Engineering Mathematics - I
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Lecture - 17
Maxima & Minima of Functions of Two Variables

So welcome back to the lectures on Engineering Mathematics I and this is lecture number 17 and today we will be talking about the Maxima and Minima of Functions of Two Variables.

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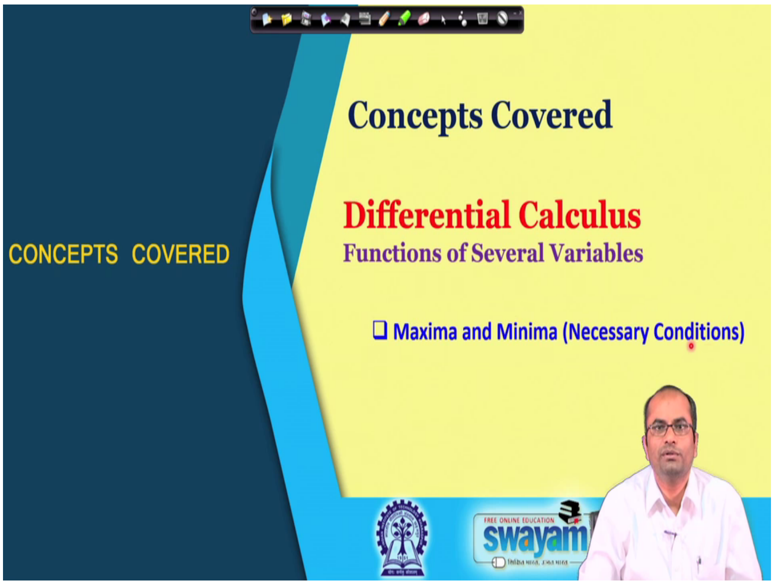

And in particular we will be talking about the necessary conditions that are required to find the maximum and minimum values of the function in a certain domain.

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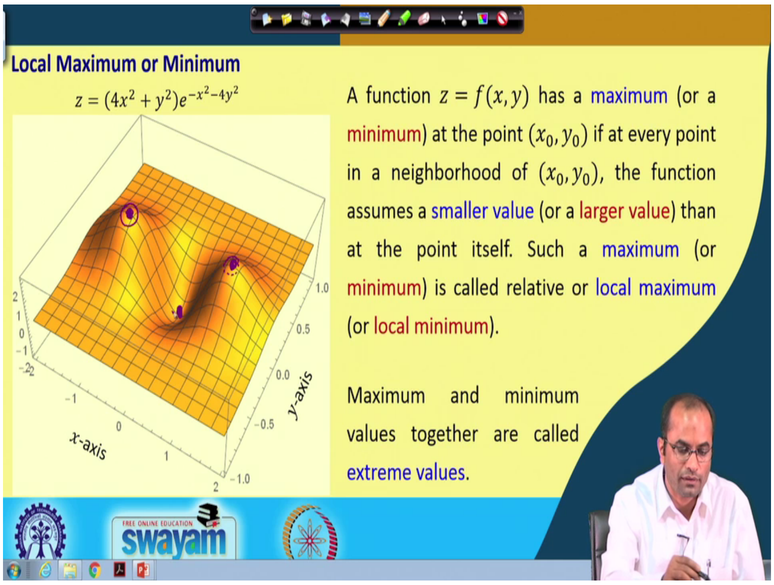

So, what is local maximum and minimum we will define here. So, a functions z is equal to f x y has a maximum at the point x 0 y 0 if at every point in a neighborhood of this point the function assumes a smaller values then the point itself. So, meaning here we are calling that the function is taking maximum value at this x 0 y 0 point if at all other points in the neighborhood of this point function is taking smaller values. So, naturally the function has attained local maximum at this point and similarly we can define for the minimum also. So, if a function has a minimum at this point in that case all the points in the neighborhood function will take larger values than the point itself, then the value of the function at that point itself.

And such maximum or minimum is called relative or local maximum because we are not talking about or we are not searching the local and maxima, the maximum and the minimum of the function in the entire domain. But we are talking about at a certain point and checking its behavior locally. So, if at all other points in the neighborhood of this function of this point the function is taking smaller values then we call that this is a local maximum and if the function is taking larger values in the neighborhood of a certain point then we call that this is a point of local minimum. And these maximum and minimum values together these are called the extreme values.

So, here the a plot or the surface of this z is equal to 4 x square plus y and e power minus x square minus 4 y square and we can see here clearly. So, at this point for example, at this point here the function in the neighborhood of this point is taking a smaller values at that point itself. So, here this is a maximum locally. So, we call this local maximum here also at this point the same situation is happening that all the points in the neighborhood of this point the function is taking smaller values. So, here also there is a local maximum of this function.

Similarly, at this point here, but this is other way around that all the points in the neighborhood function is taking more values than this point itself then the value of the function at this point itself. So, this is a local minimum.

