Engineering Mathematics - I
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Lecture – 16
Taylor's Theorem for Functions of Two Variables

So welcome back to the lectures on Engineering Mathematics I and today this is lecture number 16 and we will learn the Taylor's Theorem for Functions of Two Variables.

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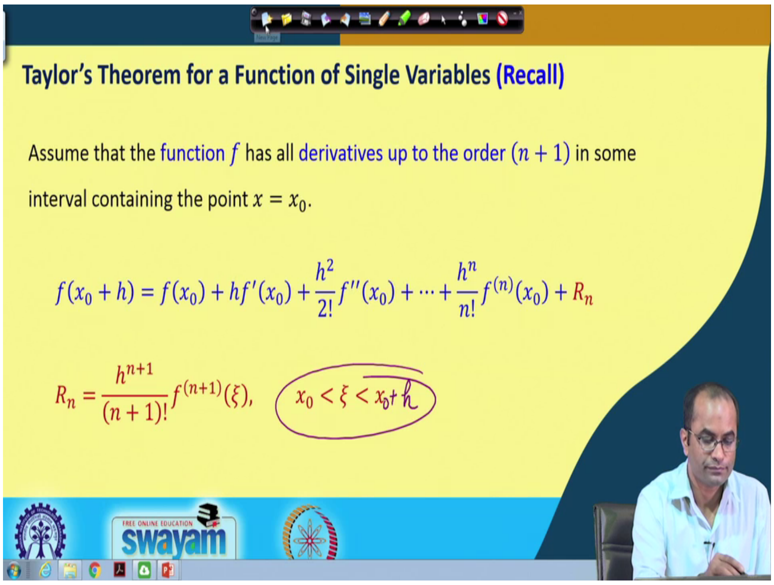

So, what is the Taylors theorem of function of single variables we need to just recall. So, assume that f has all derivatives up to order n plus 1 in some interval containing the point x is equal to x 0. And in that case we can write down this f x 0 plus h. So, this is a point in the neighborhood of 10 x 0 and we can express this as f x 0 plus the h the first order derivative at x 0 h square by factorial 2 the second order derivative and so on.

So, in terms of the these higher order derivatives plus the remainder term because we have the equality here. So, this is this is called the Taylor's polynomial and then this is the remainder the error term in this Taylor's polynomial which is denoted by the h power. So, it is a continuation again so h power n plus 1 over n plus 1 factorial and then we have the n plus 1 and derivative at some point xi which lies between these two points. So, x 0 where we have expanded this around and the point which we are considering here x 0 plus h.

So, or x yeah this is x 0 plus h means the x the point in the neighborhood of this x 0. So, the this point here xi where we have evaluated this n plus 1 a derivative it is not known, but it exists and it is somewhere between x 0 and x 0 plus h. So, precisely we can also write here x 0 plus h. So, this is the point here between x 0 and x 0 plus h.

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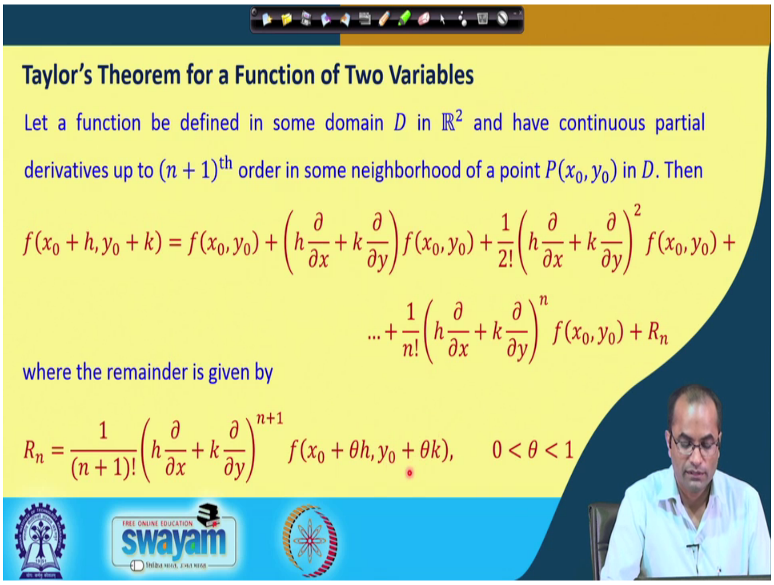

So, now the Taylor's theorem for a function of two variables so we have again that function is defined in some domain D and we have continuous partial order derivative up to n plus 1th order in some neighborhood of a point here x 0 y 0 in the domain, then we can express here this f x 0 plus h y 0 plus k. So, again this is a point in the neighborhood of this x 0 y 0 and we can get this function value at this some other point in the neighborhood by the by this expression here.

So, f x 0 plus y 0 x 0 y 0 and then these are the first order terms, now we have the partial derivative. So, these are the first order partial derivatives. So, h or the partial derivative with respect to x and with k the partial derivative with respect to y and then we have one over factorial terms similar to the single variable case and then here we have this higher order term. So, we will get in the proof now what do we mean by this h, the partial derivative with respect to x plus k the partial derivative y and then whole square. So, do we need to wait a little bit here.

