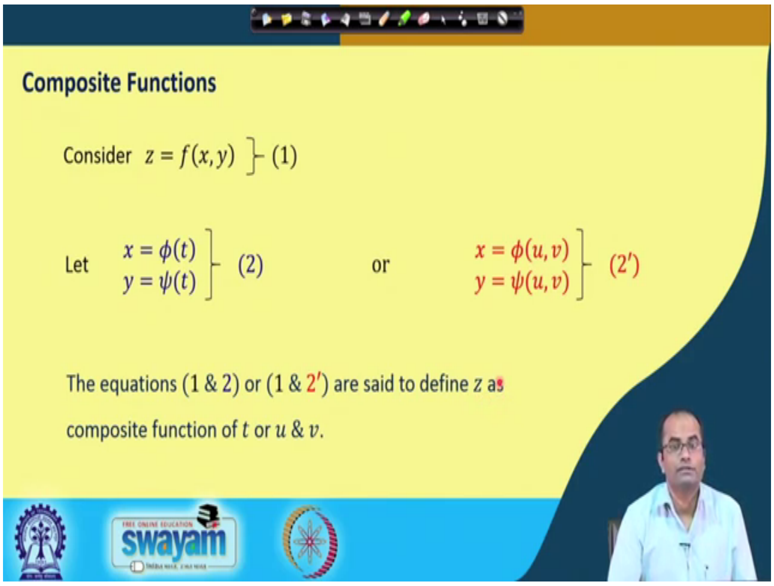
Engineering Mathematics - I
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 15
Composite and Homogeneous Functions

So, welcome back to the lectures on Engineering Mathematics-I. And this is lecture number 15. And, today we will be discussing about a Composite Functions and Homogeneous Functions.

(Refer Slide Time: 00:28)



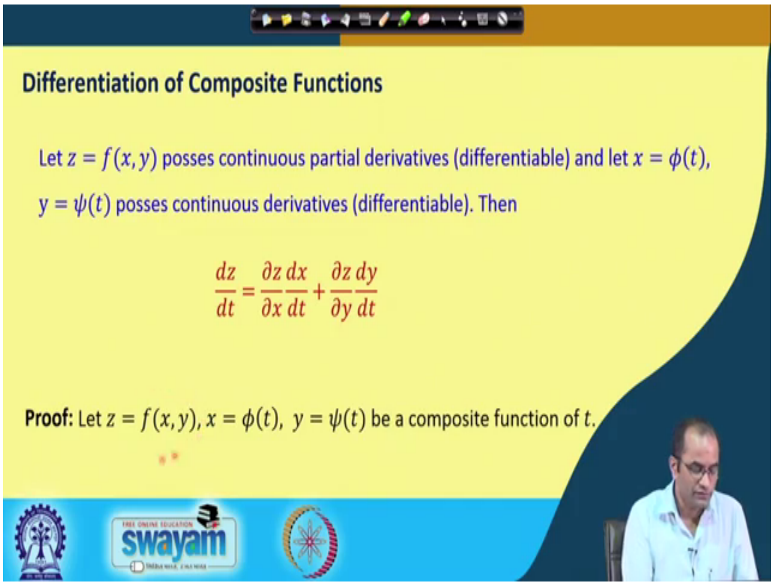
So, what are the composite functions? So, if we consider here function z is equal to f x y a function of 2 variables x and y and we let that this x depends on t. So, x is a function of a phi t and here y is also a function of t which we are denoting by psi t. Or this could be that this function x is a function of 2 variables again u and v and y is a function again of 2 variables u and v.

Let us call this equation number 1, here the equation number 2 and this is equation number 2 prime. Then the equations here 1 and this 2 together or 1 with this 2 prime together are said to be to define z as composite function of t or in this case composite function of u and v. So, here what basically we will be discussing here, if we want to get for example, the derivative of this z function with respect to 2 t. Because in this first case when we are taking equation number 1 and equation number 2 then this z depends on x and y, but the x and y again depends on t.

