties of the Rolle's theorem and we can apply the Rolle's theorem to this function and we got the desired result of the Lagrange mean value theorem.

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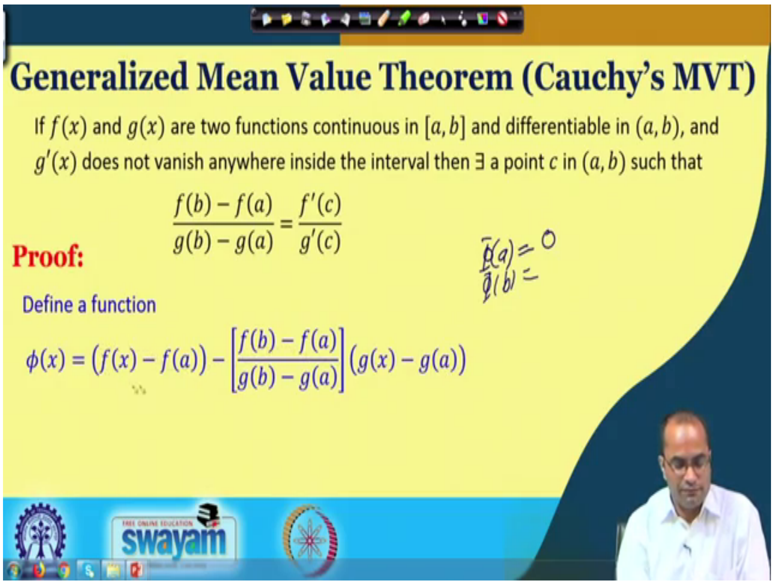

So, there is another one the generalized mean value theorem which is also called the Cauchy mean value theorem. So, here we will consider two functions instead of one. So, if f x and g x are two functions continuous in closed interval a, b and differentiable in open interval a, b and there is another condition on g that g prime the derivative of g does not vanish anywhere inside the interval then there exist a point c in the open interval a, b such that this Cauchy theorem f b minus f a over f g b minus g a is equal to the ratio of the derivative of this f and g at the point c.

So, the proof is again pretty similar to the earlier proof of the Lagrange mean value theorem and in this case we set this function or define a function in such a way that this phi x is equal to f x minus f a minus this quotient here which will be coming in the result of this Cauchy mean value theorem and multiplied by g x minus g a. So, again the similar argument since f and g they are continuous in closed interval and differentiable in the open interval a, b.

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So, the phi is also differentiable and continuous in the given intervals. Moreover if we see hear that what is the phi at a, that is f a minus f a here this is 0 and g a minus g a is also 0. So, everything is 0. So, the phi a is 0 and the phi b which is f b minus f a and g b minus g a. So, this g b minus g a will get cancel with this g b minus g a and then we will get f b minus f a minus this f b minus f a which is again 0.

So, in this case the phi a is 0 and phi b is 0 and phi satisfies all the conditions of the Rolle's theorem and therefore, we can apply Rolle's theorem to this function phi x.

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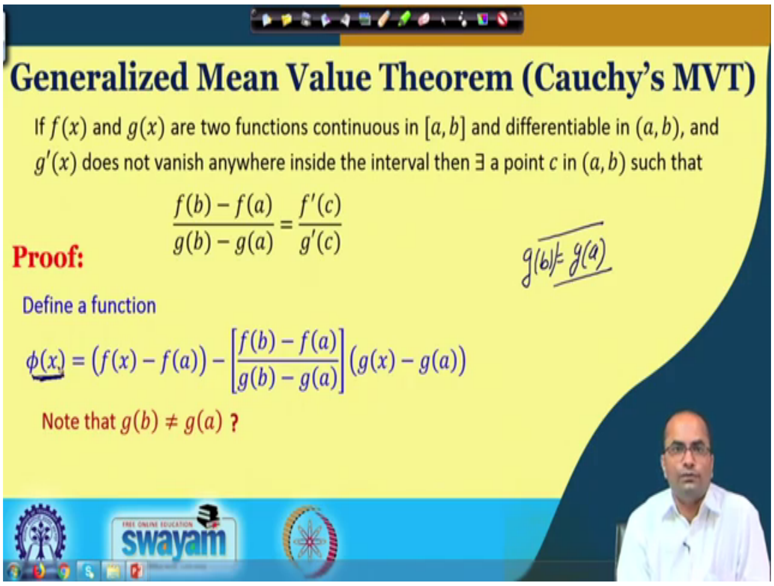

So, applying the Rolle's theorem, but before that there is a point here that we have to tell that this phi is well define because this g b minus g a should not go to 0; that means, g b should not be equal to g a. The question is why g b cannot be equal to g a? We have not made such a restriction directly in the assumptions of this Cauchy mean value theorem , but again there was an additional condition that g prime x does not vanish anywhere inside the interval.