And then this will be continued up to this n and one over factorial n and f x 0 y 0. So, these partial derivatives will be evaluated at x 0 y 0 all the partial derivatives here and then the rest term here the remainder term which is again the continuation of these terms and the only difference is that this point here the argument of the n plus 1th derivative. Which will be appearing because of this term, will be evaluated here at x 0 plus theta h y 0 plus theta k and this theta is between 0 and 1.

So; that means, again this is the point here between this x 0 y 0 and this point here x 0 plus h y 0 plus k. So, this first argument will vary from x 0 to x 0 plus h when theta varies from 0 to 1. Similarly here y 0 will vary from y 0 to or can vary from y 0 to y 0 plus k when this theta varies from 0 to 1.

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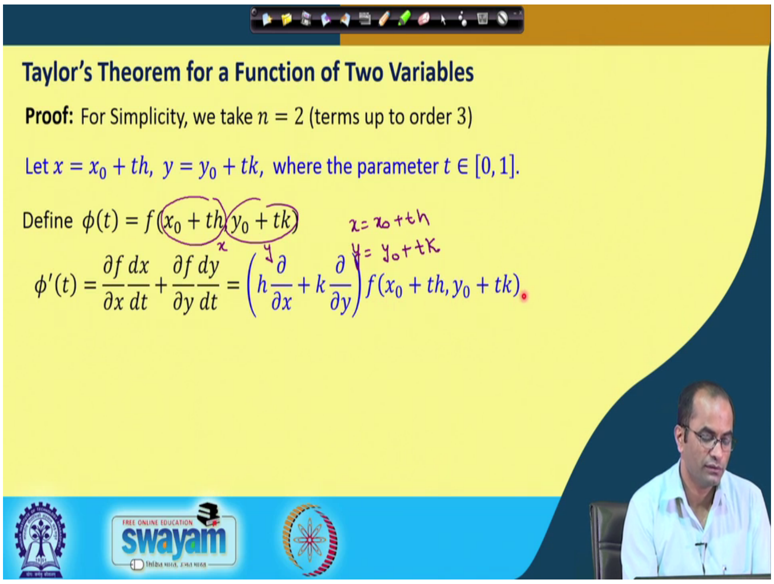

So, we will just go through the proof of this result. So, this is the Taylor's theorem for two variables and we will take just for simplicity the n is equal to 2; that means, you will consider the terms in the expansion up to order 3. So, we take this x is equal to x 0 plus th and y as y 0 plus th and t is some parameter from this interval 0 and 1, including 0 and 1.

So, having this we define a function now phi t, the same the function f x 0 plus th y 0 plus tk. So, this h and k were given already in the in the expansion which is somehow fixed, x 0 y 0 that point is also fixed and then this t varies here in this interval 0 and 1. So, this function here f x 0 plus th and y 0 plus tk having this x 0 and y 0 fixed h and k fixed we have a variable t here. So, therefore, we have defined this as function of t, function of one variable.

And for function of 1 variable we can get the derivative here with respect to t. So, this is derivative with respect to t and remember this is like composite function. So, here we have like x and here we have y where the x is x 0 plus this th and this y is y 0 plus tk. So, by that differentiation which we have learnt before we can compute this phi prime t the derivative of phi with respect to t or the derivative of this f with respect to t because this is a function of 1 variable when x and y are given in terms of t. So, that composite formula we have written del f over del x dx over dt and so on.

And now we will compute this; so, here this is dx over dt. So, x was x 0 plus th so dx over dt means only the h will remain here. So, this is h, this is partial derivative with respect to x we have written as it is then here we will get k the derivative of y with respect to t will be k and then del over del y. So, del over del y and we have just because this was del over del x on f and this del over del y on f. So, that f at x y point we have just written outside. So, let me erase this. So, we have written just this f. So, the meaning is here h and del f over del x at the point x y or this x 0 plus th y 0 t k point plus k and again del f over del y here.

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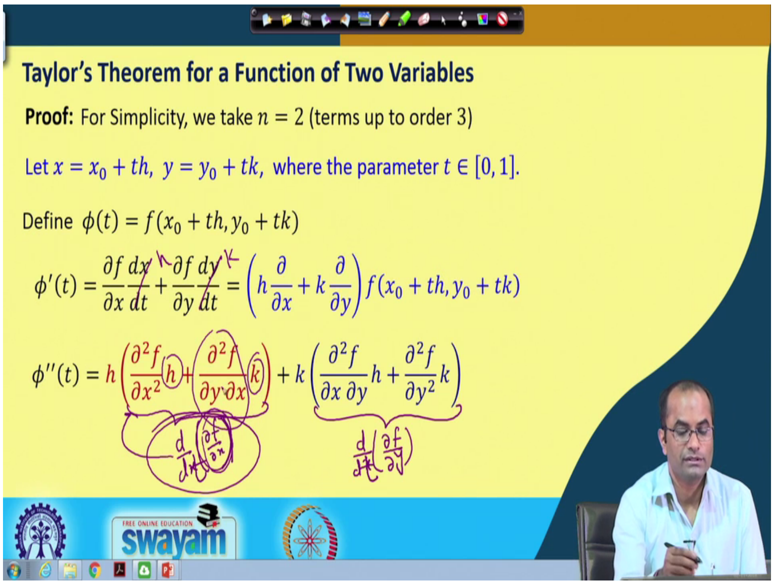

