Engineering Mathematics – I
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Lecture – 19
Maxima & Minima of Functions of Two Variables (Contd.)

So, welcome back to the lectures on Engineering Mathematics I and this is lecture number 19. And, today we will continue our discussion on Maxima and Minima of Functions of Two Variables.

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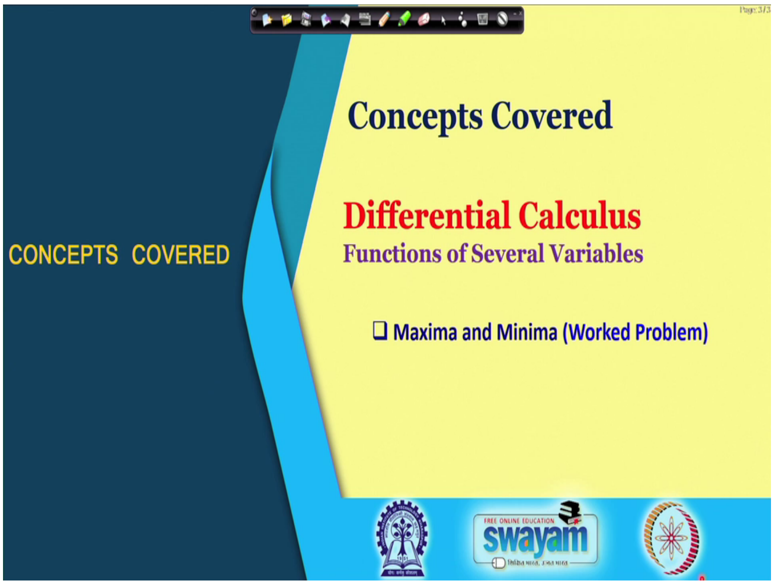

And in particular today we will see some typical problems where, we will apply the idea which was discussed already in previous lectures.

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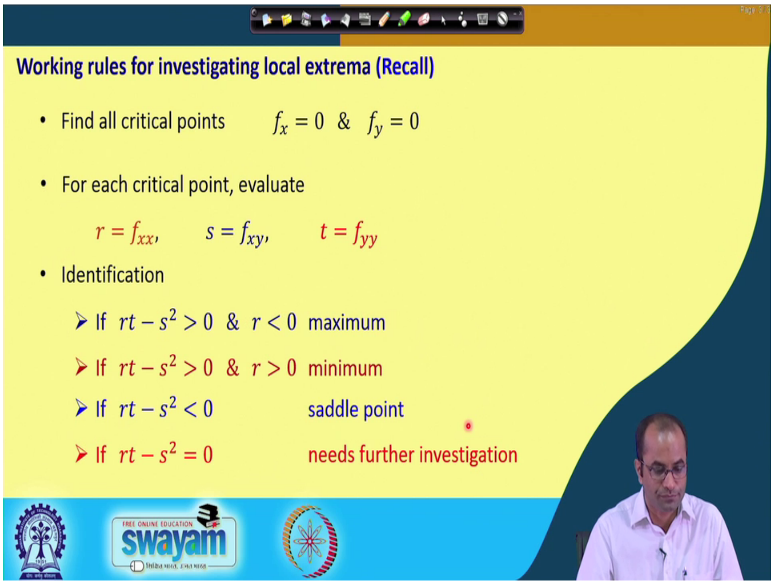

So, just to recall in the last lecture, we have investigated the local extrema and that was the sufficient conditions and necessary conditions. So, we need to find first all the critical points and those critical points we will get by solving these equations. So, the partial derivative of f with respect to x will be set to 0 and partial derivative of f with respect to y will be set to 0 and then we will solve these 2 equations to get all the points which satisfy these equations. So, those will be the critical points and then for each critical point, we will evaluate this is the notation r we have used for the second derivative with respect to x at each critical point and then s here the mixed derivative and t which is the 2 times y derivative of this a function f.

And then for the identification we have realized that if this rt minus s square, so rt and minus s square, this is positive and this r is negative, then we have the point of a maximum. And if at a point if we have again this rt minus s square positive and r is positive, then this is a point of minimum. Similarly when this is negative rt minus s square then, this comes out to be a saddle point and if this rt minus s square is 0 then, this test of the second derivatives this fails and we need to go for further investigation by some other ways.

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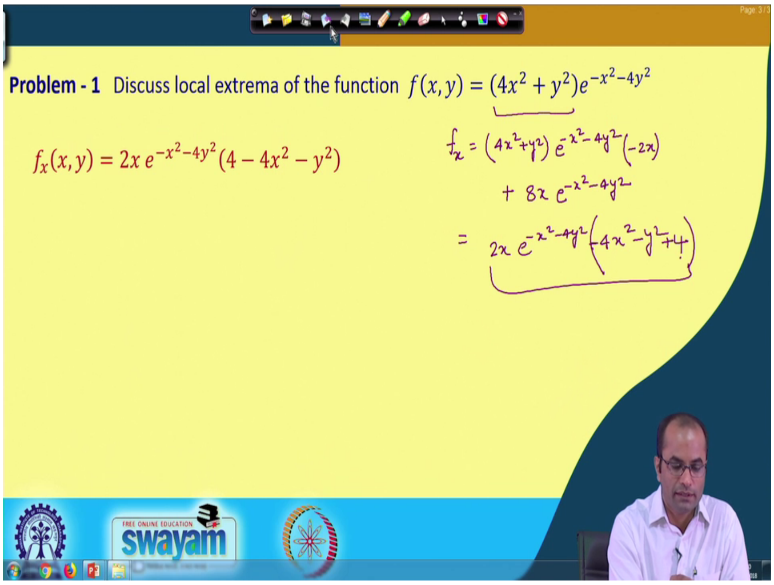

So, let us discuss the problem here. So, we want to find the local extrema of the function f x y is equal to 4 x square plus y square multiplied by e power minus x square minus 4 y square. So, we need to compute the fx. So, fx is simply if we differentiate this with respect to x keeping y constant, so we will have this first term let us keep it as it is it is a product rule and then here we will differentiate. So, e power minus x square and minus 4 y square and then the derivative of this minus x square minus 4 y square, which will be minus 2 x with respect to x and then plus.

