Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 18
Maxima & Minima of Functions of Two Variables (Contd.)

So welcome back to the lectures on Engineering Mathematics 1 and this is lecture number 18 will be talking about the Maxima and Minima Functions of two Variables.

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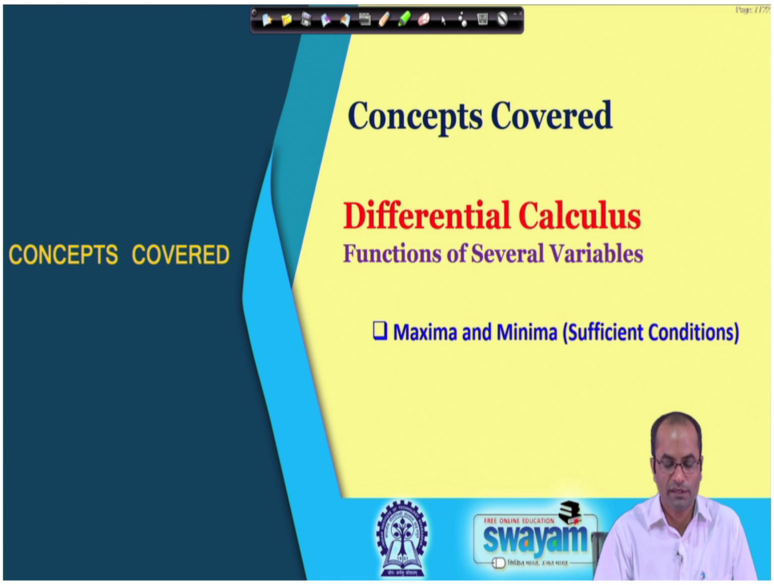

In particular today we will be talking about the sufficient conditions and that will be used for getting the or characterizing the point, the critical points whether it is a point of local maximum or local minimum or it is a saddle point.

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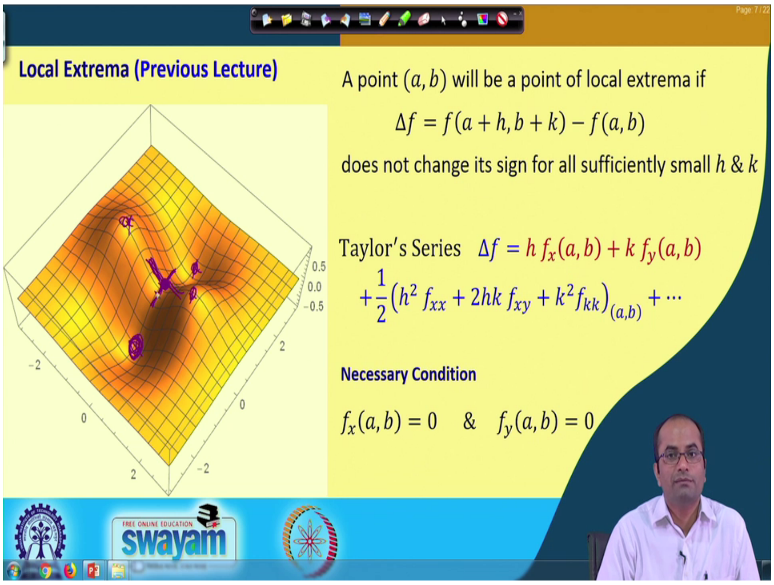

So, in the previous lecture we have seen that a point ab will be a point of local extrema, if this delta f which is the difference between the function value at the neighborhood points of this ab and minus this the function value at ab point. So, if this does not change its sign for sufficiently a small h and k then, we call that then this has a local maxima or local minima, but if it changes its sign then; that means, the point is a saddle point.

So, in that case we try to get the behavior of this delta f or rather the sign of this delta f by using the Taylor series expansion of this function fa plus h and b plus k around this ab point and from where we got the necessary conditions that to have that this point ab is a point of local maxima or minima, fx at this point ab has to be 0 and fy has to be 0. So, which we have observe that, there are for example, here 5 points, which satisfy these conditions, so there is a point here and there is a point there and also there are a total 5 points here.

And now in the this lecture, we will identify whether for example, here we can see that this looks like a point of local minimum, here also its a local minimum, these points are local maximum, but at this point here in the neighborhood of we see if we go in this direction, in the direction of this x, then the function is increasing if we go in the direction of y the function is decreasing. So, this point is a point of it is not a point of local x, local minimum or it is not a point of local maximum, but it is a saddle point.