So, basically this z is a function of t and then we can define here the derivative of z with respect to t directly. So, we will derive a formula how to get the derivative of this z without substituting this x and y here in this function. And, then making this as a function of t and then taking the derivative. But, we will find a direct formula which will use the partial derivatives of this function f which is a function of 2 variables x and y.

And also here the x and y are the functions of phi and psi both are the functions of t and we will also make use of the derivative of these functions to get the derivative of this z directly without substituting this x and y in terms of t in this function f x y. And, the same scenario we have when we have this equation 1 z is equal to f x y with equations x is equal to phi u v and y is equal to psi u v.

(Refer Slide Time: 03:06)

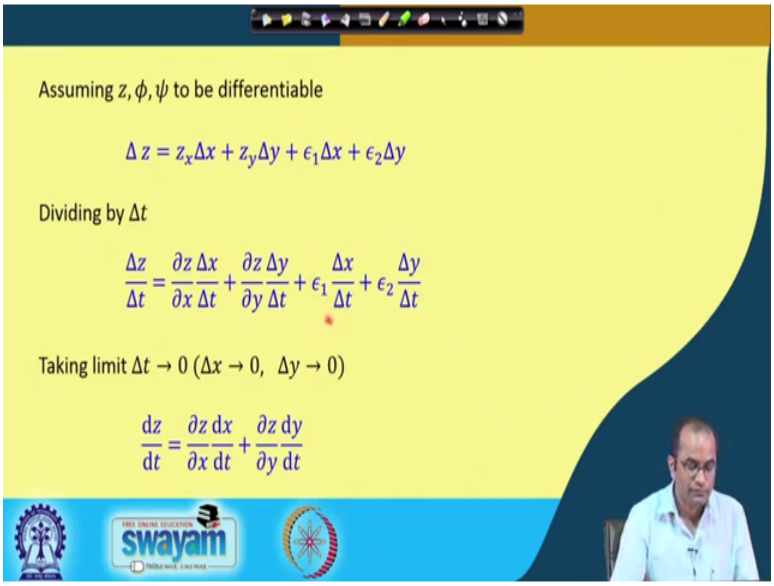


So, let us take the first case lets z posses continuous partial derivatives or we can also take this assumption that this z is equal to f x y is differentiable. So, this f x y is a differentiable function and then we also let that x is equal to phi t and y is equal to psi t, they posses a continuous derivatives or again we can assume that they are differentiable functions.

So, in that case we will derive this formula which says that we can get this dz over dt. So, the ordinary derivative of z with respect to t which makes sense now because x and y are functions of t. So, basically this z is a function of t alone. So, in this case this formula says that we can get this ordinary derivative dz over dt is equal to the partial derivative of z with respect to x. So, we are making use of that this f is a function of 2 variables x and y. So, we will take here the partial derivatives of this f with respect to x and multiplied by the derivative of x with respect to t.

So, it is again an ordinary derivative because x is a function of 1 variable t and then this is a chain rule against of plus this because z is a function of y as well. So, we will have here the partial derivative with respect to y and multiplied by dy over dt. So, we will derive this formula now. So, we take that this z is equal to f x y and x is a function of phi t and y is a function of t as denoted by psi t and this is a composite function which we call as for the definition.

(Refer Slide Time: 04:56)



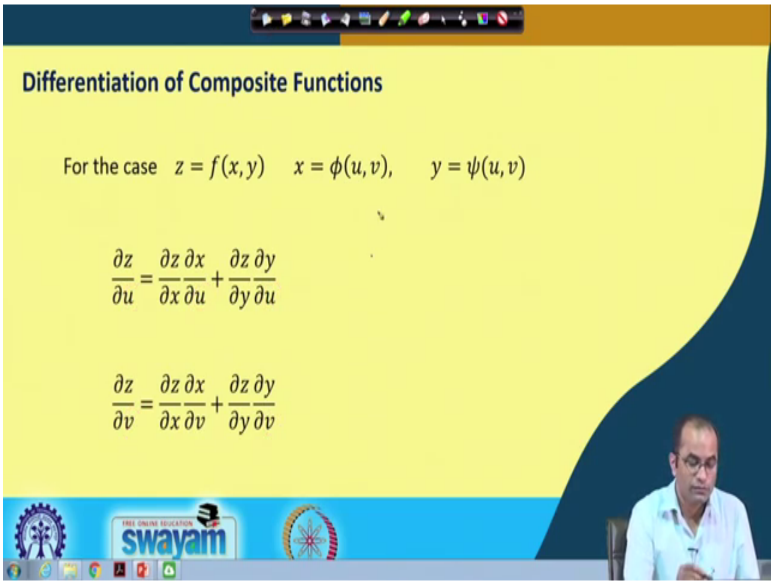
And since we have assumed already the differentiability of all these functions z phi and psi, then differentiability of z will imply that we can write down this delta z is equal to del x delta x plus del y delta y and psi 1 delta x plus psi 2 delta y. And, if we divide this expression both the sides by delta t then what we will get; here the delta z over delta t and this delta x divided by delta t. Here again we have delta y divided by delta t and it is dx and dy will be also divided by delta t.