So, here this is 8 x the derivative of this first term 8 x and then here e power minus x square minus 4 y square. And then this term we can take common with this x here; in fact, this 2 x we can take common and e power minus x square minus 4 y square, then from this first term, we will get this 4 x square and minus y square. So, with minus sign because there was minus there and then here we have taken already 2 times x and this exponential function, so we will get simply 4 here. So, that is the first derivative of this function with respect to x which is written here. So, 2 x and this exponential function 4 minus 4 x square minus y square.

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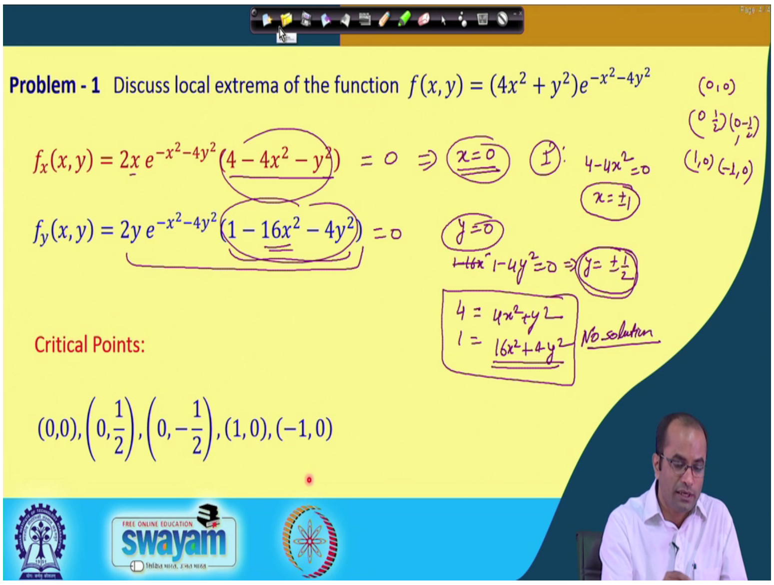

Similarly, we can get the first order derivative with respect to y. So, in now we will differentiate here with respect to y and again the product rule will be applicable. So, we will get this 2 times y now and here the same exponential function and that the extra term one minus 16 x square minus 4 y square.

And the critical points we can now get by solving these 2 equations. So, if we set these equations to 0, in that case we will get from this first equation because exponential cannot be 0. So, either x has to be 0 or this term in this bracket has to be 0. So, let us assume that x is 0. So, when x is 0 fx at whatever point along this x is equal to 0, for whatever y this will be 0. So, we have x is equal to 0 there which can set to this fx 0 and now when x is 0, so from here we get 2 possibilities to make this term 0; either y will be 0, in that case also this term will be 0 or here when we set x to 0 we are getting this 1 minus 16 x square c 1 minus 4 y square is equal to 0 because x is said to be 0 here.

So, we have 1 minus 4 y square 0, where we will get 1 is equal to plus minus 1 over 2. So, in this case, when we set x 0 there we have 2 possibilities from the second equation to set to 0; either y is equal to 0, we will make this 0 or along with this x is equal to 0, if we take y is equal to plus half or minus half that will also be 0. So, basically we are getting the 2 3 points in fact, so was a 0 0 is 1 point and then 0 plus half another point and 0 minus half another point.

So, there are 3 points here which can make this fx and fy both 0 with this possibilities and now we have another one. So, if y is 0 from the second equation which is making this derivative 0. From the first now we can get when we set y to 0 here, so we will have 4 minus this 4 x square is equal to 0, so we will get x is equal to plus minus 1. So, these are the other points with y 0, so we have plus 1 0 and minus 1 0. So, on other points will be 1 0 and then we will have a minus 1 and 0.

And now if we try to make this and this term 0, so what we will get from the first equation, we are getting 4 is equal to 4 x square plus y square, well from the second equation we are getting 16 x square and plus 4 y square correct, 16 x square plus 4 y square. So, if we take a close look, so here if we take this 4 common, so will be 1 by 4 there in the right hand side, this 4 x square plus y square. So, we cannot get any solution out of these 2 equations because first equation says 4 x square plus y square is equal to 4 while the second says that 4 x square plus y square will be 1 by 4.

So, we cannot get a solution out of this system. So, we got all these points one was 0 0 and 0 plus minus half and plus minus 1 0. So, at all these points we need to further investigate. So, as discussed, so we have these 1 2 3 4 5 these 5 critical points or the points where both the derivatives vanished and now we have to discuss for each critical point that whether it is a point of local minimum or it is a point of local maximum or it is a critical point. So, that there we have to use the sufficient conditions that were discussed before so, moving a next now.

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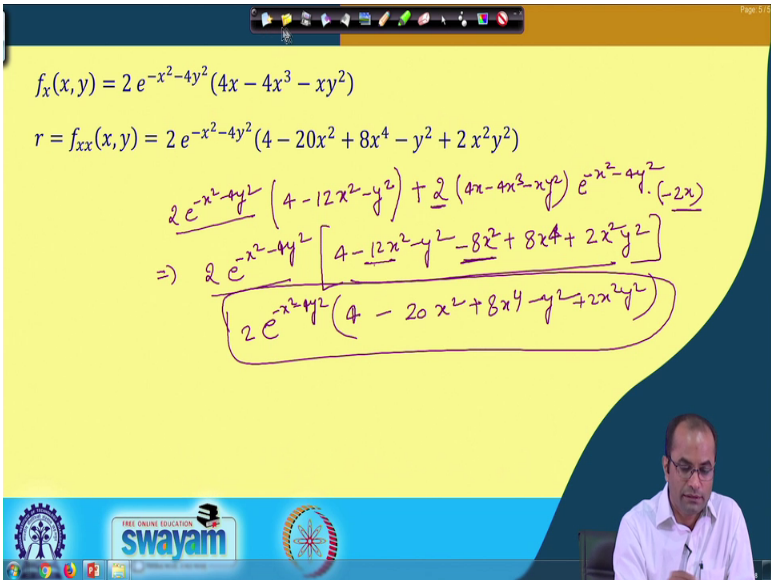

So, we have this f x and we need to get the second order derivative to further investigate these points for the extrema. So, to again we have to get the derivative of this function, so once again we have to differentiate this with respect to x and then we can get this term because there will be now many terms. So, if we can I will show you just for this one the next we will directly write. So, here again the product rule will be applicable.