So, if this g b is equal to g a in this case we can again apply the Rolle's theorem to the function g which will say that there will be a point c in the open interval a, b where the derivative will vanish. But, as per the assumption of the theorem g prime does not vanish anywhere inside the interval. So, this cannot be equal. So, there will be never such a situation that this g b will become equal to g a and this will become infinity.

So, the function is well defined the function is differentiable, it is continuous and g b phi a is equal to phi b. So, all the conditions of the Rolle's theorem are satisfied for this function phi x.

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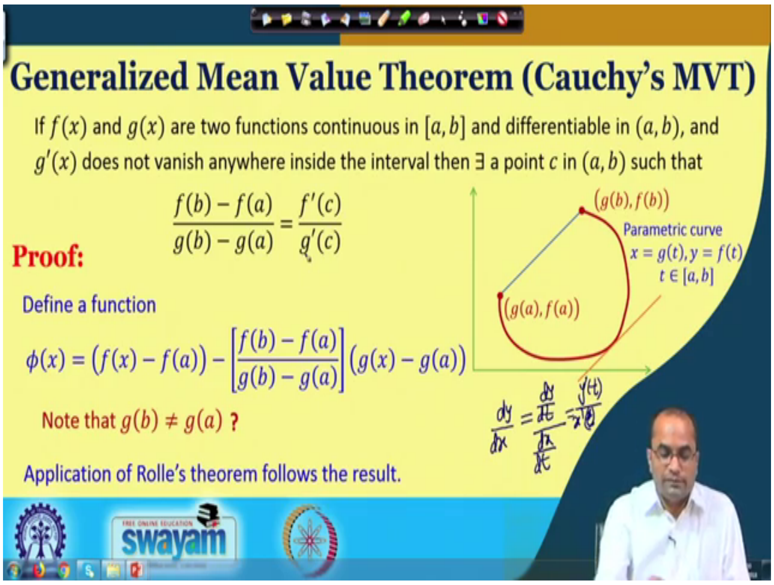

So, if we apply the Rolle's theorem now, to the function then we will get exactly the result which is given here because the phi prime x will be the derivative of f and then this is a constant here minus again this expression and the derivative of g.

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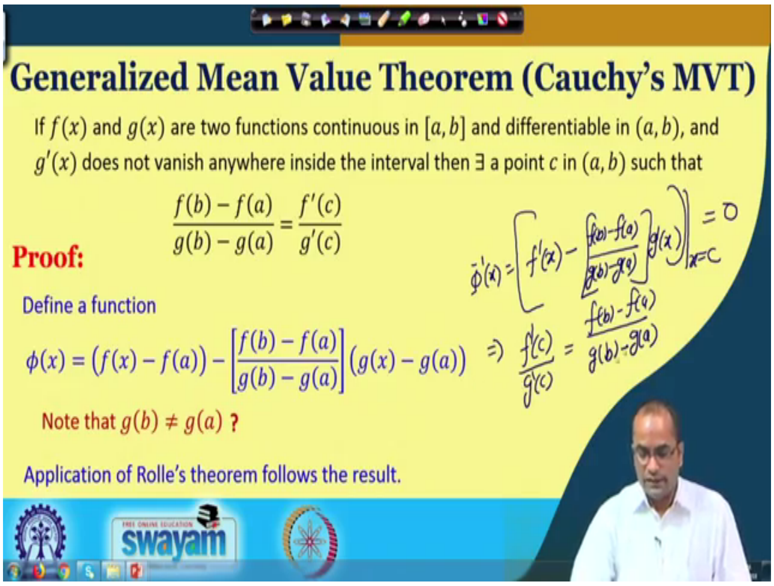

So, what will be this again? So, let me just come to this point. So, phi prime x will be the f prime x minus this 0 minus this quotient here f b minus this f a over g b minus this g a and then this g prime x and the Rolle's theorem says that at the point x is equal to c this is equal to 0.