And now if we take the limit as delta t goes to 0 then what will happen. So, delta t goes to 0 means delta x goes to 0 and delta y goes to 0 because this x and y they are functions of function of t and if there is no variation in t. So, naturally the x and y variation in x and y will be also 0. So, as delta t goes to 0 meaning delta x goes to 0 and delta y goes to 0.

So, when we take the limit now as delta t goes to 0 in this expression here. So, this delta z over delta t as per the definition of the of the derivative of z will be like dz over dt and this will remain as it is; we are not touching this del z over del x and then we have here delta x over delta t. So, again this is the derivative of x with respect to t. So, we get here dx over dt; similarly here again we have delta y over delta t which will become here dy over dt and these terms.

So, assuming again that these partial these derivatives exist and they are finite numbers. So, this when delta x and delta y goes to 0. So, this epsilon will go to 0 and this epsilon 2 will go to 0 and then these expressions here epsilon delta x over delta t and this epsilon delta y over delta t they both will go to 0. So, we have only this these three terms here. So, dz over dt we can find out directly in terms of the partial derivatives of z and the ordinary derivatives of x and y.

(Refer Slide Time: 07:26)

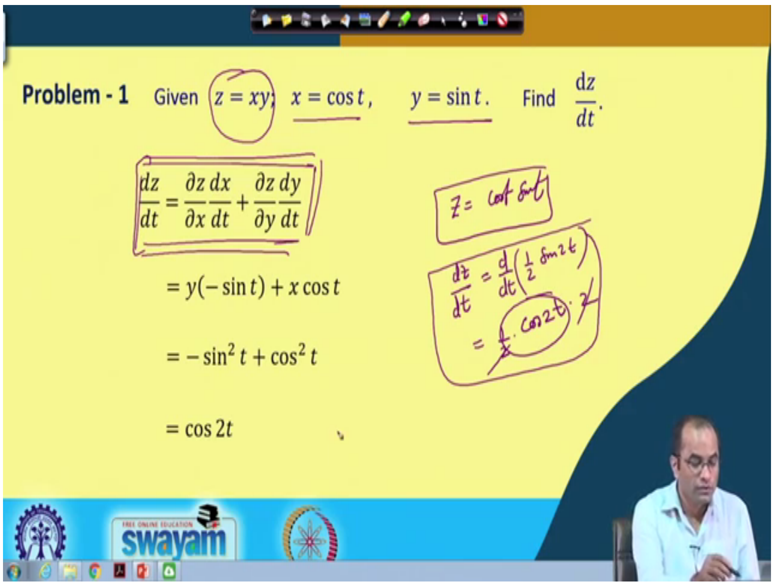


So, the differentiation of the composite functions when we have this more general case that x and y they also depend on u and v a functions of 2 variables. In that case the formula will become that so, we can prove similarly as above. So, this del z over del u so now, the partial derivatives here because the x and y they are the functions of 2 variables u and v.

So, we will have here the partial derivative of z with respect to u without substituting all these into this function and then getting this partial derivatives. We can use this formula to get the partial derivative of this z with respect to u is equal to the partial derivative of z with respect to x and partial derivative of x with respect to u plus partial derivative of z with respect to y and partial derivative of y with respect to u again because we are differentiating partially z with respect to u. So, here the u will come and here also the u will come.