So, we will do here 2 times e power minus this x square minus 4 square, we will take as the first function and then the derivative of the second with respect to x. So, that will be 4 and then we will have minus here a 12 x square and then here minus y square, so with respect to x and then plus, so here this 2 times. Again this term will remain 4 x minus 4 x cube and minus xy square and we will differentiate this first term, so which will be e power minus x square minus 4 y square and then minus 2 x term when we differentiate this minus x square minus 4 y square.

So, in this case we will take this common 2 e power minus x square and minus 4 y square, from here we have 4 minus 12 x square minus y square, here we have taken this common here 2 times the x. So, we will get this minus 2 minus 2. So, here this is 4, so minus 8 x then, we will have here 8 x cube and then we will have this plus 2 x y square and this is already taken care well.

So, here the x was there, so we have x square term and then here since this is x, so x 4 will be there and this x will make this x square y square. So, this is the term here, we have the 4, then we have a x square term. So, 8 x square and then this is 12 x square, so which makes 20 x square then we have x 4. So, 8 x 4, we have minus y square and then we have this 2 x square y square term along with this 2 e minus x square and minus 4 y square. So, this is the derivative, the second order derivative with respect to x, which is already written there. And then we need to compute the other derivatives as well, so the mixed order derivative; so, with respect to x and y and then also the derivative with respect to y 2 times.

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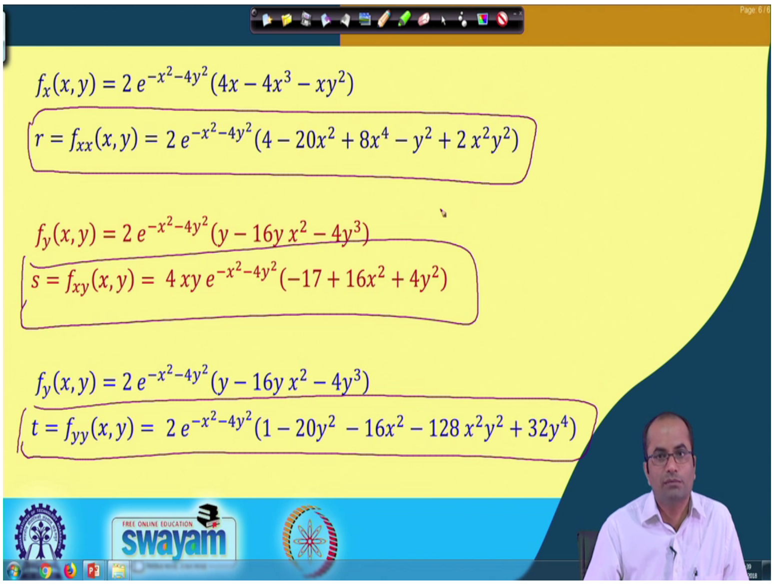

So, here we have the f y from the previous slide and then we can differentiate this with respect to x, here with the idea which we have just discussed above. So, we will get this term here and then we need to differentiate this once again with respect to y to get the f yy the second order derivative with respect to y. So, this is f y and when we differentiate this to get this t we will get this term now with exponential function and this expression 1 minus 20 y square minus 16 x square 128 x square y square plus 32 y 4.

So, we have 3 equations now, this is for 3 expressions, so here for the r and then we have the s the second order derivative the mixed derivative and then we have the second order derivative with respect to t. So, with the help of these derivatives we will investigate further whether this point is a point of maximum or it is a point of minimum or it is a saddle point.

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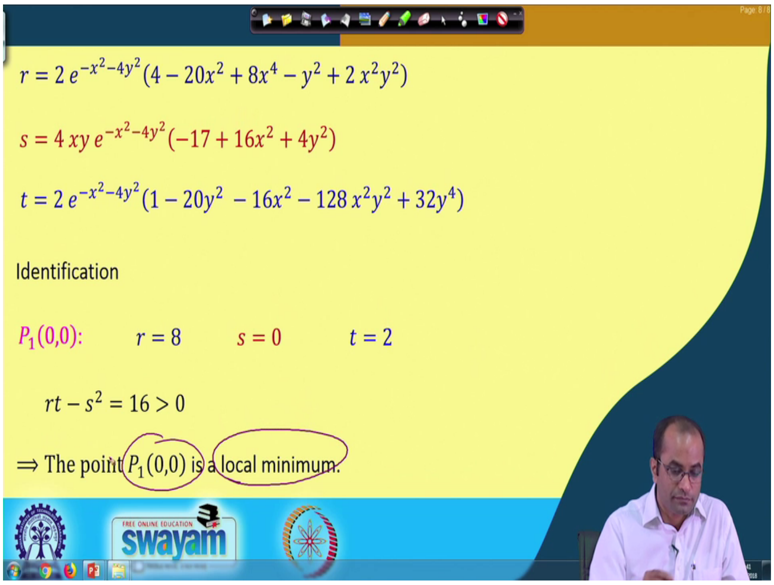

So, now as discussed we have this rs and t these 3 expressions, so we will now identify these points. So, the first point remember it was 0 0. So, at this 0 0 point, we will compute rs and t. So for r, this is when x and y both have 0. So, we have 0 term there, so we have 4, this e power 0 this becomes 1. So, we have 4 into 2 this is 8, so r is 8 and this when 0 0, we substitute here in s, so this these terms are 0 now, so we have minus 17, but here the xy is sitting in the product, so the s will become 0 and in this case again you will have some non-zero numbers. So, all these terms will be 0, we have 1 there and 2 multiplied by 1. So, this will be 2.

So, we have at this point r is 8, s is 0 and t is 2 at the 0.00, this we have taken the first critical point among all these critical because, for each we have to identify whether it is a point of a maximum, minimum or a saddle point. So, this point we have computed all these higher order derivatives and then we have to compute rt minus s square. So, r and t this product is 16 minus s square 0. So, we have the 16, which is positive. And remember in the sufficient conditions we have seen that if rt minus s square is positive, when r is positive then this is a point of local minimum. So, we got this point of local minimum. So, 0 0, the function has a local minimum and now we will consider the other points.