So, now mathematically we will identify all these points with the help of the second order derivatives.

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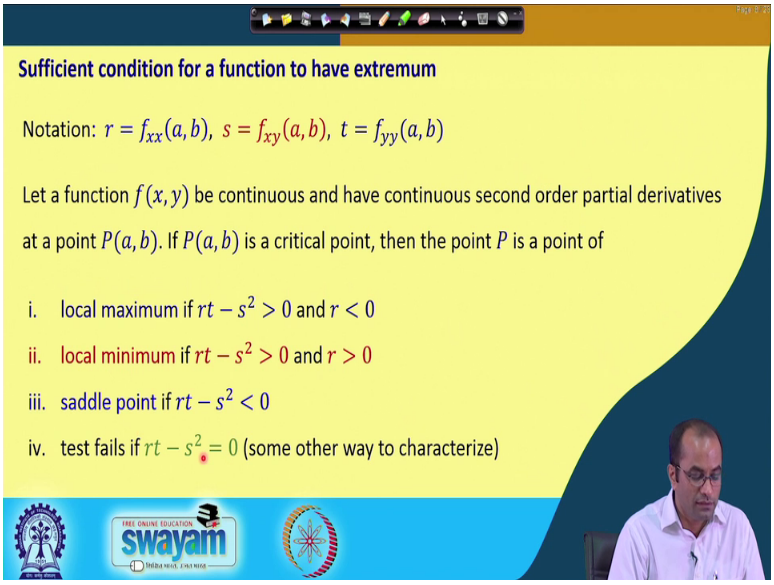

So, here this is the sufficient condition. So, we will just for simplicity we use these notations r will be fx, x at ab point s will be the mixed derivative at that point and t will be the y y derivatives or 2 times with respect to y at this point ab.

And then if this function fx let us assume that is a continuous and have continuous second order partial derivatives at this point and if this point is a critical point then at this point p there will be local maximum, if this rt minus s square is positive and r is negative. This point will be a point of local minimum, if this rt minus s square is greater than 0 and r is greater than 0, saddle point if this rt minus s square is less than 0 and the test will fail if this rt minus s square will be 0 and then we have to find some other ways to characterize the behavior of this function at this point ab.

So, here as written there if ab is a critical point so definitely because this is a necessary condition to discuss that, this point is a local point of local maximum minimum or a saddle point. So, definitely this is a critical point and then we discuss with the help of the second order test here, mainly this rt minus s square the value of this expression we can find out whether it is a point of local maximum minimum saddle point and in this situation when rt minus s square is 0 this just fails.

So, let us prove this result, how do we get these expressions.

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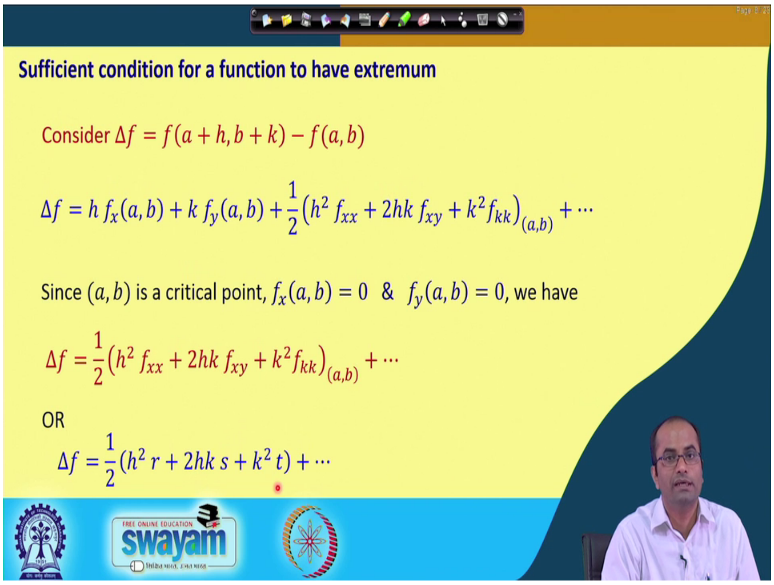

So, we consider again this delta f because, finally we need to get based on these sign of this delta f, the behavior of this f in the neighborhood of a point. So, which we have already seen that the Taylor series expansion of this will lead to this h, the first order terms and then the second order terms there.