Similarly, we can compute the second order derivative. So, we have the first order terms here with this h. So, this was h here and this was k here. So, we will compute now the second order derivative of this term. So, h will remain as it is and the derivative of del f over del x, again the same this chain formula will be used so the partial derivative of this with respect to x again. So, the double derivative and here again dx dt will term come then h then plus, the partial derivative of f with respect to y so it was already with respect to x. So, we have the second order partial derivative with respect to y and x and then this k will come because we have dy over dt term again.

So, we have used this chain rule again from this del f over del x and also similarly for del f over del y which is the term given here. So, again to make it more clear. So, this was this is the term when we take the derivative of del f over del x and then this h was already here, here the k was there and this is the term which we have computed here d over dx of del f over del f over del y. So, with respect to t sorry with respect to t

So, here we have used this chain rule again so that the partial derivative with respect to x again of this second order derivative and then dx over dt which becomes h then the partial derivative of this with respect to y here and then dy over dt which is k.

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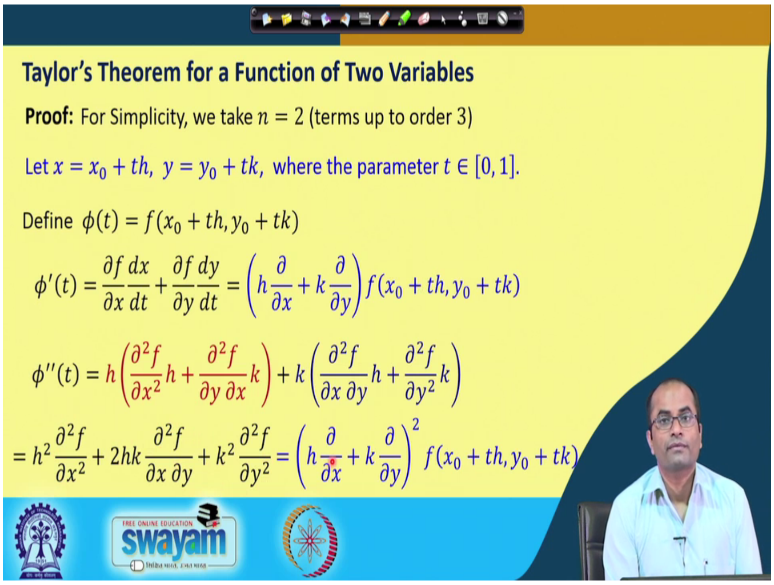

Well and then we have written again this little more simplified form so here h is square and then this hk and here also k h assuming that this second order partial derivatives are equal. So, we have the two term 2 times this h k and the second order mixed order derivative and plus this k square and the second order term here, and then we can compute the third order term as well. So, this is before we compute the third order term we have written this little more a convenient form that this is like h square two h k plus k square.

So, h plus k whole square term, and for the notational convenience we have also because here we have this like del to the second order term here also it is mix term like hk, here also the yy term. So, the second order term so that also we have written inside this these parentheses.

So, the meaning of this square will be h square the second order partial derivative is not a whole square of this, but rather second order derivative of this with respect to x, then the k square and the second order derivative with respect to y and then the two times hk and the mixed order derivatives. So, that we have to understand that this whole square is not literally the a square of this two terms, but the meaning of this here is in terms of the second order partial derivatives.

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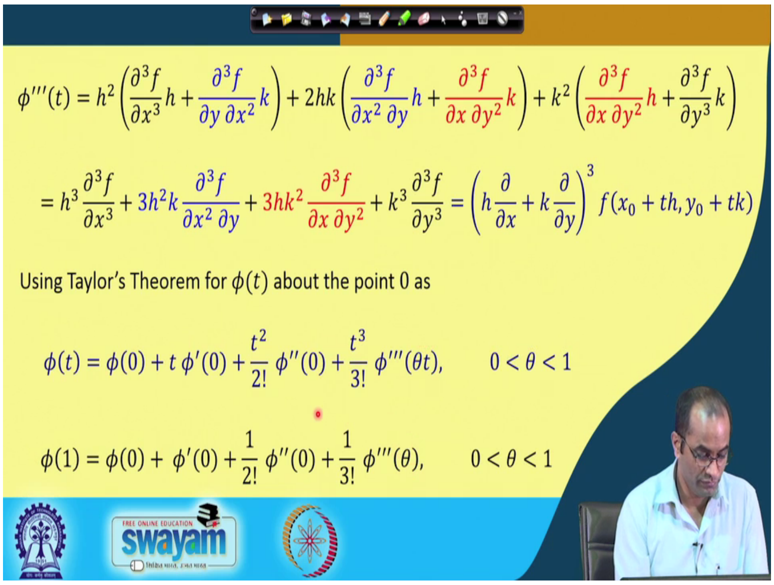

Similarly, we can compute the third order derivative. So, in I am not going to explain this now. So, here for h square term we have again here we have to use this chain rule then again here and here so, all these three terms. So, with h square this is from this first term then 2 hk and k square and so on and again the simplified form we have written this.