Similarly, for del z over del v now the similar formula, but instead of this u it will be replaced by v. So, what is important if you are differentiating here with respect to u then this u will appear both the places and if we are differentiating with respect to v here. So, then this v will appear at these places here when we are taking the partial derivative of x with respect to u and here with respect to v well; so, moving next.

(Refer Slide Time: 09:14)



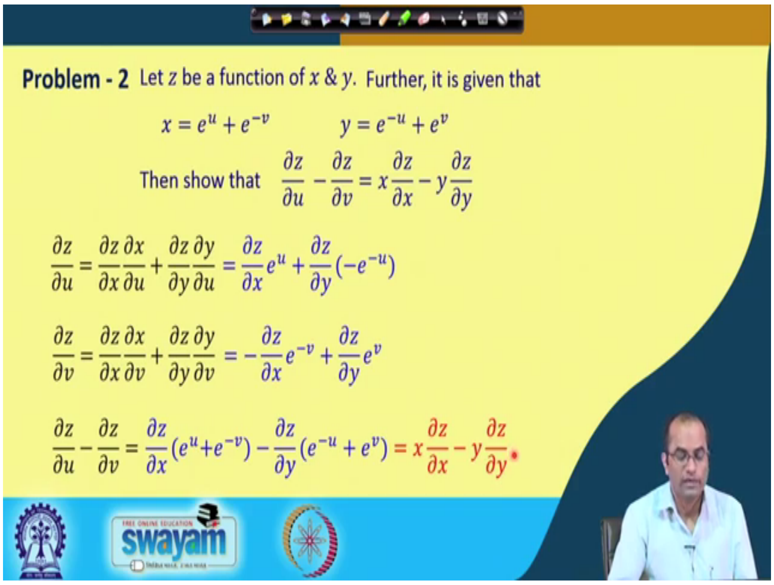
Let us solve some problem based on this. So, for example, we have this function z is equal to xy x is a function of t. So, which is defined here as x is equal to cos t and then y is a function of t again which is denoted by y is equal to sin t and then we want to find here what is dz over dt. So, one direct way would be that we substitute this x and y here in this function and then we will have z as a function of t which we can differentiate to get dz over dt. But, we will follow the idea of the composite functions that z is a function of x y and x is a function of t and y is a function of t with the formula derived above.

So, that formula says that dx over dt will be the we can have here with partial derivative with respect to x of the function z and dx dt and similarly here the partial derivative of y and then dy dt. So, what do we get here? Del z over del x. So, z was x y and when we differentiate here with respect to x then this will become only y and dx over dt so, x was cos t. So, we will get here minus sin t. Similarly, here del z over del y will become x and dy over dt which is sin t it will become cos t.

So, we get here the y is sin t so, we have minus sin square t and x was cos t. So, we have cos square t. So, this is the derivative of z with respect to t without substituting into this function and then getting the derivative. So, this we can again simplify to have this cos 2 t. As I said they alternatively we can just substitute here z is equal to x x is the cos t and then y is sin t. So, now we have z as a function of t and we can get dz over dt there. So, this we can write first as half of 2 cos t sin t which is sin 2 t and then when we take the differentiation here this will be a sin t will be cos 2 t and this 2 will get cancelled.

So, we will have this cos 2 t directly as well, but we used here the idea of this composite differentiation or the differentiation of composite function. And, with this formula we do not have to substitute the values of these x and y into the function, but directly with the help of the partial derivatives of this z we can get the derivative of z directly with respect to the function t.

(Refer Slide Time: 12:14)



So, again next problem here z is a function of x and y and further it is given that x is a function of 2 variables now. So, u and v and then we have here y is a function of again of 2 variables u and v. And, then we will show that this partial derivative the difference of this partial derivative here with respect to u and v can be written as x partial derivative of z with respect to x minus the partial derivative of z with respect to y. So, z is a function of x and y. So, with the formula derived above we have a partial derivative of z with respect to u in terms of the partial derivatives of z with respect to x and y and the partial derivative of x with respect to u partial derivative of y with respect to u again.