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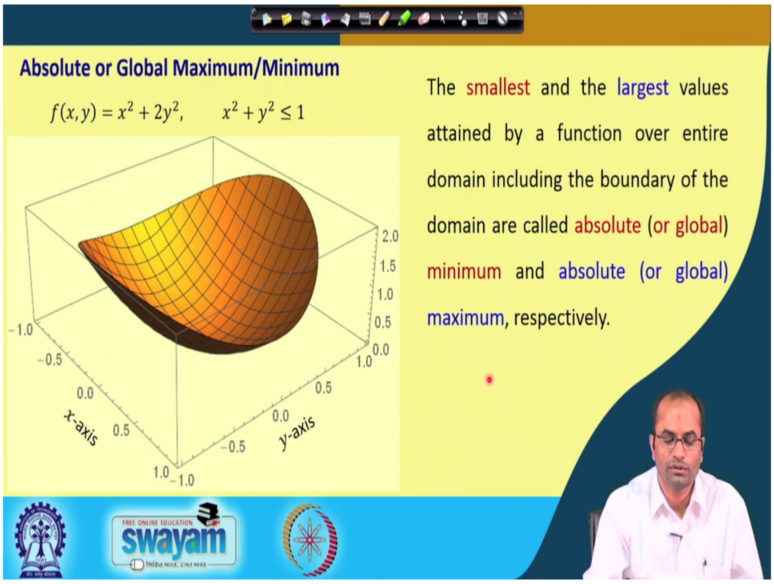

So, what is the absolute or the global maximum minimum? So, the smallest and the largest values attained by a function over entire domain including the boundary of the domain are called absolute or global minimum or absolute or global maximum respectively.

So, here we are not talking about what is happening locally around a certain point, but we are talking about the smallest and the largest values attained by the function over the entire domain. So, in that case if such a value we will call that we will call that this is the absolute minimum value of the function or the absolute maximum value of the function. And we have to include naturally the boundaries as well or the boundary of the domain to find out such a local such a absolute maximum or absolute minimum.

So, for example, if we consider this function f x y is equal to x square plus 2 y square and we have this bounded domain your x square plus y square less than equal to 2 1. So, this disk of this radius one and including this 1 so, we have the boundaries there. So, what we have to search now we have to search inside the domain so at all these points.

So, this is a surface here x square plus 2 y square so, as we can see somewhere here there will be a local minimum at this 0.00 that is local minimum, but that will be also a global minimum because we do not see any other point where the function will take a smaller value than this point and to find the maximum for example. So, at these points here it seems that the function is taking the maximum values so at though this point here or that point here.

So, we have to later on mathematically find out what is the where what is the point in the domain where the function is taking the maximum value or it is taking the minimum value so, but in this case when we consider the boundaries and find among all these local maximum local minimum and also at the boundaries we have to check whether the function is taking somewhere again the maximum value. So, among all these local maximas a local maxima and minima we have to find the largest the smallest values and then we can claim that this is the absolute maximum or the absolute minimum or the a function.

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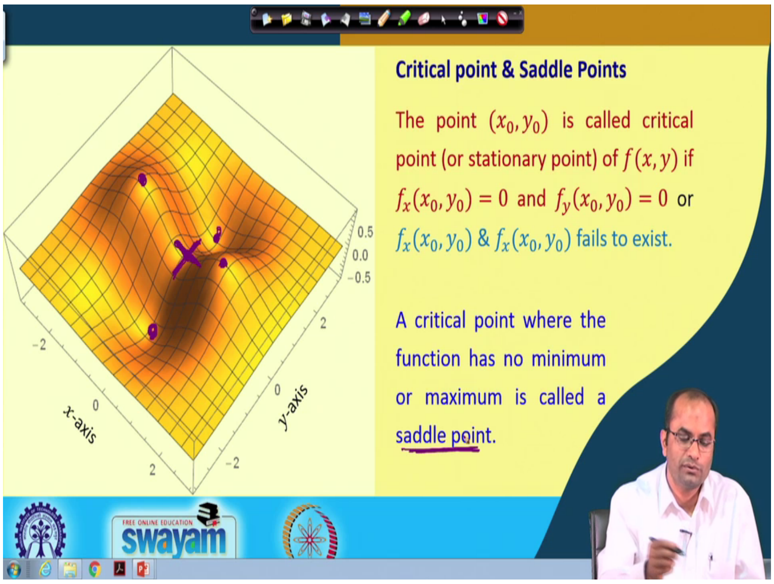

So, there will be another terminology used for used in this lecture there will be critical point and the saddle points. So, the point x 0 y 0 is called critical point or we also call like a stationary point of f x y if the partial derivative of the function f x at this x 0 y 0 point is 0 and f the partial derivative of the function f y at that point x 0 y 0 is 0 or simply they fail to exists. So, that possibility is also included. So, that is the definition of the critical point or the stationary point, and a critical point so among these all the critical points where f x is 0 and f y is 0 where the functions a critical point where the function has no minimum and maximum is called a saddle point.

So, basically what we will observe now in the next slides that if the local maximum or minimum exists. So, that will exist only at these critical points, but there will be some critical points, where the function is not taking the maximum the local maximum or the local minimum and those critical points will be called as saddle point. So, just to realize this situation here so we have many critical points which we will identify again mathematically little later.