Since this ab is a point of is a critical point, so the fx at ab and fy at ab will will vanish and then we have this delta f is equal to these second order terms, so h square fxx and so on. So, now, the behavior of this delta f, we have to get based on these second order derivatives terms or the first this leading term here, the rest all the terms in the expansion will be higher order terms.

So, with our notation we have used this fxx r and this xys and this fkk t at this point ab. So, this delta f is this a simple expression h square r 2 hk s and plus k square t.

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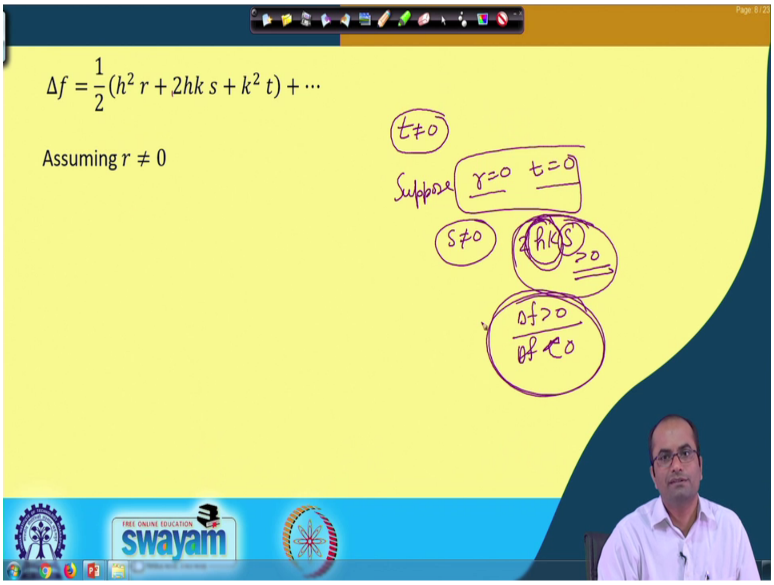

So, now we will consider that or we will assume that r is not equal to 0, here we can also assume that if this r is 0 in case then we can assume that t is not equal to 0. So, at least one of them s 0 then we can proceed in this way. In the situation when both are 0. So, suppose we have a r 0 and we have t is equal to also 0, so in that case naturally we cannot proceed in this way, but in that case directly from here we see that if these 2 are 0 r is 0 and t is 0.

So, s maybe 0, s may not be 0, so if s is 0 again the second order test will not give anything and we have to go for the higher order derivatives. So, suppose this s is not equal to 0 then what do we get here? We get this 2 hk and s term. So, s may be positive, s may be negative, let us just assume that s is positive the same thing we can argue when s is negative..

So, we have this 2 hk terms, when this product h and k whether if both are positive for example, of both are negative this hk term is positive. So, we have this delta f, the leading term is positive and we have already discussed in the last lecture that this leading term will decide the sign of this delta f. If it is positive then the sign of this delta have in the neighborhood of this ab point will be positive, if this leading term is negative then the sign will be negative.

So, here this hk if this product is positive, we have this delta f positive, if this h is negative and k is positive or k is negative h is positive, we have delta f negative. So, means this delta f is changing its sign in that case and this will be a saddle point definitely, which will be also concluded from this result, which I have shown you before. So, we do not have to discuss this separately.

So, let us assume that either r is 0 or t is sorry, r is not equal to 0 or t is not equal to 0. So, here we assume r is not equal to 0 and now, proceed so same thing we can do when t is not equal to 0.

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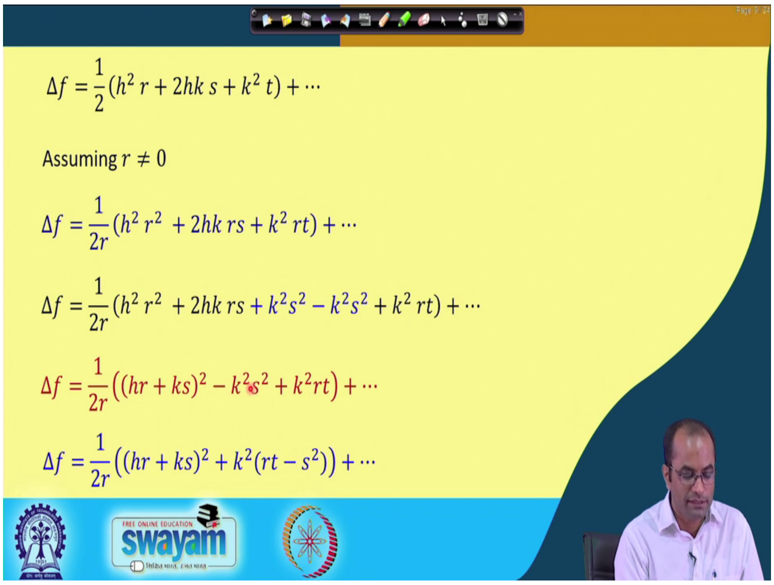

So, let us assume this r not equal to 0 we can divide by r and multiplied by r. So, we got this expression here and this delta f 1 over 2 r. So, in this case we have added here k square s square term and also subtract to this k square, s square term plus this k square rt plus the higher order terms.