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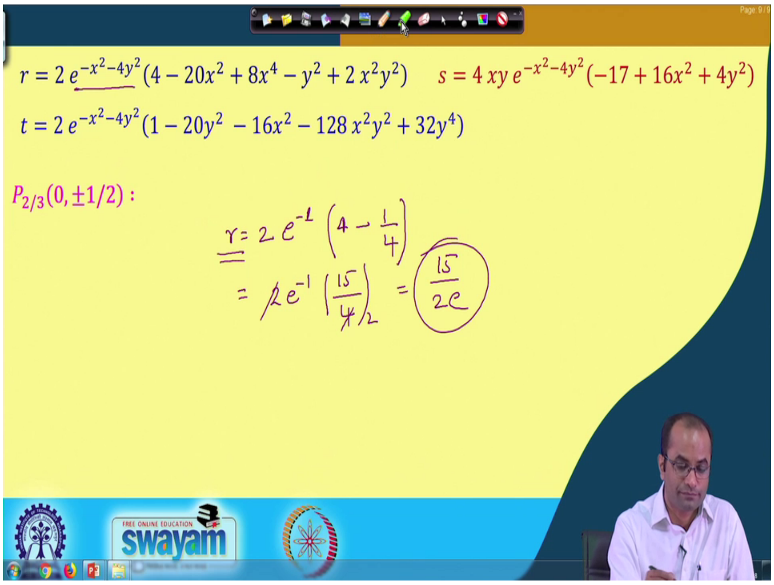

So, for the second point this was second and third points, so 0 plus half and 0 minus half. Since this y appears in all these expressions in the even power. So, we can deal this together because, so whether we take the plus sign there or the minus sign the y power even will have the same number.

So, we can consider this to these 2 points together, so we are now discussing point number 2 and point number 3, so 0 and plus minus half. So, in this case we need to again compute this r, so x is 0, so these terms, so what we will get here 2 times. So, r will be 2 times and then we have exponential function x is 0 and y square is 1 by 4.

So, here you will get e power minus 1 and then here we have 4 then minus this is 0, here also 0 x is there only this y square term will survive. So, here we will have 1 over 4 and then here again x that will become 0. So, what do we get here, e power minus 1 and then here 15 over this 4. So, this 2 also gets cancelled, so we get 15 over 2 e, it is 15 over 2 e that is a value of r at this point.

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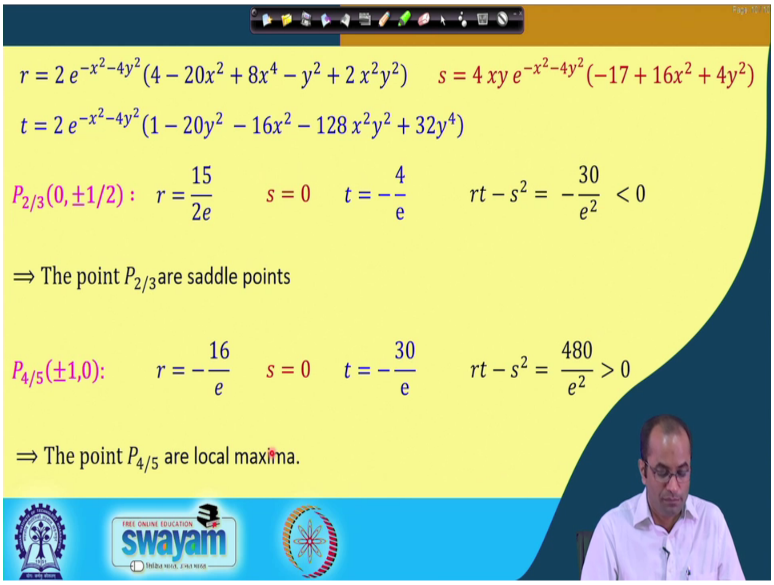

So, that is a 15 over 2 e similarly, we can compute now this s and also t just by substituting this 0 and plus minus half. So, when we substitute in this s, since there is a term x and y here in the product when x is 0 the whole term will become 0. So, the s is the straightforward 0 and the t again we need to substitute this y square here and all these terms these 2 terms have x, so they will vanish and here we have this y 4, so these terms, when we solve this we will get this minus 4 over e and this rt minus s square.

So, rt when we make this product, so since the minus sign is there with the t we will get this minus thirty by e square. So, this time now this rt minus s square is negative. So, negative means, as per the sufficient conditions now, this will be a saddle point. So, these point number 2 and point 3, 0 plus half and 0 minus half they both are the saddle points, with our sufficient conditions we are able to identify that these 2 points are the saddle points.

Now, the last here the 4 5 we have plus 1 and 0 and minus 1 0 these 2 points again we can deal together because the x also appears in the power either square or we have the 4. So, this value whether we take plus and minus will remain the same. So, when we compute r, so we need to substitute y to 0 here. So, these 2 terms will go to 0 and then we have here again this e power this is 0 and then you have 1 there, so e power minus 1, again the same, so which will come in the denominator and then after simplifying this we will get 16.

So, here r is minus 16 over e square and the s is 0 because of here y is there in the expression. So, we will get 0 and t will be minus 30 over e from this expression here. So, again this rt minus s square we have to compute and in this case, so rt here we get this positive again. So, 480 over e square and now based on this sign of r, we can decide whether this is a point of maximum or minimum. So, in this case the r is negative and remember when r t minus s square is positive and r is negative then we have a point of maximum. So, in this case these 2 points plus 1 0 and minus 1 0 these are the points of local maximum.