So, what do we get then? The phi prime c and divided by this g prime c is equal to the f b minus f a over this g b and minus g a. So, that is the Cauchy mean value theorem or the generalized mean value theorem.

So, if you now we discuss the geometrical meaning or the of this Cauchy mean value theorem. So, here now we consider this parametric curve which is given by x is equal to g t; g is the function the given function there, but I have introduced this parameter t which is commonly used for the parametric curves and the y is equal to the other function f t and t varies from a to b in this close interval.

So, this parametric curve you can trace by varying the values of t. So, if for example, t is equal to a, then we have here the x co-ordinate g a and the y co-ordinate f a of this point. So, this point is g a, f a and then if we vary t we will basically move on this curve we will trace this curve and till we reach the end point here, t is equal to b which is given by g b, f b.

So, now the geometrical meaning is similar to the earlier result on Lagrange mean value theorem. So, they will if we join these two points by this line segment, then this theorem says; so, first of all this the slope of this line segment will be given by this f b minus f a over g b minus g a because of the same argument as we have discussed earlier. The height will be f b minus f a and this the base of this triangle will be g b minus g a. So, this is the slope of this line segment and then the right hand side here says that there will be at least one point on this curve where the tangent will be parallel to this quotient line.

So, if you take a close loop this f prime c over g prime c is nothing, but the slope of the tangent line at some point c, because the slope will be calculated as at some point here the dy over dx is equal to; for the parametric curve, so, this will be dy over dt and divided by dx over dt or the y prime t divided by the x prime t. And, this y is basically the f, so, here you have the f prime t over g prime t and this theorem says that there will be a point somewhere in the interval. So, t is equal to c. So, we will get this slope here of this tangent line as f prime at c divided by g prime c.

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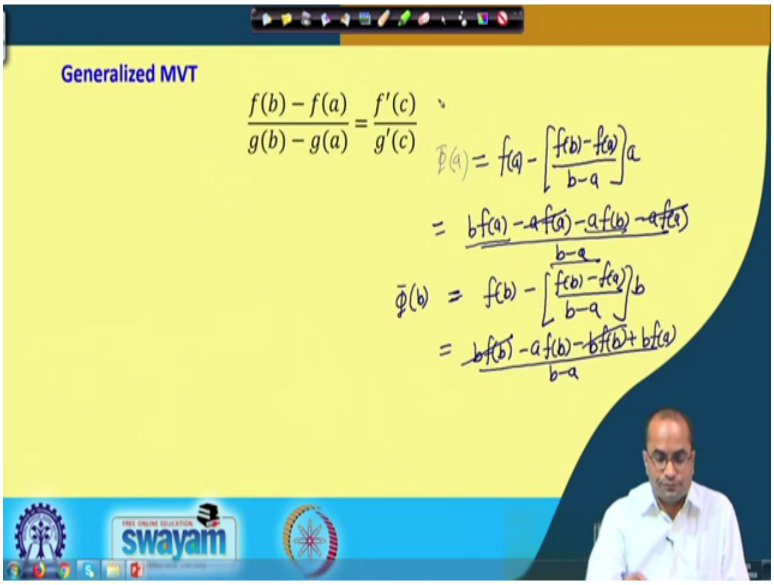

And, now let me just quickly summarize at this point what we have learnt today. So, we discuss the generalized mean value theorem which was this f b minus f a over g b minus g a is equal to f prime c over g prime c.

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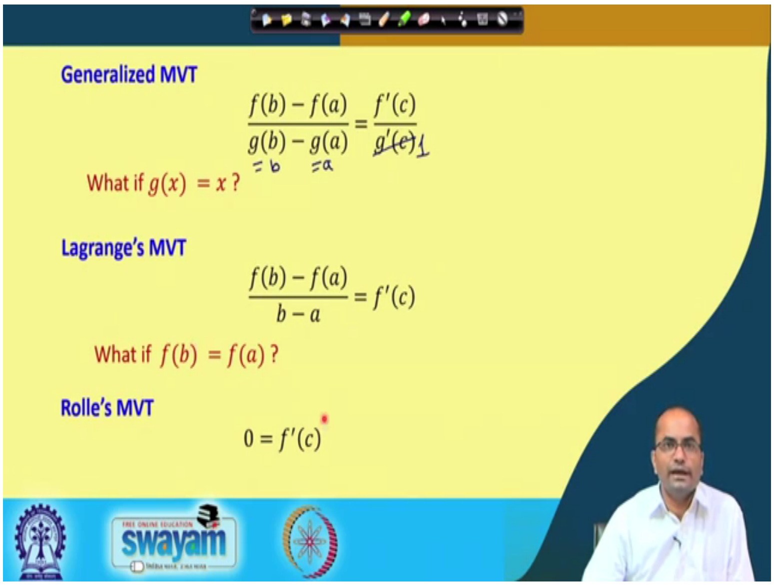