So, this here for example, will be the point where the partial derivative will be 0 with respect to x and also with respect to y there will be a point here where both the partial derivatives would be 0, there will be a point here where the partial derivatives will be 0. So, basically the tangent plane will be parallel to the xy plane, here also there will be a point where f x and f y will be 0 and there will be a point here as well where f x and f y will be 0.

So, all these points for example, the critical points and for this will be like the local minimum here also local minimum there will be local maximum here again this is local maximum. But at this point if we see that in the neighborhood it is not showing the same behavior because if we go in the direction of y. So, here the function is taking lower values, but if we take in go in the direction of x then the function is taking the more values or the larger values then this point itself.

So, this point we cannot identify as the local maximum or local minimum though the partial derivatives here at this point was 0 with respect to x and with respect to y but. So, this is a stationary point, but this is not a local maximum or local minimum and such a point, such a point will be called as the saddle point. So, that terminology we will use in today's lecture.

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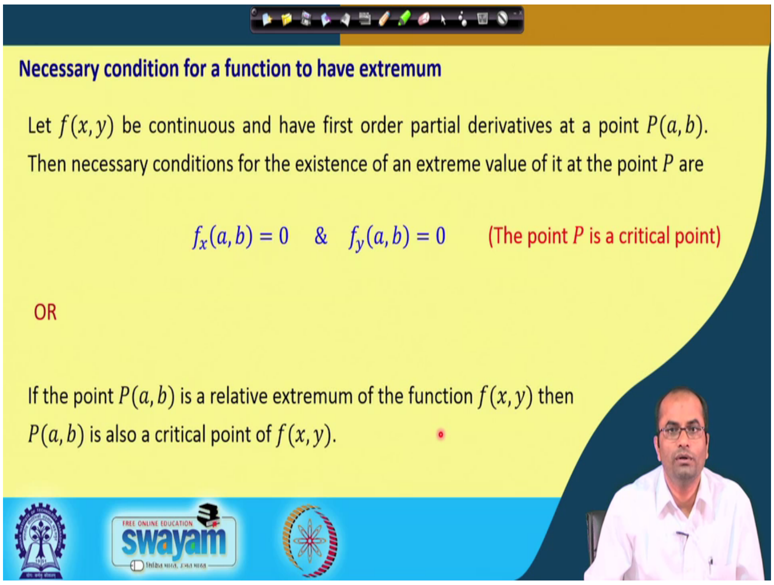

And then we come to the necessary conditions or condition for a function to have extremum. So, what are these conditions let f x, y be a continuous and have partial order derivatives at a point a b, then necessary conditions for the existence of an extreme value. So, extreme value means the maximum and the minimum value or rather the local we are talking about local maximum and local minimum values of it at this point p.

So, these are the necessary conditions what are the necessary conditions that the partial both the first order partial derivatives. So, with respect to x and with respect to y at that point they should be 0. So, this is what we are calling necessary conditions; that means, without these conditions we cannot have the local maximum and local minimum at this point. So, if a point a b is a point of local maximum or local minimum, then definitely these partial derivatives must vanish at that point.

But this is not sufficient to say that yes definitely there will be a if we have a point where f x and f y both are 0 then we cannot claim that at this point certainly there will be a local maximum or local minimum. Because this is a necessary condition, necessary conditions means that this is the first condition we have to check where and or in other words these points will be the candidates for the local maximum or minimum. They may be there may be local maximum minimum there may not be a local maximum or minimum. So, if there is a local maximum minimum we will identify those points local maximum and minimum and among these which are the critical points as per the definition we have defined.

So, at these critical points we will identify if there is no local maximum and minimum that point will be called as the saddle point. So, this point P is a critical point as per the definition we have discussed or in other words if a point P x y is a relative extremum of the function f x y then this is a critical point of f x y because yes as we said that this is the necessary condition to have the extremum minimum. So, if we have a point a b where the relative extremum exists of this function then definitely that has to be a critical point because those x, local extremum and local minimum will exist only on the critical points.

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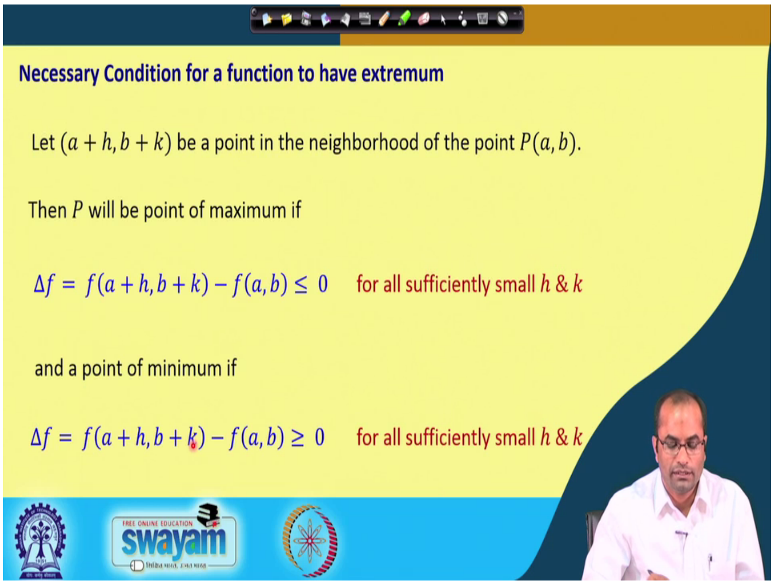