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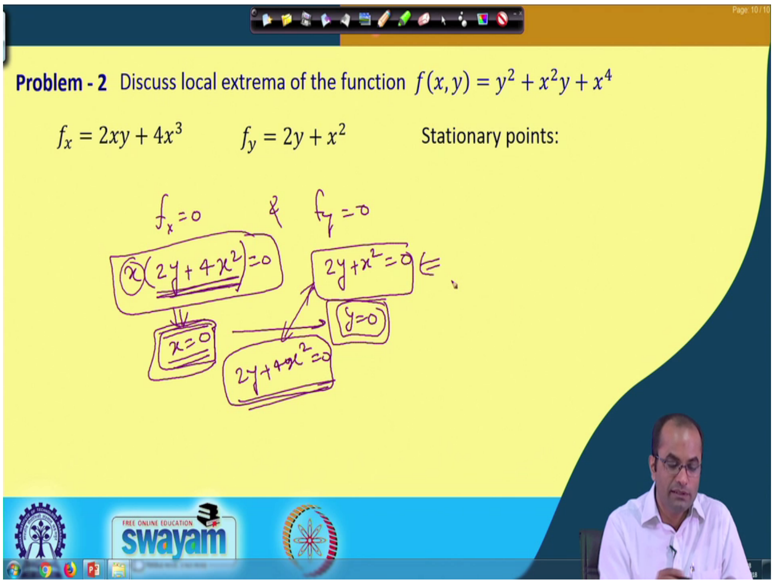

And now we move to the next problem which is the function here f x y is equal to y square plus x square y and plus x 4 we want to discuss 4 local extrema. So, in this case we will again compute the first order partial derivative which is f x is equal to here 2 xy and 4 x cube, will come and then we have the partial derivative with respect to y. So, here 2 y and plus x square, so we have the partial derivative and to get the stationary points, so we need to solve these 2 equations; that means, the f x is equal to 0 and this f y is equal to 0.

So, from here we have this x into 2 y plus this 4 x square and f y is equal to 0, we will get when we set to zero; that means, 2 y and plus x square is equal to 0. So, out of these 2 equations we need to get all the points which satisfy these 2 equations. So, from the first immediately we see that x equal to 0 satisfies at least there is a point here x is equal to 0 which satisfies this one. So, corresponding to this when x is equal to 0 here what will be the value of y from the second equation because we are looking for all the points which satisfy both the equations together.

So, here x is equal to 0 makes this f x 0 and then when x is 0, so y has to be also 0 from the second equation. So, we got this 1 point, now x is equal to 0 and y is equal to 0. Now we have to also look for any other possibility which can make these 2 terms 0, so here either x is 0 or the second term here is 0; that means, that 2 y and 4 x square is 0 and together with this 2 y plus x square has to be 0. So, again out of these 2 equations we see that the only number which satisfy only point which satisfy these 2 equations 2 y plus 4 x is x square is equal to 0 and this 2 y plus x square is equal to 0, the only point is 0 0 again. So, we are not getting any other point then 0 0, so a stationary point in this case is 0 0.

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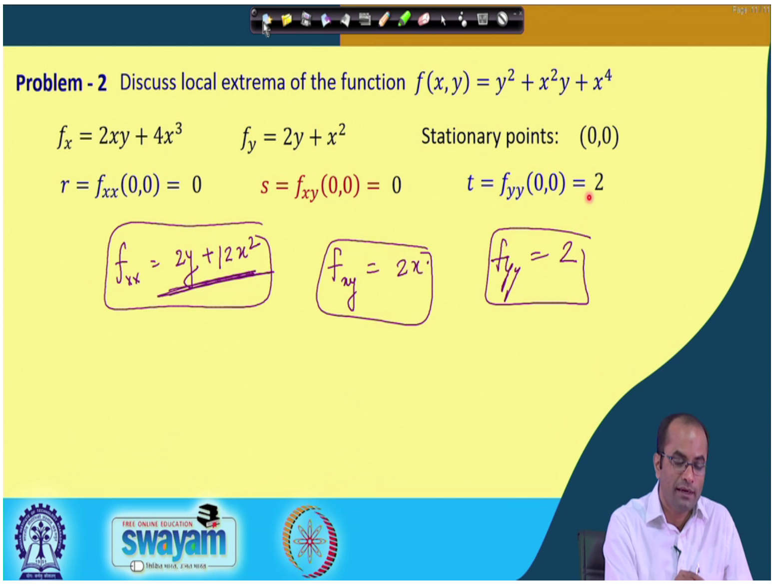

So, at this 0 0 point, we have to now discuss for the identification whether this is a point of local maximum or it is a point of local minimum. So, for that we need to compute now the second order derivatives. So, f x x at 0 0 points, so what will be f xx now, so this was f x, so this f xx will become with respect to x again. So, we will get 2 y and we will get here 12 x square. And at 0 0 points, so again here we have the x and y term, so this will become 0 and if we compute here f xy which also needed, so with respect to y here we will get 2 x term and then f yy we will get 2 because here when we differentiate with respect to y we will get 2 there.

So now, if we compute this r at 0 0 point this will be 0, the s at this the mixed order partial derivative at 0 0 point, this will be also 0 because x is sitting here and then at this point again this f yy, this is 2 there is no x and y term, so this is 2.

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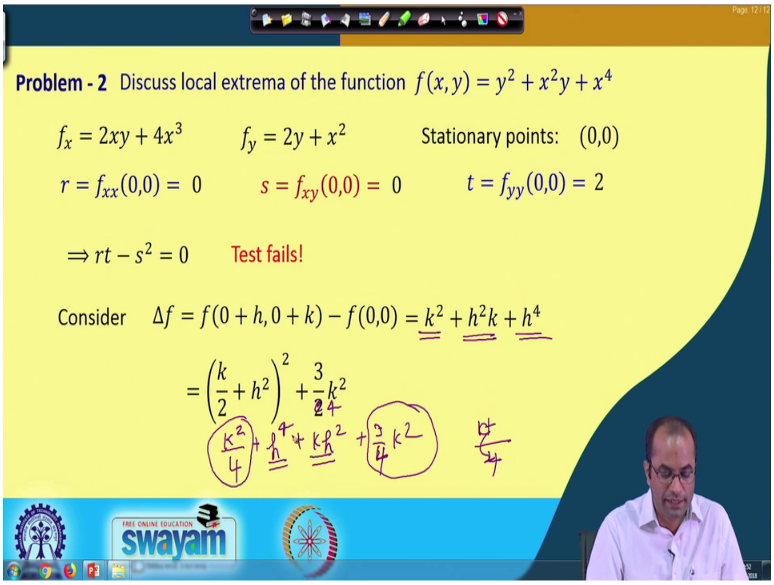

And then so what we have rt minus s square we need to compute rt minus s square, so this is r and t here, so, this will be 0 and minus s square 0. So, we have this rt minus s square and in this case the test fails. So, we cannot conclude anything based on this rt minus s square or the second order derivative test and we have to find some other ways to conclude if we can that what is this point 0 0. So, this test fails, but we can easily identify this point whether it is a point of maximum minimum or a saddle point, if we compute now directly the idea was this delta f. So, based on the sign of this delta f, we can decide whether this is a point of maximum point of minimum.