Now, the first 3 terms will make a whole square; that means, hr plus this ks whole square these first 3 terms minus this k square s square and then we have k square and r t term. So, here from the last 2 terms also we take k square common. So, we have this rt and minus this s square.

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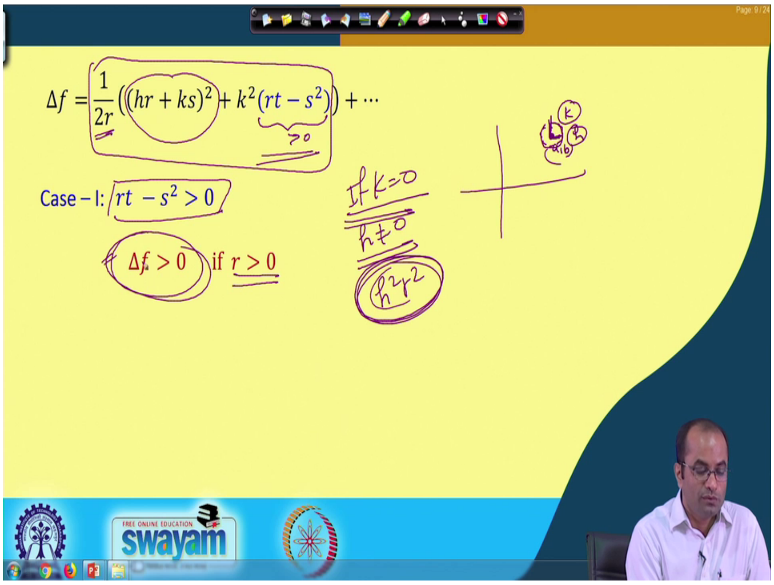

So, this is the expression the leading one this hr plus ks whole square plus k square and rt minus s square. So, we will consider this case 1 now, where we assume that this rt minus s square is positive, that is the case 1. So, in this case what will happen, when this is positive here then we have this whole square term and we have this k square term. So, we have to now see how this delta f is positive when r is positive.

So, r is positive we further assume that r may be negative, r may be positive it is the second order a derivative at this ab points. So, depending on that ab and the function but it will be a definite sign whether it will be positive or negative. So, if this r is positive, we will see now here that delta f will be positive why?

So, if we have this expression here this is positive, so this one is strictly positive. Now for the neighborhood either h will be 0 then k will be non 0 or if k is 0 then h has to be non 0, both cannot be 0 because both 0 means we are at the point ab and we are looking at the neighborhood point. So, here for example, we have this ab point. Now in this neighborhood of this point, either in this increment in this direction is denoted by h increment in the y direction was denoted by k.

So, to have a point in the neighborhood one of them has to be non zero both of them have to be non zero. So, if both are non zero means this k is non zero, then we have a positive term here this strictly positive term; does not matter what is this term here, we have the overall positive number here. So, in that case it is fine. Suppose this k is 0, so if if k is 0 then h has to be non zero, h has to be non zero to have in the neighborhood.

So, in that case this term is 0 and here the k is 0. So, we have h square r square. So, again this h square, r square, we have the positive term, because h cannot be 0. So, in any case whatever the situation is, if this rt minus s square is positive then, this delta f will be positive for r, positive and when r is negative just the sign will change because this r is sitting here; otherwise the rest everywhere we have the s square there.

So, what do we get now that this delta f is positive if r is positive 0 and this delta f is negative when r is negative.