So, here the necessary conditions again to just to have a simple proof that how do we care that these partial derivatives must be 0 at these at the points a b if a b is a critical point. So, we consider this a plus h and b plus k this is a point in the neighborhood of the point a b. So, this is a neighborhood because h and k can take any value. So, then this a plus h and b plus k will be the neighborhood of this a plus b a comma b point. So, there is no restriction on h and k if they can take any value to have a point in the neighborhood.

Then P this point p a b will be a point of maximum, when it will be a point of maximum? When the values of the function in the neighborhood of this point are lower; so, it takes the or the function attains lower values in the point of the neighborhood of this a b point. So, for example, if we define this difference delta f; that means, f a plus h b plus k the point in the neighborhood and minus this f a b.

So, if it is a point of local maximum; that means, this f a b will be bigger will be the large value then this these values in the neighborhood of this function. So, this value will be less than equal to 0 if this is a point of maximum for all sufficiently small h and k. So, whatever h and k we take in the neighborhood of this a b point. So, this value will be less than equal to 0 or in other words there will be definitely a neighborhood of this point a b where this delta f will be less than equal to 0 for all values of h and k.

And this point will be called a point of minimum if this delta f will be greater than equal to 0. So, in other way round that this f a b will be a lower value than this f a plus h, b plus k for all sufficient values of h and k.

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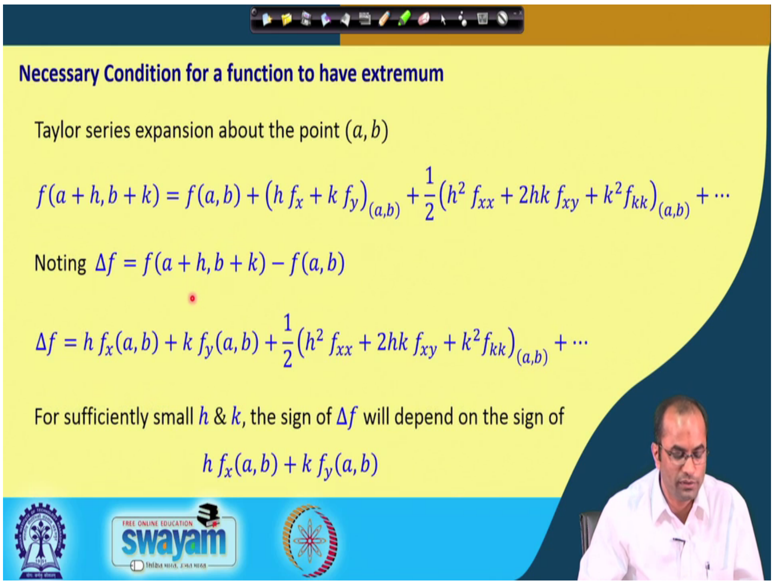

So, if we use this Taylor series expansion about this point a b. So, what will happen here f a plus h and b plus k will be equal to f a b. So, we have already discussed this in the last lectures. So, h f x and this is the first order derivative here the second order derivatives at a b and so on we can write down this expansion about this a b point as long as these derivatives exist.

And this noting that this delta f which we have defined in the previous slide the difference of this a plus h and this minus f a b. So, that will be equal to so this h and this is the function first order derivative at a b k into f y a b and these higher order terms.

So, for sufficiently small h and k, h and k the sign of delta f will depend on the sign of this here h f x plus k f y a b. So, what we will also see in the detail in the next slide that the sign of this expansion here in the right hand side the sign of this delta f basically we will depend on the sign of this first term, these are the leading terms here in terms of h and k. So, therefore, sufficiently small h and k the sign will be determined by the sign of this term, if this is a positive term delta f will be positive. If this is a negative term delta f will be negative irrespective of whatever we have here because these are the higher order terms in terms of h square there is a product h and k. So, at least we can find small h and k here sufficiently small so that the sign will be determined by this factor only in the close neighborhood of this point a b.