And, now what will happen if g x is equal to x. So, g x is equal to x meaning that here you have this g b will be just equal to b and this g a will be equal to a and g prime x here will be just 1. So, what do we get in this case the Lagrange mean value theorem because that conclusion will be f b minus f a or b minus a is equal to f prime c. So, in this particular case when we take g x is equal to x we will get the Lagrange mean value theorem.

And, what will happen to this Lagrange mean value theorem if we put f b is equal to f a, the additional condition what we have for the Rolle's theorem. So, f b minus f a this quantity here will become 0 and then we will get f prime c is equal to 0. So, this is the generalized mean value theorem and as a particular case if we take the function g x is equal to x we will get the Lagrange mean value theorem and again if we add another condition that f b is equal to f a we will get the Rolle's mean value theorem which is f prime c is equal to 0.

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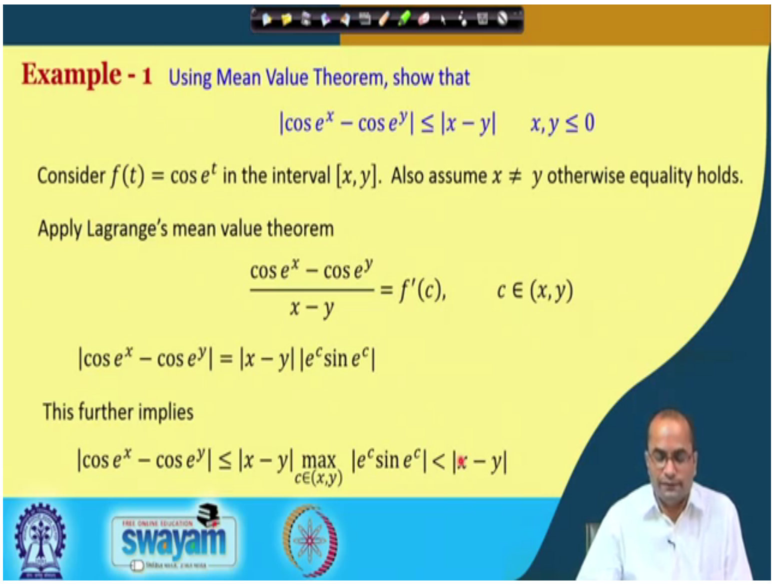

Now, we go to the we will go to this some examples the first one that using mean value theorem we will show that this inequality cos ex minus cos e power y is less than equal to x minus y for x, y less than or equal to 0. So, first we note that when both are equal x, y are equal then naturally this cos e power x minus cos e was 0 and is equal to 0. So, then in equality is naturally satisfied when x and y both are same. So, we will consider the case when they are not same.

And, now we consider the ft another function cos e power t because clearly we can see that we want to prove this cos e power x minus cos e power y using mean value theorem. So, if you consider this function ft is equal to cos e power t in this interval x, y and naturally we have assume that x is not equal to y and now, we apply the mean value theorem to this result what we will get? Cos e power x minus cos e power y x minus y is equal to there will be some point in this interval open interval x, y and the value of this quotient will be is equal to f prime c.

So, this is the Lagrange mean value theorem and now we will estimate this derivative because the derivative we can compute the f t is cos e power t. So, taking the absolute value both the sides we get this cos e power x minus cos e power y and this absolute value will take to the right hand side. So, the x minus y absolute value and the absolute value of these f prime. So, f prime is nothing, but the sin minus sin e power t into e power t. So, because of the absolute value we have not taken this minus sin into consideration. So, we have e power c sin e power c because this the derivative has to be evaluated at point c, ok.