So, 3 times h square k is appearing here we have h square k and then here also we have h square k and assuming the continuity of these third order derivatives. So, we can assume that this partial derivative with respect to x 2 times and then partial derivative with respect to y or first partial derivative y and then 2 times with respect to x they are equal.

So, treating them equal we have these terms and again we have written for simplicity because this is like a cubic term in terms of h cubed 3 h square k 3 h k square and k is cube. So, we have combined this again these third order derivatives terms as well in this cubed term and this f at this point.

Now, we will use the Taylor's theorem for phi t phi is a function of one variable. So, it is a 1 variable case again for the Taylor's theorem which says that phi t will be phi 0 t phi prime 0 and so on. And up to the third order term so t cube or factorial 3 so, we can extend this to any order term. So, phi the third order derivative and theta t and now we will substitute this t is equal to 1 here because t varies from 0 to 1 it is this is true for any value of this t. So, we have taken the t is equal to one and then we get here just only theta. So, now we will substitute this phi which we have computed.

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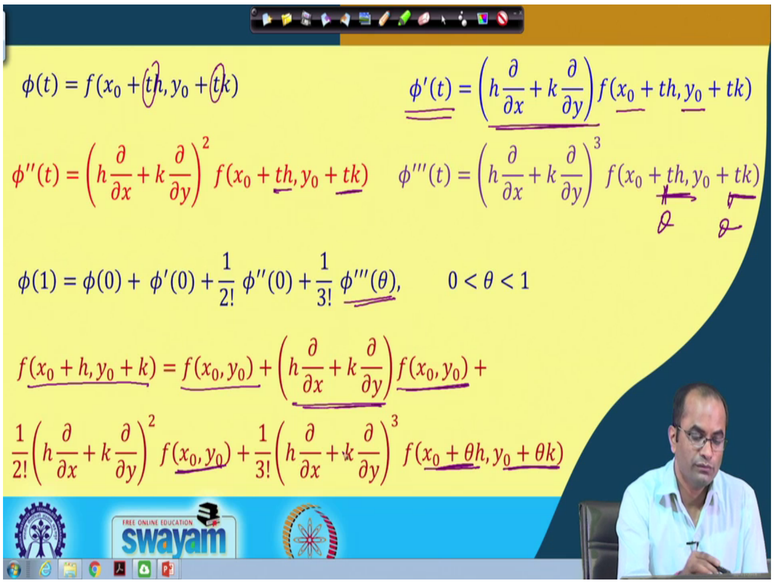

So, the phi t was this phi prime phi double prime and so on and this was because of the Taylor's theorem of one variable and then we can substitute now. So, the phi at 1, so phi at 1 means the t will be substituted as 1 here. So, this t will be 1 1 so we will get phi f x 0 plus h and y 0 plus k this term, phi 0. So, when t is 0 so we will have x 0 y 0 term and then here phi prime at 0. So, phi prime is here and at 0 means x 0 and y 0 point. So, this parenthesis here h and the partial derivatives the first order partial derivatives and this evaluated at x 0 y 0 point.

Similarly, for the second order term again t will be set to 0. So, we will have this x 0 y 0 point and for this one the here the third derivative will be computed as theta point. So, here the t will be replaced by this theta, here also by theta and then we have this third order terms. So, x 0 plus theta h and y 0 plus theta k and this is the result which we want to, we want to prove for this Taylor's theorem for the functions of two variables.

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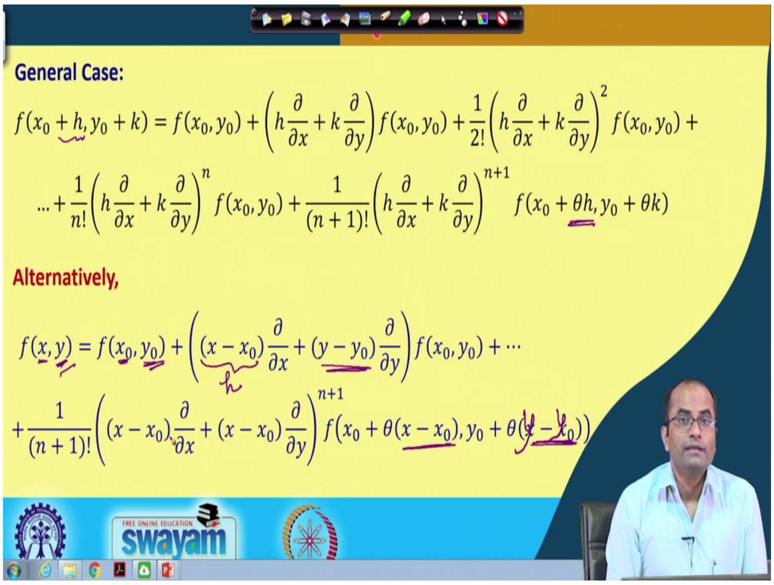