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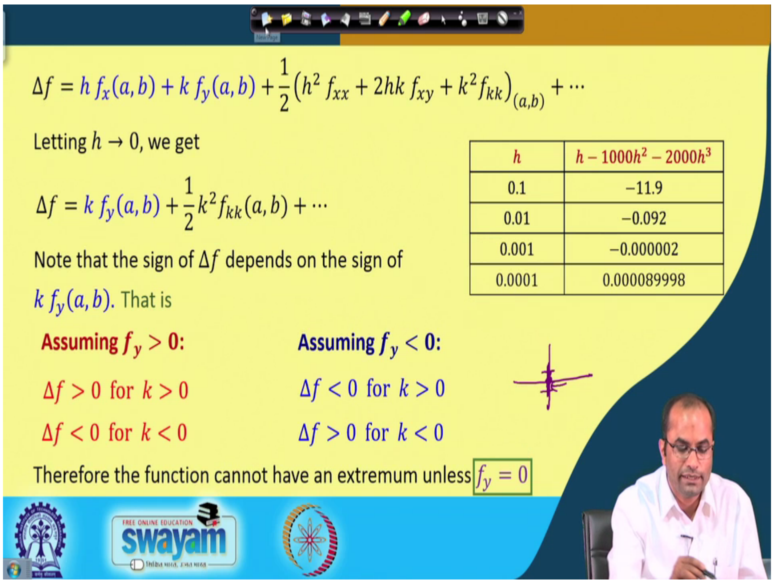

Let us explore this little bit more. So, we have this delta f and first let this h tends to 0 here because we are a still neighborhood we are we are just letting h tends to 0 and then they will be k in the direction of y. So, if we let this h tends to 0 we are still in the neighborhood of this a b point, then what we are getting here delta f will be k into f y and then here this k square k q terms will be coming there.

So, let me just again repeat the argument we had in the previous slide. So, note that the sign of this delta f will depend on again this leading term here of this k, the rest all these terms are k square k cube and so on for very small k these terms will be like negligible and the sign will be determined again by this term just for example, we take an example here that you have h and this expression here h minus 1000 h square minus this 2000 h cube.

So, what we will notice here the sign will be determined by this h only not by these terms here, but we have to go to a sufficiently small h close to 0 to see that this will be determined by the sign of this one. For example, we take h is equal to 0.1 this is too large value and we are getting this negative number because these are the dominating term for this h 0.1. But if you go pretty close to h to 0 for example, h is equal to 0.1. So, this is a still negative we have taken a much smaller h now. So, this is a still negative because these are the last term, but what we see here now we have taken this point 0001 and in that case now we can see that the, since h is positive.

So, this value of this expression is also positive and that is a point here. So, for sufficiently small h the sign of this term here; however, large these numbers are the sign will be determined by the first term which is which one can see here. So, same argument we are making here that there will be a sufficiently small k for which the sign of this expression will be determined by this first term which is k fy a b and that is the point here. So, if we assume this f y a b is positive because either this is a derivative, partial derivative with respect to y at a b. So, it has it must have some determined sign. So, if it is positive in that case this delta f will be positive for k positive and this delta f will be negative when we have k negative. So, this k we are moving in the direction of.

So, for example, this is h is equal to 0 and k is equal to 0. So, this we already let this h to 0 here. So, we are in only the neighborhoods points are in this direction of y. So, either when k positives, we are in the neighborhood of upper side when the h k is negative we are in the neighborhood somewhere here. So, then we have that for k positive. So, in the neighborhood a point where delta f is greater than 0 there is a point in the neighborhood where delta f is less than 0 and if you assume again if delta y is less than 0 because we do not know what is f y at this point ab. But if it is negative then we have this delta f less than 0 and delta f greater than 0 again in the points in the neighborhood. So, what we observe here that whether the f y is positive here or f y is negative this delta f is changing its sign, the delta f was the difference between the value at the point a b and the value in the neighborhood. So, this is changing time, but changing its sign, but to have the local maximum or local minimum it should not change its sign, because then only we can determine this is a point of local maximum or local minimum when all the points in the neighborhood it behaves us with the same sign.

So, here this delta f is positive delta f is negative for some points in the neighborhood and that will conclude that the function cannot have an extreme value extremum here, unless this f y is 0 because k is we are moving in the neighborhood of this point. So, if this f y is not 0, if f y is positive it is changing sign if f y is negative it is again changing sign, but if this is a point of local maximum then this should not change its sign and for that the f y must be 0, because if f y is not 0 then we have observed that it is changing sign. So, this is necessary that f y has to be 0 otherwise it will not be a point of local extremum. So, moving next now similarly what we can do.

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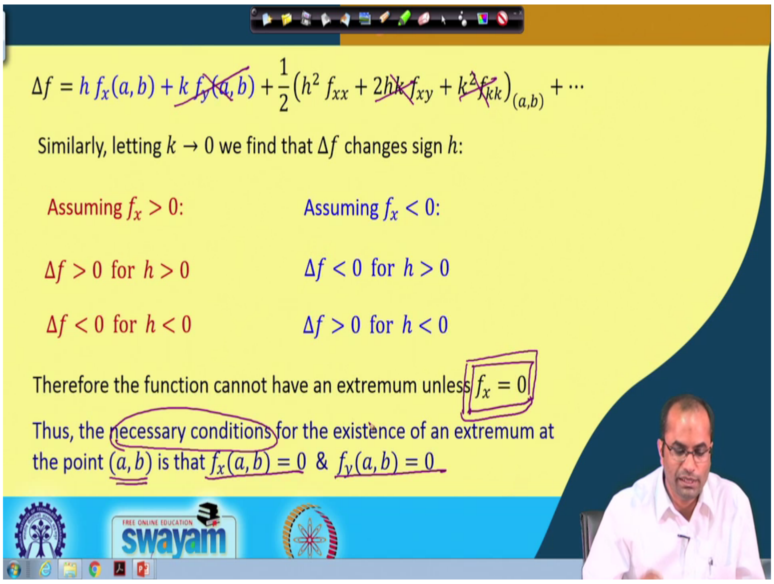