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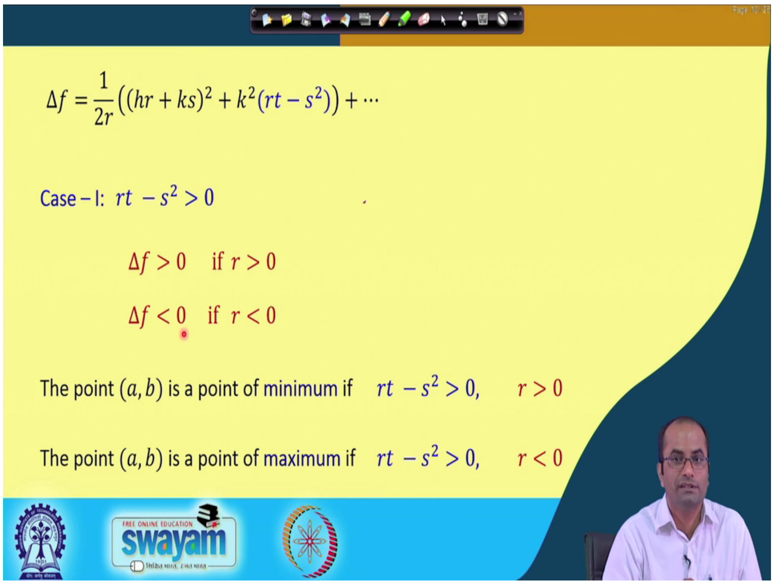

So, we can conclude now that we have the same sign because r will have either positive or negative. So, if r is positive delta f is positive and then we have this point is a local minimum in this case and when r is negative this point will be a point of local maximum when this rt minus s square is positive and r is negative because having this delta f a negative means that, this ab point is taking more larger values than the points in the neighborhood at; that means, this point is a point of local maximum.

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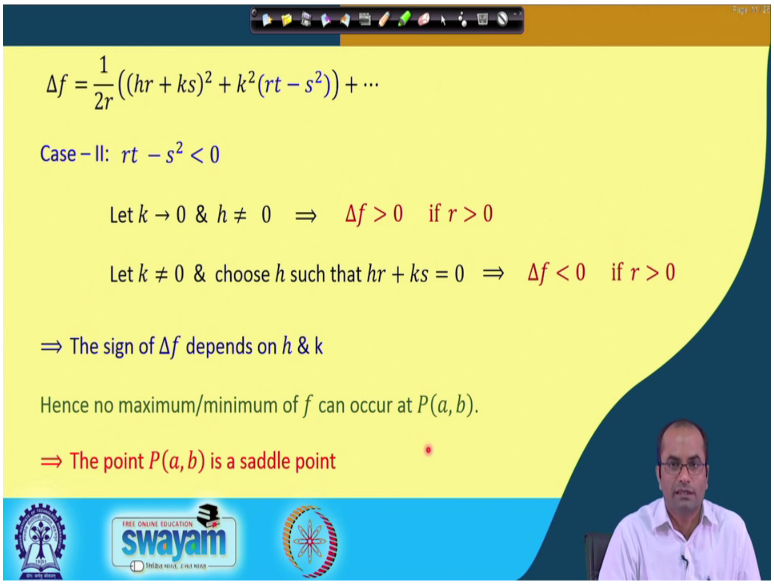

Moving further to the case number 2, when we take this rt minus s square is less than 0 what will happen this case, when this rt minus s square is less than 0? So, we have to again carefully look at, so we take this possibility that is let this k to 0 first and then h is non zero as I discussed before that one has to be non zero. So, you are letting at k to 0 and the h non zero. So, k to 0 means this is 0 and h is non zero. So, from here, we will conclude that this delta f will be positive, if r is positive. So, let us assume here r is positive, r can be negative. So, this delta f will be negative in that case nothing else will change.

So, here r is positive, so this delta f is positive because, h is non zero and k is 0. So, we will have here h square r square term, which is positive and then the second observation we will take let us take k is not zero. So, k is not zero and this rt minus s square is negative.

So, we have something negative sitting here now and we choose this h now such that hr plus this ks is equal to 0. So, we choose our h, so that this relation holds for given k whatever is small you can take, you can we can choose this h here as minus ks over r. So, we choose h as minus ks over r for any value of k.

So, on this point here, when h is chosen from this minus ks over r for whatever k, this first term this hr plus ks this will be 0 because, we have chosen our h in such a way that, this hr plus ks is 0. So, this term is 0 and here this k square is positive, but this is negative this is less than 0. So, the overall, this first term for r positive, which we are considering at this moment, this will be less than 0 delta f will be less than 0 because of this term, there is no first term if we are in the neighborhood which satisfies all these points.

Then this delta f is less than 0 if r is positive and we are not restricting for the neighborhood that we have to be a far at some point where this relation holds, we you can be as close as to this ab point by choosing this hr plus ks is equal to 0 for any small value of this k and not equal to 0.