And in general also we can extend that expansion in this Taylor's theorem of one variable and then we can have for the two variables as well. So, in general we have this result that x 0 plus h y 0 plus k and we can keep on this continuing. So, a squared from the cubic term and then we have this nth order partial derivatives terms and then this is the remainder term where we have this general variable this x 0 plus theta h y 0 plus theta k where theta is between 0 and 1. So, this argument here varies from x 0 to x 0 to this x 0 plus h and here also this varies now from y 0 to y 0 plus k when theta varies from 0 to 1. And alternatively we can also write down this expression in terms of like f x, y is equal to f x 0, the only difference is here it was x 0 plus h and now we have x y point here. So, this difference again. So, this h was nothing, but the difference here x 0 plus h and x 0.

So, here also now the h will become as the difference of x and x 0; that means, x minus x 0. So, exactly we have this is like h here. So, here also we have this h and because this was just the difference of this x 0 plus h and this x 0 trump. So, that h was appearing there, but in this case when we have taken x y point in the neighborhood of x 0 y 0 then this institute of h we will write down the difference here x and this x 0. So, x minus x 0, here again this y minus y 0 from here and so on so, the rest everything will be the same. Only this h will be like here the it was theta h so theta x minus x 0 and y 0 plus theta k so this is y minus y 0 term, the rest everything will remain the same.

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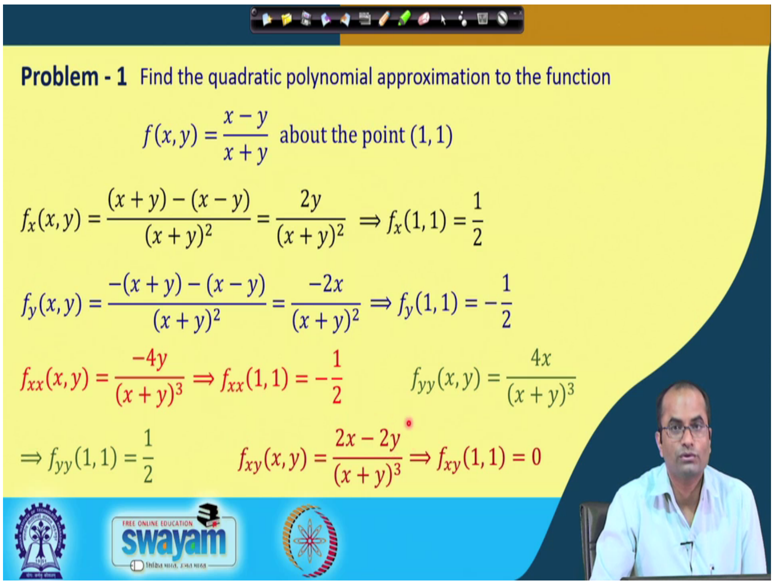

So, we have now some problems based on these Taylor's expansion the first one is find the quadratic polynomial approximation of the function this fx y is equal to x minus y over x plus y. And we want to expand this only the quadratic polynomial around this point 1 1. So, we are not writing here the whole Taylor's theorem, but only the quadratic polynomial term we want to approximate that.

So, when we take the in our formula if we just go back. So, up to this one if we write down at this x 0 y 0 point. So, this is like a Taylor's polynomial up to this point here and if we include this one then we call this the Taylor's theorem or Taylor's result including the remainder term. But if we leave this remainder term then this will be the approximation of this f at this point in which is in the neighborhood of this x 0 y 0 point. So, that is the Taylor's polynomial if we fix this n and do not write this r n term.

So, here we are interested in a quadratic polynomial of this function and about this point x 0 y 0. So, we need to get the partial derivative of this with respect to x; that means, we will be differentiating this with respect to x treating y is constant. So, this quotient rule here the whole square term there and this term as it is and the partial derivative of this numerator term with respect to x which is 1 and then we have the minus term x minus y and this partial derivative of this term with respect to x which is again 1. And when we simplify this so this is a 2 y term here; so, 2 y over x plus y a whole square and then at this point 1 1 we can evaluate this, this is 2 over 2 squares this is 1 over 2. So, we have the one over 2 term here.

Then the partial derivative similarly with respect to y; so, again we will get here minus 2 x over x plus y square and we can compute this at 1 1 point which will give us minus half. Similarly, we need because for the quadratic term we need at least we need to go to the second order term so f xx. So, we have to take the derivative again with respect to x here treating y is constant. So, the 2 y will be treated as constant.