So, we had this expression already we have seen; now similarly we will add that k tends to 0 and then we will find that delta f changes sign for h. So, here for example, if we take f x positive and then we will realize that this delta f is positive and then when h is negative delta f is negative when we let this h k tends to 0. So, this term is anyway is removed here and this all these terms were the k is there we will go to 0. So, we have h f x and so on.

So, here assuming this f x is less than 0. So, the sign will depend on now this h. So, if h is positive with the delta f is negative h is negative then delta f is positive. So, again the same argument what we have observed that in the neighborhood of again of this point a b point this is changing its sign, whether if delta f is positive it is changing sign if delta f is negative it is again changing sign.

So, again to have that this a b is a point of local extremum we must have that this f x is equal to 0. So, this f x must be equal to 0 earlier we have observed that f y must be equal to 0. So, we can conclude that the necessary conditions so here the necessary conditions for the existence of an extremum at this point a b is that f x should be 0 and also the f y should be 0. These two derivatives must be 0 this is what we have observed, if these are not 0 this delta f was changing sign and then that point cannot be a point of local maximum or local minimum.

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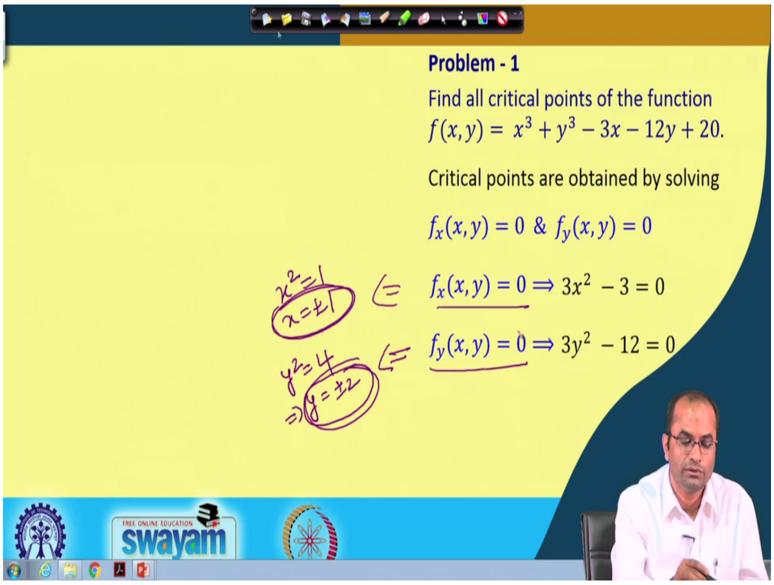

So, now let us just go through this problem number 1. So, we here we will find all the critical points of this function f x y is equal to x cube plus y cube minus 3 x minus 12 y and plus 20. So, what we have to do now to find the critical points, we have to compute the partial derivatives with respect to x and with respect to y and set them equal to 0 and out of these two conditions we will find out how there are how many points which satisfy f x is equal to 0 and f y is equal to 0.

So, this f x is equal to 0 and f y is equal to 0 will give us; so, here the 3 x square and then with respect to x. So, here minus 3 and that we are setting to 0 to find the critical points similarly when we take f y here. So, we will have three y and minus this 12 so 3 y square. So, here three y square yeah 3 y square and this minus this 12 is equal to 0. So, out of these what we get so from this first condition we are getting like x square is equal to 1. So, x is plus minus 1 and from the second condition we are getting y square is equal to 4 and this implies y is equal to plus minus 2. So, we have all these points when x is equal to plus or minus 1 and y is equal to plus and minus 2 these f x at x y all these points this is 0 and f y is also 0.

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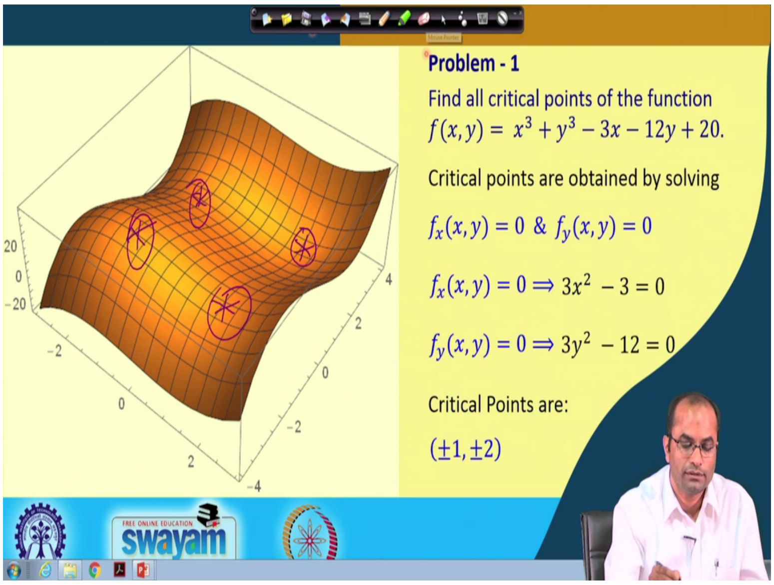