So, what we have realized here that the delta f is strictly positive when r is positive delta f is negative when r is positive. So, we assume if we take r negative then naturally, this delta f will be negative delta f will be positive. So, the point is that this delta f is changing its sign in the neighborhood and that is exactly the case of the critical point. So, this is the condition if this rt minus s square is negative then we can conclude immediately that this will be a point of this will be a point of a saddle point.

So, this is a saddle point, it is not a point of maximum or minimum because this delta f, the sign of this delta f depends on h and k. So, there are points in the neighborhood where the delta f is positive and there are points in the neighborhood where this delta f is negative. So, it is changing sign in the neighborhood of this ab point and; that means, this is a saddle point.

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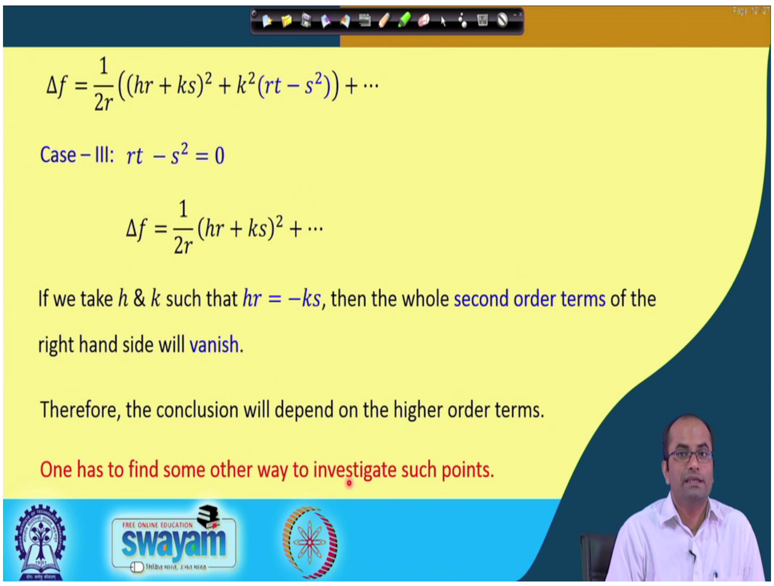

The third situation we will take when this rt minus s square is equal to 0. So, in this case there is no the second term here rt minus s square is 0. So, the behavior will be discussed with the help of the first term hr plus ks square. At a first glance, we will see that this ok, this is a positive greater than equal to 0 term hr plus ks is equal to 0.

It k hr plus ks whole square, so this is definitely greater than equal to 0, but there is a possibility of having equal to 0 because if this term the second order term becomes 0 then we cannot identify the behavior of this delta f because, then the behavior will be determined from the other higher order terms which can make this delta f to negative as well because, if this term is 0 the next term will decide the sign because that might be the case that delta f is negative. So, we cannot conclude out of this immediately by having this that this is here greater than equal to 0.

Now, we have to have a strict sign here then only we can say this will be the whole delta f will be of that sign. So, here if we can prove that this is strictly greater than 0 then it is fine, we can conclude about the point but here this is now greater than equal to 0 why? Because if we choose our hr and ks in the neighborhood such that this hr plus ks is 0. So, in that case at all those points in the neighborhood, this term will become 0 and the behavior will be determined from the next higher order terms.

So, let us write down this here. So, if we take this h and k such that; this hr plus ks is 0, that means, hr is equal to minus ks. So, all these are the points in the neighborhood where delta f is 0 plus this third order terms, then the second order terms of the right hand side vanish and then therefore, the conclusion will depend on the higher order terms. So, we cannot conclude anything in this situation about the sign of this delta f, it may be negative, it may be positive. So, one has to find some other ways to investigate such points.