Now, this implies, so, if I we take the maximum value of this expression here. So, the c varies from x to y. So, we have taken the c from this x to y and we will take the maximum value of this one. And, note that the c belongs to this x y open interval, so, it is basically a negative number the c is less than 0 because x and y both are less than equal to 0 and therefore, the c will be strictly less than 0 in the open interval.

So, the sin is always founded by one. So, we have less than equal to one the sin function and the e power c the exponential function for this negative argument c will be always less than 1 because e power 0 is 1 and o for all a negative values it takes value less than 1 for positive values it will take more than 1.

So, this is strictly less than 1, this is less than equal to 1. So, this expression here or the maximum value of this derivative is bounded by a strictly bounded by 1. So, we got this inequality cos e power x minus cos e power y is less than the absolute value of x minus y which we want to prove in this result.

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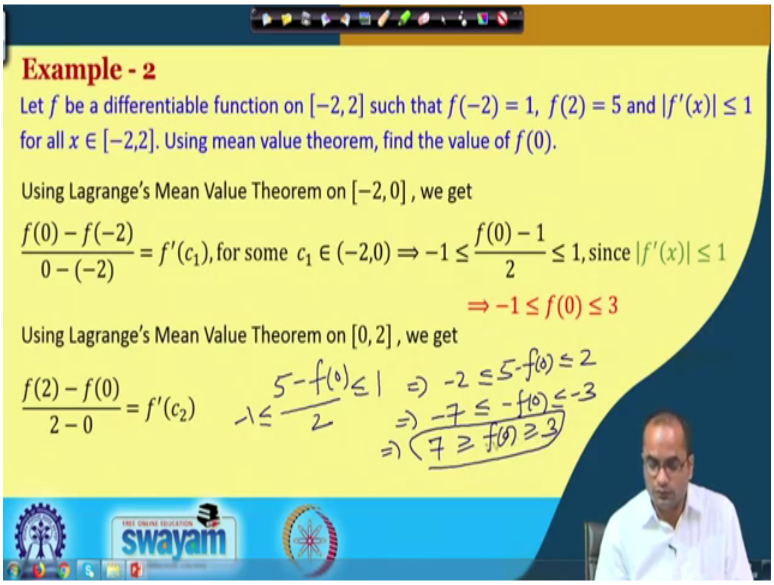

The second example we will consider that this f is the differentiable function on the closed interval minus 2 to 2 and such that the value is given as minus 2 is equal to 1, f is given as 2 and there is another information here that f prime x; f prime x is bounded by 1 for all values of x in this interval minus 2 to 2. And, using mean value theorem we want to find the value of the function at 0.

So, if we take a look at this problem and we want to find the value of f 0, so, we need to apply the mean value theorem or Lagrange mean value theorem in the interval minus 2 to 0 and 2 to 0 and then we will get some estimate on this f 0. So, if we apply the Lagrange mean value theorem on minus 2 to 0 interval what we get the f 0 minus f minus 2 divided by 0 minus 2 is equal to there will exist some c 1 in the open interval minus 2 to 0 so that this value will be equal to the derivative at that point c 1.

Now, this derivative here f prime c 1 is bounded by 1. So, we know the estimate of this f prime c 1. This is always between minus 1 and 1. So, what is this expression here? The f minus 2 is 1. So, f 0 minus 1 and divided by 2 f 0 minus 1 divided by 2 lies between minus 1 and 1 because this is equal to the derivative and the derivative is bounded by less than equal to 1 the absolute value. So, this expression here lies between minus 1 and 1.

Now, if we multiply this 2 to both the sides or that we can multiply 2 to this inequality here we will get minus 2 less than equal to f 0 minus 1, less than equal to 2 and then we can add this 1 to the inequality. So, we will get here the 3 less than equal to f 0 and less than equal to so, here minus 2 was there plus 1, so, minus 1 and then here 2 and then plus 1 we will get 3. So, out of this inequality we will get that f 0 lies between minus 1 and 3. Again if you use the Lagrange mean value theorem in the interval 0 to 2; in the interval 0 to 2 we will get f 2 minus this f 0 over 2 minus 0 is equal to the first derivative at some point c 2.

So, again here the f 2 is known the f 2 is 5; so, 5 minus f 0 over 2. So, what do we have here? We have f 2; f 2 is given as 5 and minus this f 0 divided by 2 and this value again is bounded by minus 1 and 1. So, we got this 1 here minus 2 and less than equal to of 5 minus this f 0 and minus 2. So, this implies that this minus 7 minus f 0 and this will be minus 3. So, if you multiply by minus 1 here, so, the inequality will change; so, 7 less than f 0 and then less than 3.