So, we have 1 over this x plus y whole square and when we take the derivative with respect to x. So, this will be minus 2 over x plus y cube. So, minus 2 and this 2 y will become minus 4 and x plus y power 3 and again at this point 1 1 this will become minus half similarly the f yy when we differentiate this with respect to y. So, we will get here 4 x over x plus y cube whose value at 1 1 is is again half, then we substitute x and y as 1 1. So, we will get half there.

And the mixed order term so, whether we can differentiate this f x is equal to 2 y over x plus y square term with respect to y or we can differentiate this term here with respect to x to get this term 2 x minus 2 y and x plus y power 3 and again we need to compute this at the point 1 1. So, we will get as 0 because when we substitute 1 1 there 2 minus 2 that will become 0.

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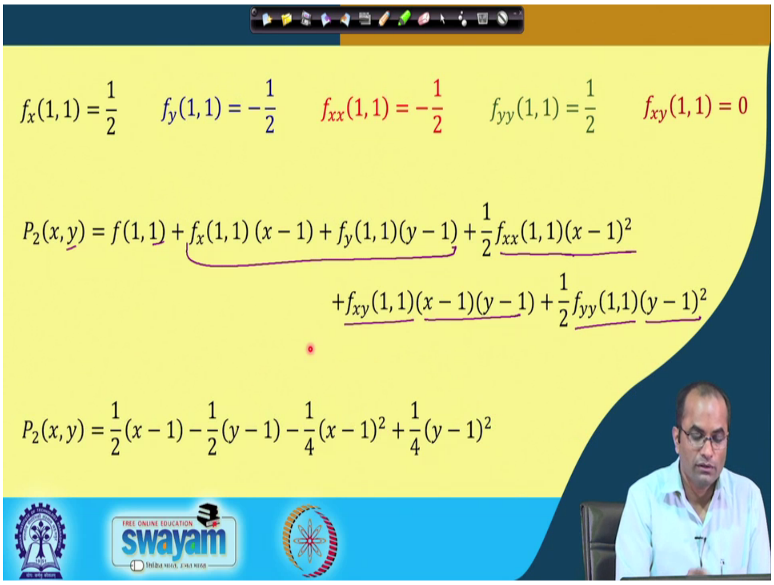

So, here we have now the f x we have evaluated at 1 1 which is 1 by 2 f y at 1 1, it is minus 1 by 2 f x x the second order term its minus 1 by 2 and so on. So, the second order polynomial as I said we will just go up to the second order term. So, we have f 1 1, the partial derivative with respect to x at 1 1 and then x minus 1 from here to here these are the first order terms. So, the f x at 1 1 x minus 1 so, this difference again in terms of h f y 1 1 and then we have y minus 1 again the difference here.

Then we have the second order term with this 1 over factorial 2, which is 1 over 2 there is a higher order term f x 6 and then this a h square term. So, x minus 1 is square and this is 2 times, but with that half it will become 1 now. So, x minus 1 x minus y minus 1 term and then here the second order term with respect to y and the way we have y minus 1 whole squared term.

So, we can substitute these values here f 1 1 will be 0, because of that function and here f x at 1 1 was half. So, we have half x minus 1 here again we have minus half and y minus 1 1 by 2 and f xx minus half. So, will become minus 1 by 4 x minus 1 whole square, this will be 0 because this mixed order term is 0. So, this will become 0 and then we have 1 over 2, again here we have 1 over 2 so 1 over 5 and y minus 1 square.

So, this is the quadratic polynomial which is approximating the function in the neighborhood of this 1 1 point and the accuracy of this polynomial will depend on how far we are from the point 1 1 if we are in a very close neighborhood of 1 1 this will give us a very good approximation of the function.

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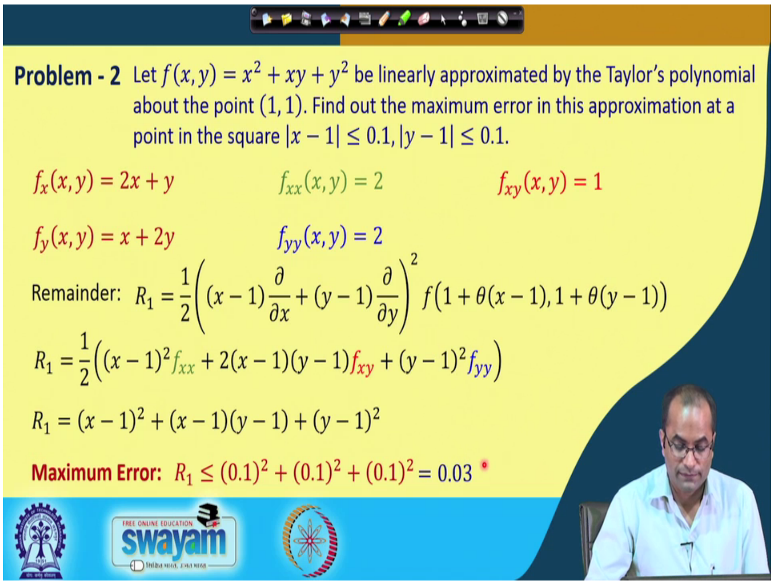

So, another problem when we have the function here x square plus xy and plus this y square be linearly approximated by the tailors polynomial. So, it is given here that this function is linearly approximated by the Taylor's polynomial about this point 1 1. And now we want to find out the maximum error in this approximation at a point in this square here x minus 1 less than 0.1 and y minus 1 less than 0.1.