So, if it is negative, if it is negative; that means, this f 0 0 is larger than the points in the neighborhood, so we have the local maximum at the in this case. If this delta f is positive in the neighborhood then this is a point of local minimum naturally and if we changes sign in the neighborhood of this 0 0 point, then we will conclude that this is a saddle point. So, we will try here now to observe the sign of this delta f directly without going through the second order derivative test. So, in many cases this works, but sometimes this is difficult to realize the sign of delta f directly from the function.

So, in this case perhaps it is possible. So, when we have this f h k minus f 0 0, f 0 0 is 0. So, this f h k will be just simply this y will be replaced by k and this is h square k and then here h will be replaced by this h, so here we have h power 4 for this x 4. So, we have this value of this delta f in the neighborhood of this point 0 0 and then we can make we can rewrite this term as so the first term if we look at, so we have k by 2 and h square whole square.

So, what we are getting out of this whole square, k square by 4. So, there is a term here, let us discuss. So, the k square by 4 and we have h 4 term and then we have a k h square term and then this is 3 by 2 k square. So, this k square by 4 and then here we have this k square was 3 by 2 k square. So, these will be, so here we can combine these 2 terms to get this 4 and then you have 1 there and this will become, so this would be a 3 by 4. So, this is here 3 by 4, instead of 2 we have the 4 there.

So, 3 by 4 and this 1 by 4 will make exactly this k square there and we have h 4 and we have k into h square. So, this is just 3 by 4 k square there. So now, we will 3 by 4. So, now, we can try to get the behavior based on this term here, which is clear because when h and k both are non-zero.

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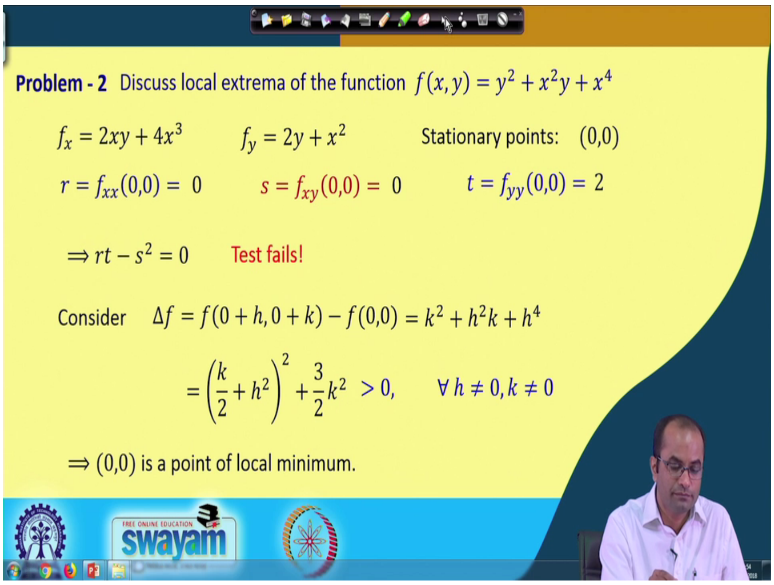

So, 1 can be 0, we will be still in the neighborhood, but we cannot set both to 0 together. So, if 1 of them is non-zero for example, k is non-zero. So, here we have something positive and here also we will have positive this is a square there. If h is non-zero again this can be 0. So, h is the again this will be a positive term. So, for as long as both h and k are not 0, they can be negative positive does not matter but this term here for h either h is non-zero or k is non-zero or both are non-zero, this term will remain positive and what does that mean that in the neighborhood whatever point we take and; however, small this neighborhood is, this delta f is positive, delta f is positive.

So, in this case if this is positive; that means, in the neighborhood the function is taking more values, the larger values and then this 0 0 has to be a point of a local minimum. So, this 0 0 is a point of local minimum. So, it was easier in this case because the test fails, so but it was a sufficient condition. So, we could not get any sufficient condition, which can tell us about this local minimum, but this function was very easy to discuss directly, the behavior and we realized that whatever point be taking the neighborhood the function this delta f will be positive and hence, this is a point of local minimum.

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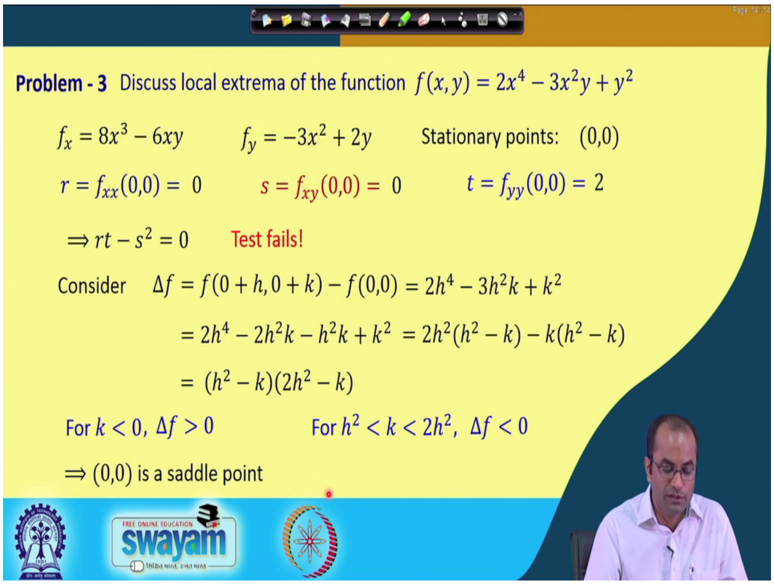

So, we will discuss one more example here f x y is equal to 2 x 4 minus 3 x square y plus y square. It is a similar to the earlier example, so we will compute f x f y and then the stationary point again and we will find that there is only one stationary point that is a 0 0 in this case also similar to the earlier problem and when we compute this f xx from here we will at 0 zero this will become 0 because the term will x or y term will survive there and when we compute this f x y.