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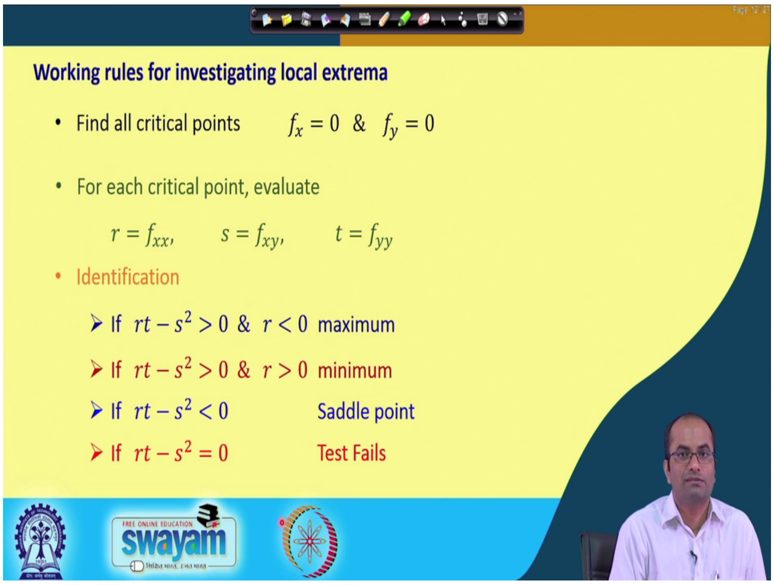

So, what is the working rule now for investigation of local extrema we have to find all the critical points and these are the points by solving these equations fx is equal to 0 and fy is equal to 0 we will get and for each critical point we have to evaluate r s and t, the second order second order derivatives. So, fxx, fxy and fyy, so at each critical point we will evaluate these second order derivatives and for the identification we will use this rt minus s square, if it is positive and r is negative, then this is a maximum point of local maximum, if rt minus s square positive, but r is positive then we have minimum, the this is a point of local minimum and rt minus s square is less than 0, then this is a saddle point and if this rt minus s square is 0 then the test fails.

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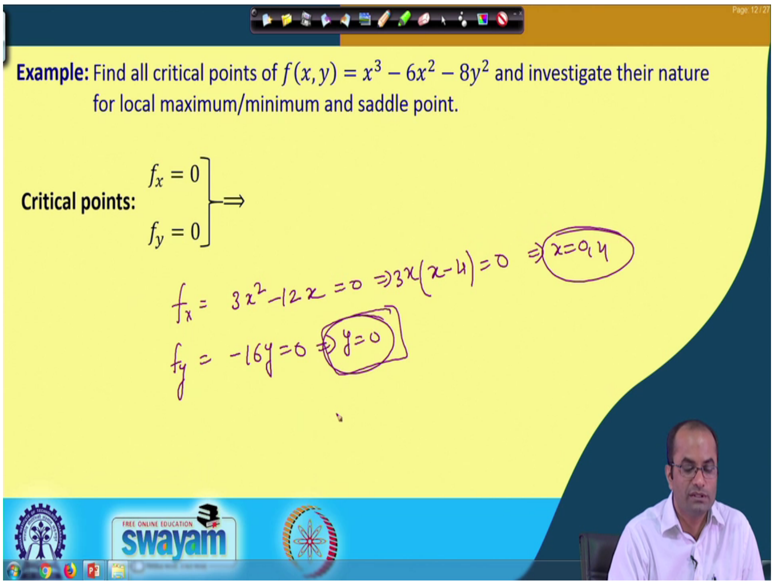

So, we will consider this example now to discuss find all the critical points of this and investigate their nature for local maximum minimum and saddle points. So, we will compute first all the critical points and for each critical point we will investigate that rt minus s square term to discuss the behavior of those critical points in the neighborhood. So, here we have the critical points now the fx is equal to 0 and fy is equal to 0, so what do we get, fx is equal to 0 is what is fx, fx is 3 x square minus this 12 x.

So, in this case this will be set to 0 and this fy will be minus 16 y is equal to 0. So, from here we will get y 0 and from the first equation we will get 3 times x and x minus 4, so is equal to 0. So, we will get x is equal to 0 and 4 from the first equation, from the second equation we will get y is equal to 0. So, we have the 0.00, we have the 0.40 these are the critical points.

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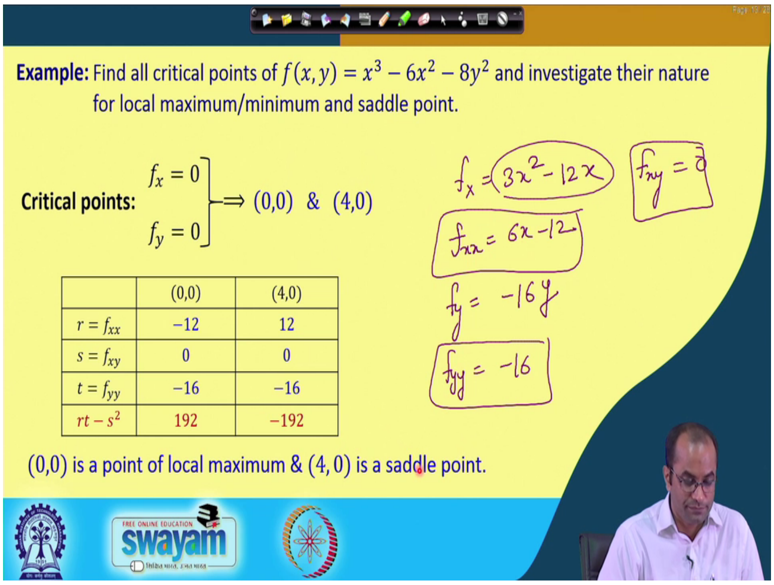