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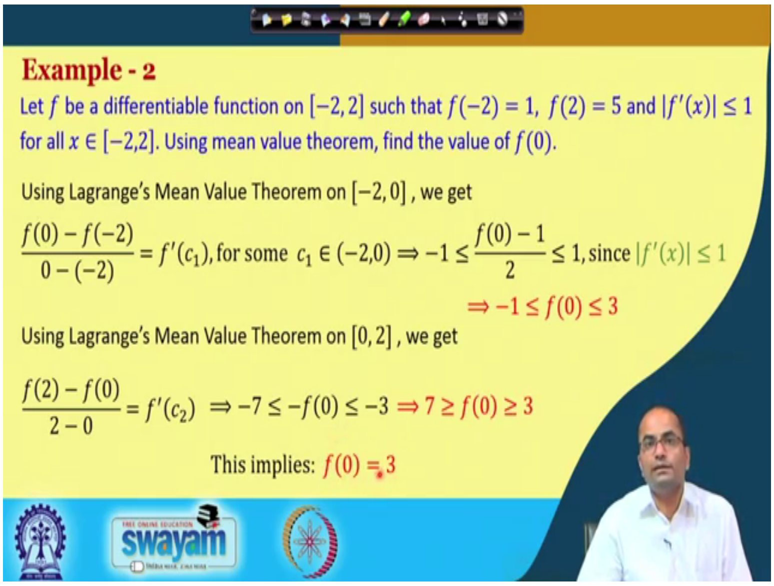

So, this inequality we will get now that f 0 is greater than equal to 3 and less than equal to 7 this one which says that the f 0 is greater than or equal to 3, but less than equal to 7. The earlier inequality says that f 0 is less than or equal to 3.

So, by these two inequalities here f 0 is less than equal to 3 and f 0 is greater than equal to 3 what we will conclude that f 0 has to be 3. So, f 0 has to be 3. So, we got the value using the mean value theorem of the function at 0 given that those derivatives and the end points value was given.

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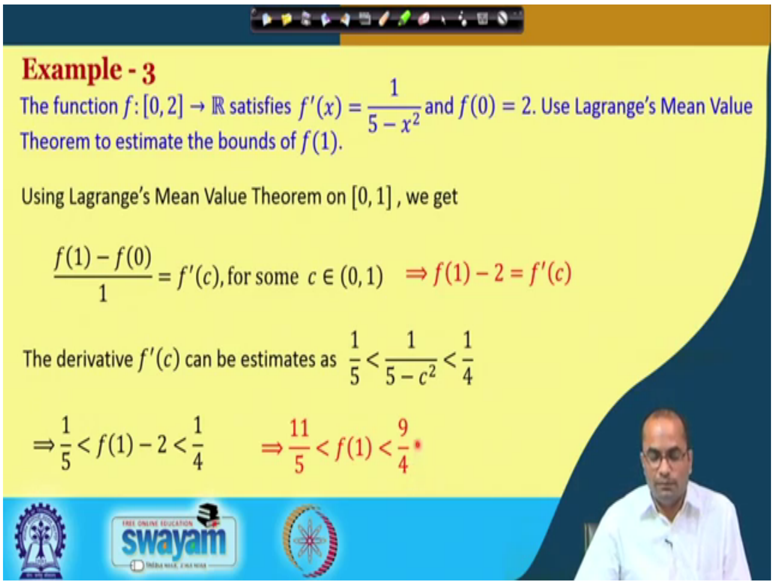

The last example here, the function f which satisfies now that the derivative is 1 over 5 minus x square and f 0 is 2. Now, we want to use the Lagrange mean value theorem to estimate the bounds on f 1. So, in this case the exact value of f 1 is not possible so, we will estimate the lower and the upper bound for f 1.

So, again if we use the Lagrange mean value theorem in the interval 0 to 1 because you want to estimate 1. So, 0 to 1 then we will get f 1 minus f 0 divided by this difference 1 and there will be some point whose value will be equal to the derivative at that point. So, out of this inequality e f 1 minus this f 0 is 2 is equal to f prime c. Now, what is the derivative? Derivative is 1 over 5 minus x square. So, f prime c is 1 over 5 minus c square.