So, again here this 6 x will come and at 0 0 point this will be 0 and when we compute f yy, so we differentiate this with respect to y again, so we will get 2. And similar to the previous case we have rt minus s square, so this test fails again and we will use the same idea a similar idea what we have done before. So, we will compute this delta f now, the point in the neighborhood minus this f 0 0 which will come exactly that function 2 h 4 minus 3 h square k plus this k square and in this case we will do again little manipulation here.

So, minus 3 h square k, we have written this minus 2 h square k and minus h square k and then we take common from the first 2 terms the 2 h square. So, here we will get h square and minus k and then minus k from here we will get h square minus k. So, we have the product of h square minus k into 2 h square minus k and now we will discuss how to identify the sign of this delta phi whether we have a definite sign either positive or negative at all the points in the neighborhood of this point or the sign changes.

So, first possibility we will take that if this k is negative for example, So, if k is negative irrespective of whatever our h is, so as long as this k is negative, this term will be positive here minus k term, here also minus k term will be positive. So, whatever h is h may be 0 or h may be non-zero, but when k is negative this product is going to be positive. So, we have this delta f positive, in this situation when k is negative. Now we will see the other situation actually in this case that the delta f can be also or delta f is negative in the neighborhood, how whatever close you go to the h is equal to 0, k is equal to 0 and this delta f will be negative.

So, here we have seen that for k less than 0 delta f is positive and in the other possibilities, we will choose our h and k such that; the k is bigger than h square and less than 2 h square. So, by choosing this, so we are not restricting our neighborhood, we can go as close as to this 0 0 point meaning h 0 and k 0 point having this path here h square this k should be greater than h square and less than 2 h square. So, if we choose our h and k point in the neighborhood which satisfies this inequality, what will happen? This h square is smaller than the k.

So, k is larger this is a negative number and then 2 h square is bigger than k, so this is a positive number. So, this negative positive will make it negative now. So, in their neighborhood when this h and k satisfies these points then we have this delta f negative. So, what we have observed this is interesting that in the neighborhood of this 0 0 point, we realize that this delta f is positive and also this delta f is negative. So, it is changing sign in the neighborhood of this 0 0 point and that means, this 0 0 is a saddle point.

So, again we were able to identify this point here the 0 0 point and this comes to be the saddle point and which was not possible with the second order test, but the direct observation we can find that this is a saddle point.

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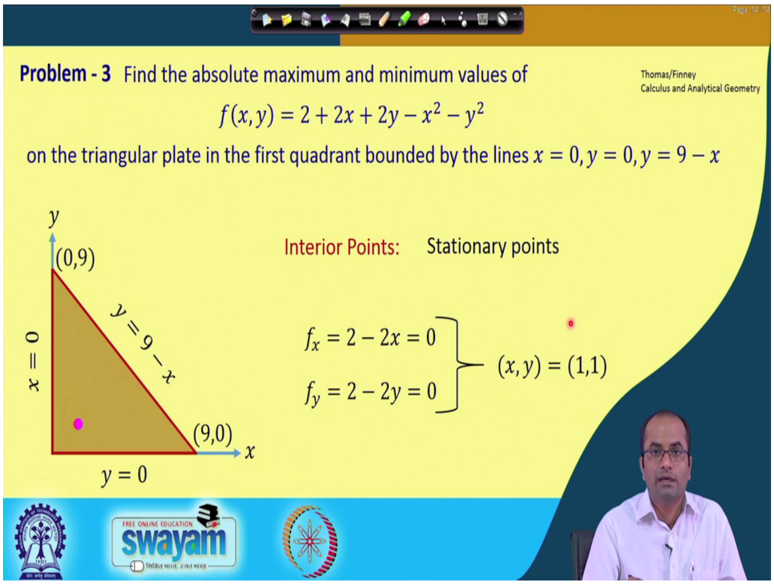

Well so, this is again a simple problem of this maximum and minimum of this function over a triangular plate in the first quadrant which is bounded by this 0, x is equal to 0, y is equal to 0 and y is equal to 9 minus x line.

So, this is the domain of the problem the bounded domain here because these x is equal to 0 y is equal to 0 and y is equal to 9 minus x, they are the boundaries here this is 9 0 point this is 0 9 point. So, we have to now in this case we have to consider the interior and also the boundary. So, when we have the bounded domain we have to consider because this maximum minimum may exist at the boundaries. So, this interior points, so that means, leaving the boundary now.

So, at interior points we will search for the stationary point and separately we will handle or we will look for the extrema at the boundary. So, the stationary point for this problem we can easily compute f x and f y and they come to be 1 and 1. So, this here is the stationary point in the domain or in the interior of this domain excluding the boundary. So, we have to consider this point because we can have local extrema at this interior point, but in this problem we our interest is to find absolute maximum and minimum. So, definitely we will test at this point what is the value of the function later on.

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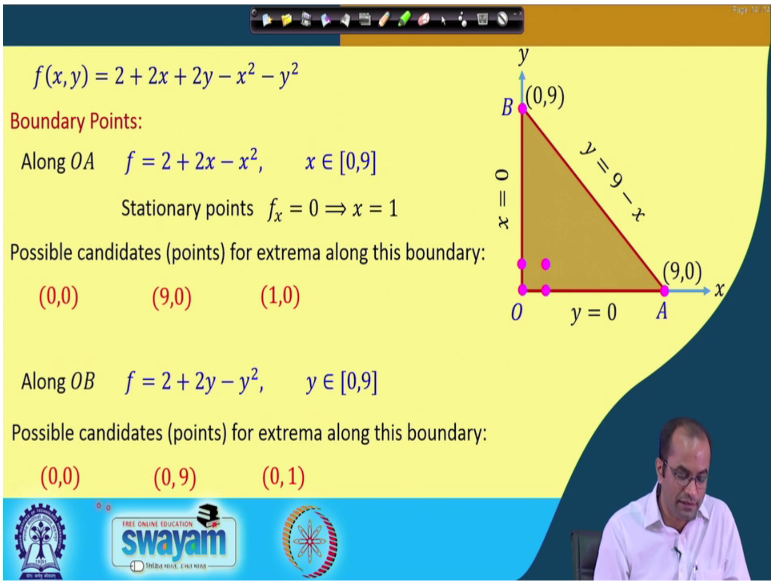

So, we have one point and none the boundary points we have to look. So, along this OA line here, so y is 0. So, we can set in our fx as y 0. So, f will be 2 plus 2 x minus x square and x varies from this 0 to 9. So, this is a 1 variable problem and we can look again for the critical point were just by differentiating this function with respect to x so which comes to be at x is equal to 1 there might be a point of local maximum or minimum.