So, these are the critical points now here we can take one and then 2 then we have minus 1 2 or we have the plus 1 minus 2 and. So, all these four possibilities are there and if we can see here in the surface of this function. So, there are the points for example, 1 and 2. So, here in x we have 1 and then we have 2 there. So, here there is a point where there will be a critical point, again here the plus 1 and then we have a minus 2.

So, there will be a point here and there will be minus 1 and plus 2. So, there will be a point here and minus 1 minus 2 there will be a point here. So, all these 4 points are there which are the critical points. So, here the function derivative with respect to x and with respect to y both are 0 at these points.

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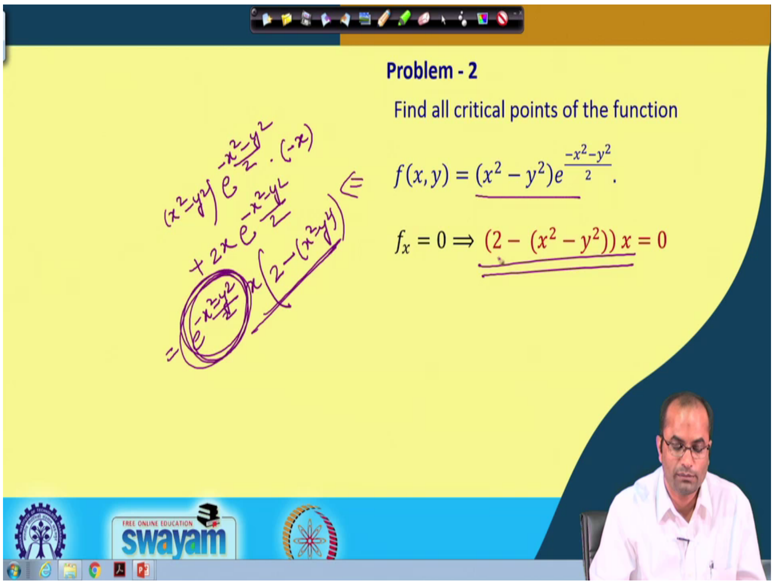

So, now moving to the next problem here again we will find all the critical points of this function, f x y is equal to x square minus y square exponential minus x square minus y square by 2. So, again proceeding with this f x is equal to 0. So, we will get what is f x is equal to 0 in this case. So, we are differentiating this. So, we will get this x square minus y square and the derivative of this term again the same x square minus y square by 2 and then there will be a term minus 2 x divided by minus x plus the derivative of this with respect to x and then we have e power minus x square minus y square by 2 term.

So, if we take this e power minus x square and minus y square by 2 term common and also we can take this x common. So, here we will have 2, then here we will have minus x square minus y square yeah. So, and this cannot be 0 this exponential of minus x square minus y square. So, we will have that this is equal to 0 this is the condition precisely here that this x multiplied by 2 minus x square minus y square must be 0

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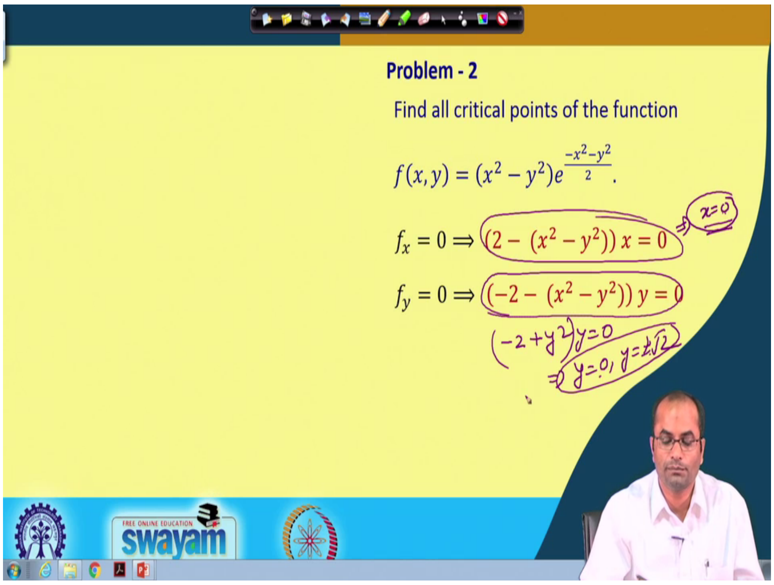

So, out of this condition and now for f y also similarly we will get minus 2 here because of this minus 2 will come and this y is equal to 0. So, now, we can solve this. So, for example, we said here we take here x is equal to 0. So, from this first equation if we take x is equal to 0 irrespective of y there this f x will be 0. And if we take x is equal to 0 and in this case we will get minus 2 and then plus this y square. So, this x is 0 and then we have y is equal to 0.