So, what you want to discuss here that if we have this 1 1 point for example, and we are expanding this function around this point by a linear approximation. So, only the linear terms are considered in the approximation and we want to find out the maximum error in that linear approximation, if we take any point here in this square around this point 1 1 by this point. So, this x so, this is 0.1 here this is 0.1, 0.1 or the whole h here is 0.2 from here to here.

So, if we take any point in this neighborhood square neighborhood around this point and what will be the maximum error in that linear approximation of this function we want to evaluate. So, in that case we will use make use of the remainder term directly. So, we have f x x y as we can compute from here it is 2 x the partial derivative of this f with respect to x this would be 2 x plus y and f y x plus 2 y again the second order terms. So, here will be just 2 and there with respect to y it will be again two there with the mix term we will have 1 and then the remainder which is written after this linear term.

So, the quadratic term will be coming the remainder. So, this is exactly the remainder which we have discussed before and which we can write down in this form. So, x minus 1 whole square and this will be the second order term f xx 2 times f x minus 1 y minus 1, the mixed order term and y minus 1 is square and then we will have here again the second order term with respect to y.

So, all these second order derivatives we have already computed and they are actually constant in this case. So, we have here 2 and here we have 1 and then 2 again. So, after substituting this we have this remainder term and now that is given that this x minus 1 is less than 0.1 and y minus 1 is less than 0.1. So, we can write down those terms here now to get this approximation that R 1 will be less than equal to we have written down the maximum error here which is 0.1 this x minus 1. So, 0.1 square here again this product will come and 0.1; so, 3 times this 0.1 square which will be 0.01 03.

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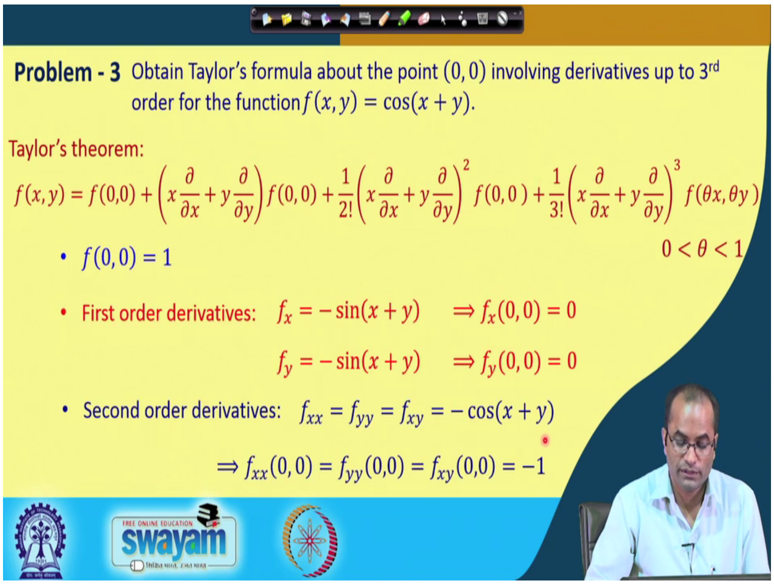

The last problems here so we have Taylor's formula about this 0.00 involving derivatives up to these third order terms of this function cos x plus y. So, again this is the Taylor's theorem including this remainder term and we have considered these third order terms here. So, we need to compute this f 0, 0 which is in this case 1.

So, a cos 0 will be 1 and then the first order derivatives we have to compute for this f x simply it is a minus because this cos will give minus sin x plus y and then again at 0 0 this will be 0. Partial derivative with respect to y will be again sin and this will become 0 there.

So, similarly the second order derivatives here the sin will becomes the cos and then we have minus cos x plus y which we can evaluate at this 0 0 point and this will give us minus 1.