We do not have to actually go for the identification, so there will be few points where the maxima minima can appear and then we will get the function value at those points. And, from there we can conclude which is the maximum value and which is the minimum value.

So, here this is a stationary point in this one dimensional problem. So, the possible candidates for this extrema because again for this problem now; we have the boundary points 0 and 9. So, we have to also consider 0 0 points in 9 0 point and 1, this at x is equal to 1, which is a stationary point for this problem. So, we have 3 points, we have the 0 0 points, which is the boundary of this one dimensional problem and here we have the 9 0 point and we have the 1 0 point. So, these are the 3 points; similarly along this ob line, we have to search now here along this line x is equal to 0, we can set in f xy function.

So, again this is a problem of one variable we have to look for the stationary point and along this boundary here we have the 0 0, this boundary point which was already taken earlier we have 0 9 point and from here by differentiating this we will get the stationary point as 0 1. So, we have these 3 stationary points I mean these 3 candidates for extrema.

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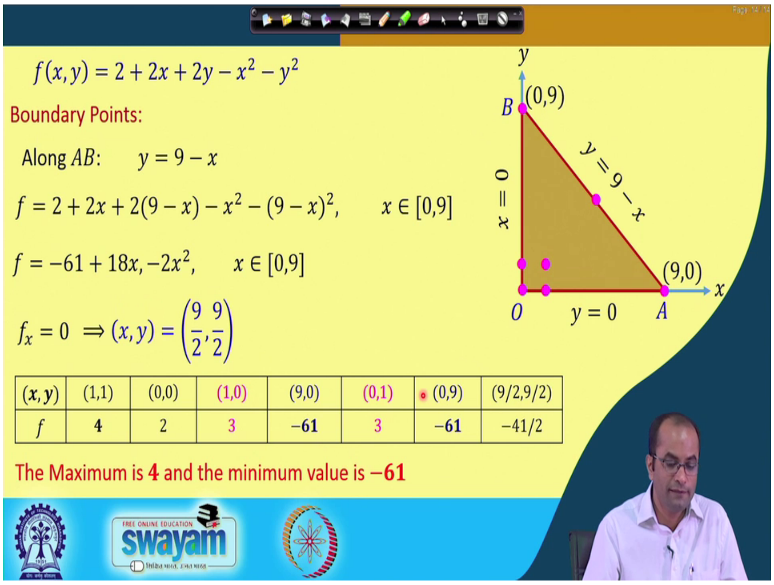

Similarly, we have to do again this third boundary y is equal to 9 minus x. So, we will substitute y is equal to 9 minus x into f x y. So, we will have again a problem of 1 variable, we have to look for the interior point there. So, f x is equal to 0, we will get only 1 point which is 9 by 2 and 9 by 2. So, this is the point there and obviously, these boundary points which are already being taken care by the earlier problem. So, we have so many points here the 4, 5, 6, 7, 7 points where the function may take the maximum or the minimum value. So, we have to consider the interior and we have to also consider the boundaries when we have a closed and the bounded domain.

So, here these points we will evaluate simply the function because our interest is to find the global maximum minimum, we do not have to identify each of this point for local maximum or minimum. So, at 1 1 the function f x y is taking value 4 and 0 0 is taking 2 and so on. So, here we have computed all these values at these points and then we realize this minus 16, which is the which the function takes at 9 0 and 0 9 point is the minimum among all and this 4 is the maximum 1 at 1 1 point.

So, the function takes maximum at the interior here at 1 1 point and it takes the minimum at 9 0 or 0 9 point which is the value is minus 61. So, this was a problem with bounded domain the earlier problems were in the open domain. So, where we have not considered the boundary because, there were no boundaries in the problem, but this here we have to if there is a boundary the domain is closed. Then we have to also look for the boundaries because maximum and minimum may occur on the boundaries which is the case here like minimum is at 9 0 or 0 9.

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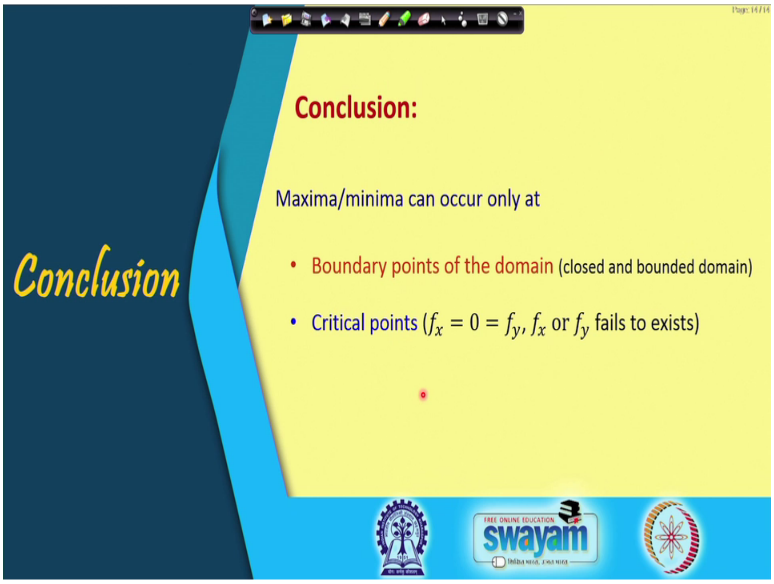

So, the conclusion here that maximum and minimum can occur only at the boundary points of the domain; that is a one and the second the critical points which we have to look for the possible candidates for maximum and minimum.

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So, these are the references.

Thank you very much.