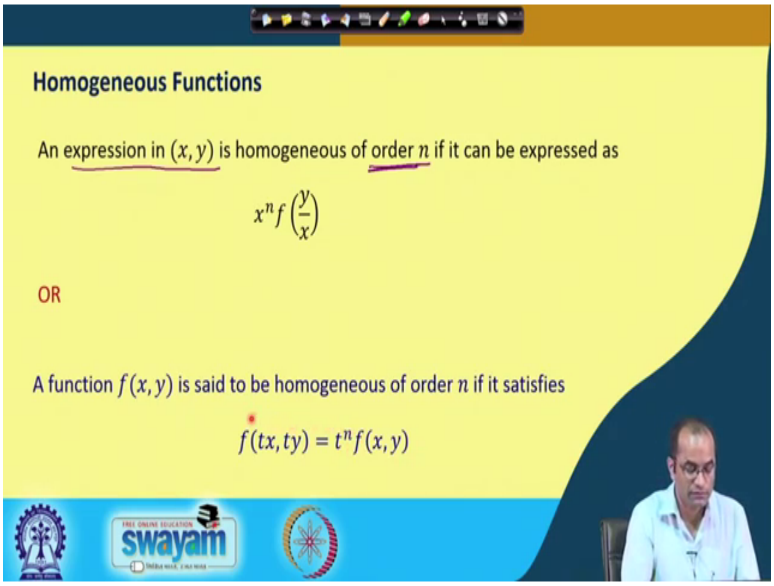
So, if it is u here it will be u there in both the terms here with respect to u of the derivatives x and y. So, in this case so, del z over del x because this is not given function here z is a just function of x y. So, this partial derivatives z with respect to x will come and del x over del u. So, what was x here? e power u plus e power minus v. So, if we take that partial derivative with respect to u; that means, treating v as constant so, this term will be treated as constant. So, the partial derivative of x with respect to u will become simply e power u which is written here and then del z over del y and del y over del u.

So, the y is here e power minus u plus e power v and when we take the partial derivative here with respect to u. So, it we will get here minus e power minus u which is the term here and now the partial derivative of z with respect to v again the similar formula, but instead of u we have v here and then when we take the partial derivative of x with respect to v. So, it is e power minus v. So, we will get minus e power minus v and then we have here y with respect to v.

So, this will become e power v which is written here and now since we want to get the difference of these 2 partial derivatives. So, we will take this difference here and then so, del z over del x here this will become plus. So, del z over del x when we take common it will be e power u and plus e power minus v; similarly here when we take these 2 terms common with minus sign.

So, again e power minus u plus e power v will come as written here and then this e power u plus e power minus v is x and here e power minus u plus e power v is y. So, we get precisely the desired term here. So, this is x del z over del x and then minus this is y and del z over del y.

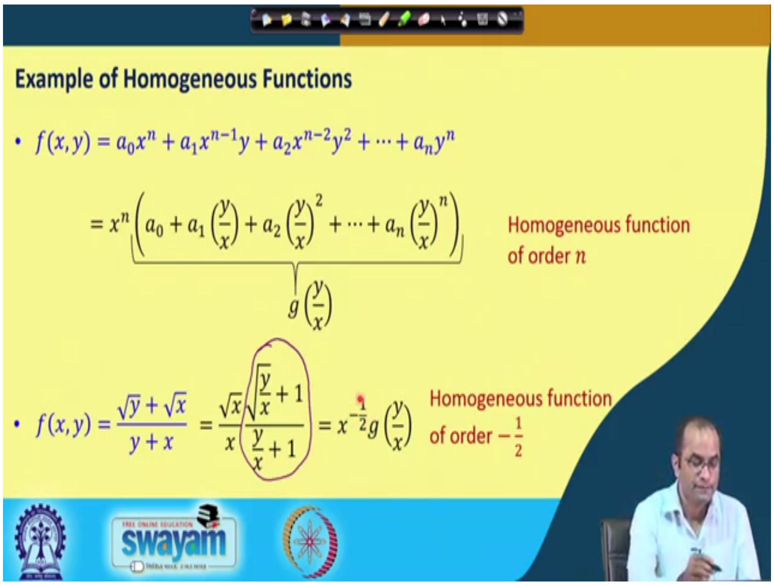
(Refer Slide Time: 15:25)



So, another part of this lecture is to define the homogeneous functions. So, what are the homogeneous functions we will learn now? So, an expression in xy is homogeneous of order of order n there. So, any expression here in x and y of order n, if it can be expressed in this form that x power n n some function of y over x then we call such a function of homogeneous function of order n.