So, from here we got y is equal to 0 and y is equal to plus minus 2. So, corresponding to x is equal to 0 we have the point x 0; so, here 0 0 and 0 plus square root 20 and minus square root 2. Now, similarly what we can do we can set like here y 2 0 first and then because this for y is equal to 0, this will be also 0 and then setting here y is equal to 0 you will find the values possible values of x. So, in this way we can find all the critical points by solving these equations.

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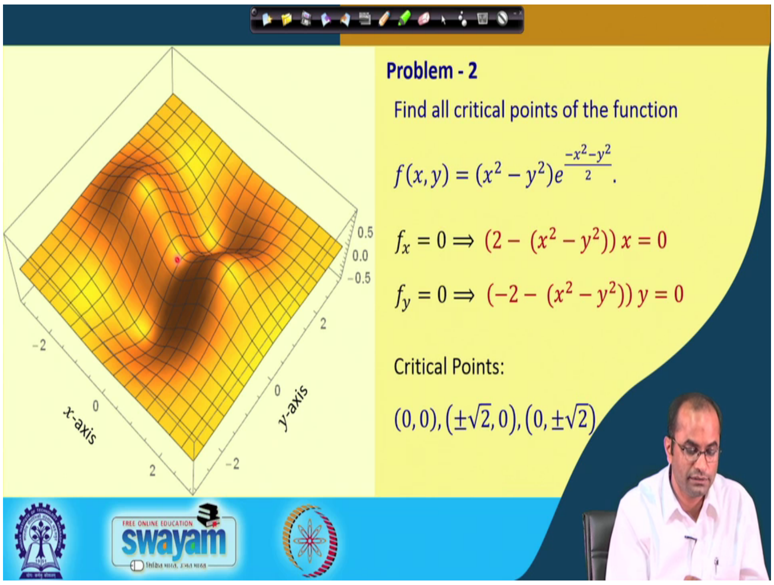

And what do we get? So, we have the critical points is 0 0 we have the square root 2 0 minus square root 2 0 and then 0 plus square root 2 0 minus square root 2. So, all these are the critical points which we can see here in this problem.

So, these are the critical points. So, we can see like this one this one this one this one and again here there is a critical point. So, these are the critical points of this problem.

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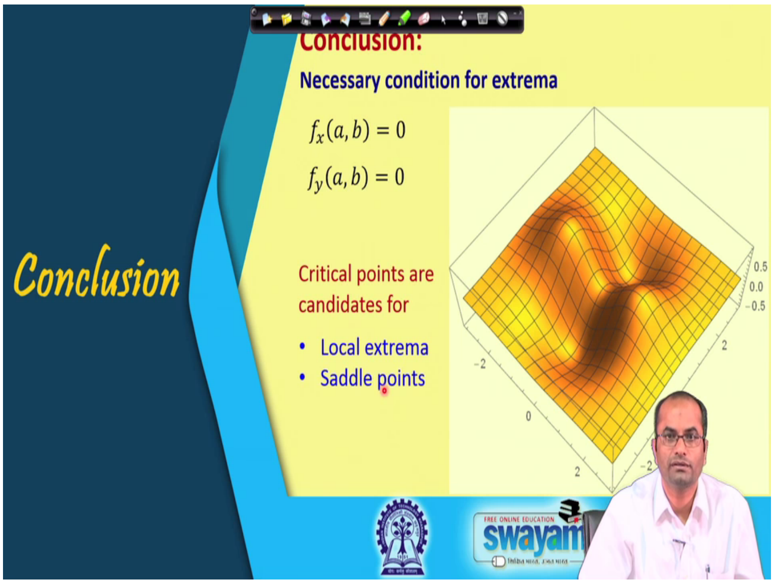

So, what is the conclusion now, we have found the necessary conditions for the extrema f x at a b should be 0 and f y a b should be 0 if this point a b is a point of local maximum or local minimum and these critical points are the candidates for the local extrema and the saddle points. So, because the saddle points are those critical points where the local extrema do not exists. So, these are the points called saddle points.

So, we have already seen that this is for example, the point of local extrema here, we have a local maximum. Here also local maximum these are the points here with local minimum and there is a point at this place here where we do not have a local maximum or minimum and then this will be called a saddle point.

In the next lecture we will learn now, how to identify whether it is a local maximum or local minimum because necessary conditions will give us the candidates for the local extrema and also for the saddle points, but then we have to identify whether a given point a given critical point the function is taking local maximum local minimum or it is a saddle point. So, that will be the content of the next lecture.

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These are the references.

Thank you.