0 0 and 4 0 these are the 2 critical points of the problem and in this case, now we will compute the behavior of the of this rt minus s square, the sign of rt minus s square at all these points. So, at 0 0 point this second derivative which we have to be computed here. So, this fx is 3 x square minus 12 x and then fxx. So, fxx will be 6 x minus 12 and this fxy will be 0 because there is no term of y here. So, fxy will be become 0 and this fy was minus 16 y, so, this f yy will become minus 16.

So, based on these 3 now, we will compute these r. So, fxx at 0 0, so will be minus 12 will be minus 12 and then s s is 0, so this is 0 and t here this is t. So, t is minus 16. And at this 4 0 point, so r here, so 6 into 4, so 24 minus 12. So, this will be 12, s is 0 and fyy is constant here minus 16.

So, if rt minus s square this is a product of these 2 negative number we have plus 192 and in this case we have minus 192. So, here when we have this rt minus s square positive and r is negative. So, this is a point of local maximum. So, 0 0 is a point of local maximum and in this case this is rt minus s square is negative less than 0. So, it will be a saddle point, which we have discussed in the in the sufficient conditions.

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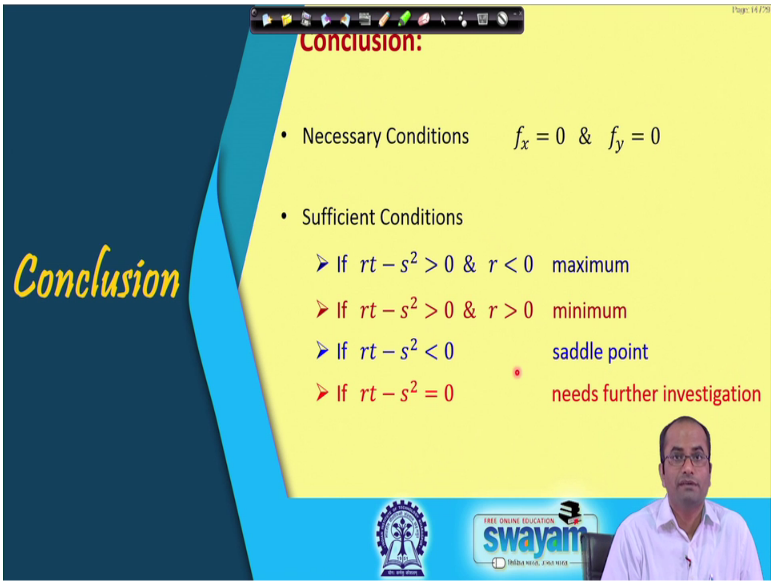

So, coming to the conclusion here, so what are the necessary conditions we need to know the critical points because, these critical points are the candidates for the local maximum and minimum. So, we have fx is equal to 0 and fy is equal to 0 that will gives give us the necessary conditions for the extrema. So, these points will be the candidates, which we need to investigate for the behavior the local behavior of the function at that point.

So, the sufficient conditions help us to identify these points for local maxima, minima or the saddle point. So, for that we need to compute rt minus s square, if it is positive and r is negative in that case, this comes to be a local maximum. If this rt minus s square is positive and r is also positive then, such point will be a point of a local minimum and if rt minus s square is less than 0 then this will be a saddle point. So, it is not a point of maximum, it is not a point of minimum and it is a saddle point as per the definition we use.

And here rt minus s square will be 0 in that case we cannot identify the behavior of this point and then this test at least the second order test fails and we have to find some other ways; either they the higher order terms we have to investigate or we have to directly investigate the behavior of this function at that point, but it is certainly needs further investigation.

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So, these are the references we have used to prepare these lectures and in the next lecture now we will see more such examples to identify the local the behavior for the local maxima minima and the saddle point.

So, thank you very much for your attention.