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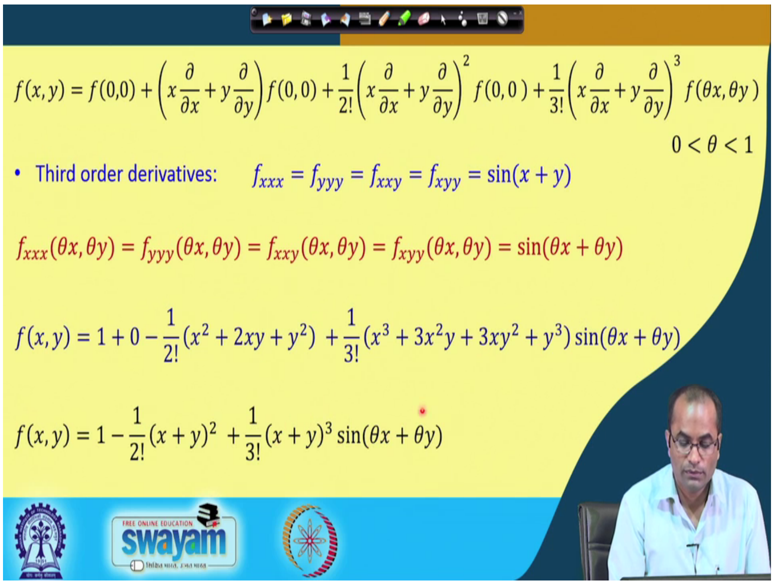

So, the third order derivatives similarly we will get back to again the sin x plus y and for the third order derivative we need to compute at theta x point. So, at theta x point this will become as theta x plus theta y and now we can substitute here in this expansion. So, we will get 1, this was the 0 the first order derivative terms then again here one was the value and then we have the sin the third order derivatives in this term.

So, we have the sin theta x plus theta y and after the simplification. So, we will get 1 minus this x plus y whole square this is x plus y whole cube and then this is sin because it was common in also sin theta x plus theta y.

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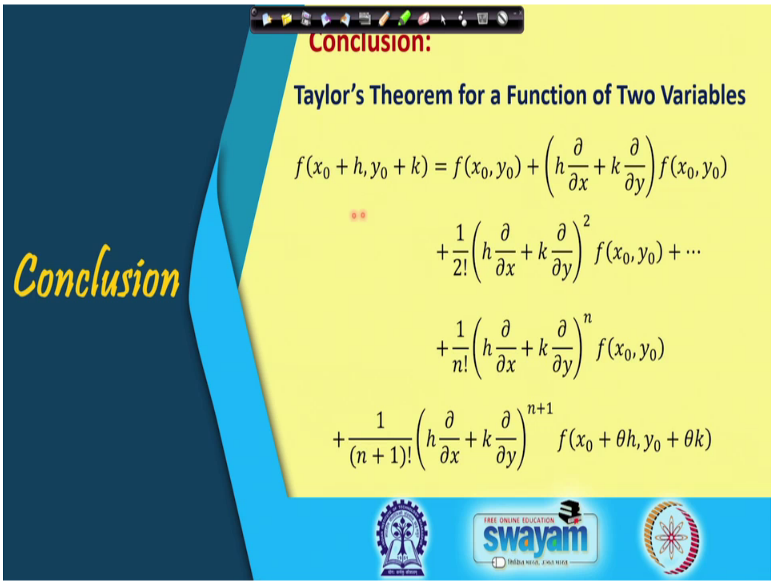

Well so, we have learned this Taylor's theorem for a function of two variables and in this case what we have what we have observed that it is just the extension of these functions of 1 variable. So, this is the point in the neighborhood of this 1 and we can expand this function or get this value or write down this value in terms of this expansion here.

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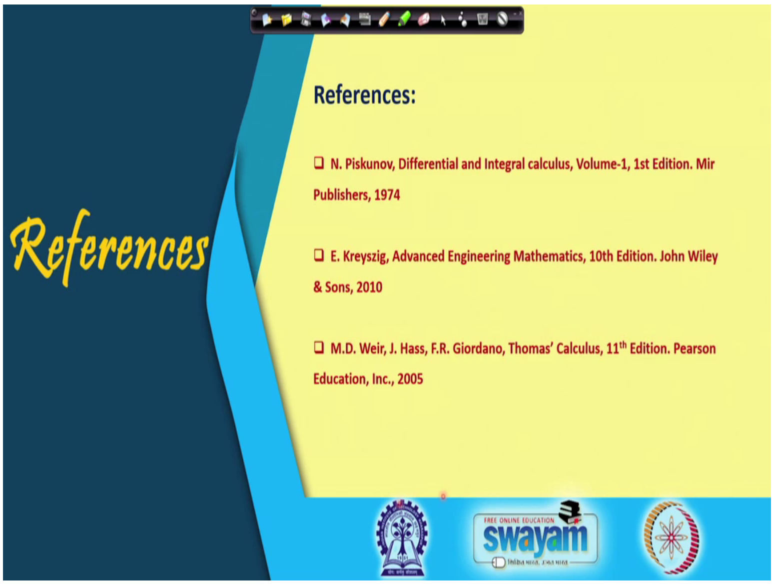

So, these are the first order terms here we have the second order terms with h square k square and 2 hk and similarly we will have the higher order derivatives there. Up to this n and this is the remainder term which is often useful to get the error or the estimation of the error in the approximation. So, these are the references used for the preparation of this lecture.

Thank you very much.