Now, just note that c here is between 0 and 1. So, the lower bound of this 1 over 5 minus c square will be obtained when we then the c this c approaches to 0; that means, this value is always greater than 1 over 5 and when the c approaches to this maximum value in the interval as 1 in that case this will become 4, and the maximum value of this 1 over 5 minus c square will be 1 by 4. So, now we know that the derivative lies between 1 by 5 and 1 by 4.

So, what is the derivative f prime c is here? f 1 minus 2. So, with this we got the inequality that f 1 minus 2 lies between 1 by 5 and 1 by 4 and this implies. So, to we can take to the other side and also it has to be added to the right side here. So, this we will get

11 by 5 less then f 1 and then if we add 2 here this will be 9 by 4. So, we got the estimate on f 1 that it lies between 11 by 5 and 9 by 4.

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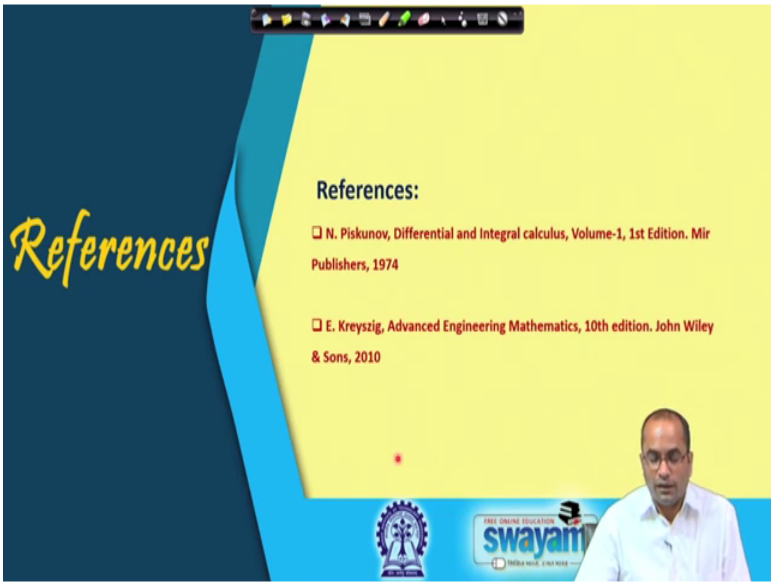

So, these are the references we used here. The Piskunov, Differential and Integral Calculus and Kreyszig, Advanced Engineering Mathematics.

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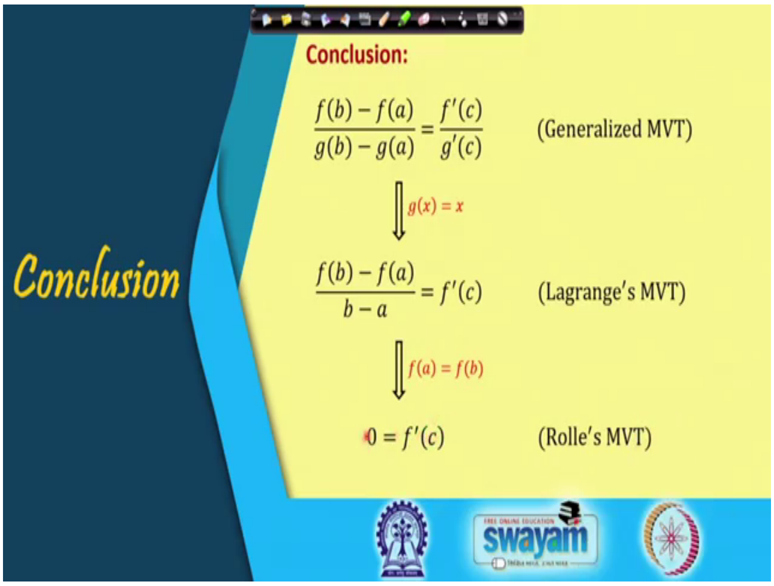

So, the conclusion for today's lecture that we have learnt the generalize mean value theorem and as a special case when we substitute this g x the other function the second function as x, we will get the Lagrange mean value theorem and if we take another assumption that f a is equal to f v then this will be the Rolle's theorem which says that f prime c is equal to 0, ok.

Thank you, for today's lecture.