So, this is important this n here. So, if we can bring this x power n and then the rest term can be written as a function of y over x, then we call such expression as a a homogeneous expression of order n or in terms of the functions we can define. So, a function f x y is said to be homogeneous of order n if it satisfies so, we replace here the argument x and y by t x and t y. And, if we can bring this t power n out of the function; that means, this t power n. And again the function of x y remain then we again have this the concept of the homogeneous function of order n. So, we will call this such a function a function a homogeneous function of order n.

(Refer Slide Time: 16:55)



So, these are the examples of the homogeneous functions. So, we consider for example, this f x y is equal to a 0 x power n a 0 x power n minus 1 into y again here x power n minus 2 y square and so on a n y power n. So, what we observe that this is a function this is a homogeneous function of order n. Why order n, because if we take this x power n from all these terms outside then what will we will get here this will be a 0 because x power n we have taken out.

And here we have again x power n taken out so, 1 x we have to divide. So, a 1 and then y over x it will become. So, here we have taken x power n out again and then this x power minus 2 will remain; that means, a 2 and then y over x power 2 and so on. Here a n and x n we have taken out. So, this x power n will be in the denominator term and in this case again we have here y over x power n.

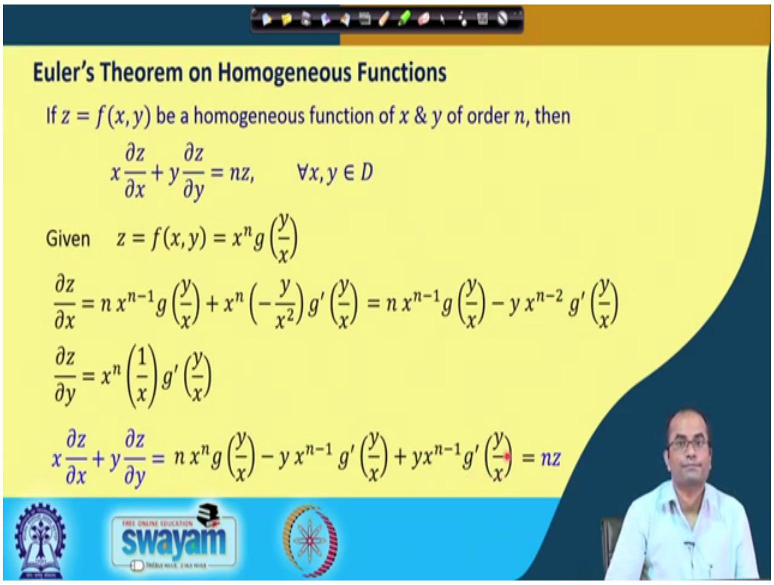
So, all these terms here or this function we can denote as the function of y power x, because this y power x appears together in all the terms. So, we can consider this function here as a function of y over x and then x power n is sitting outside. Therefore, this is a function of homogeneous function of order n. Similarly, if we consider this 1 f x y is equal to square root y plus square root x over y plus x.

In this case also we can write down this as x power something and the rest we can consider as the function of y over x. And, this is simple because we can take here square root x out and here we will take x out. And in this case so, here square root x over x and then inside here we have a square root y over x plus this 1 and here also y over x plus 1.

So, now this one here, this expression we can consider as the function of y over x and then what is together here x power minus half. So, this we can write down as x power minus half and some function of y over x because in all these terms y over x comes together. So, we can say that this is some function of y over x and in this case as we see here x power minus half. So, this is a homogeneous function of order minus half. So, what is the importance of these homogeneous functions: we will see in the next slide.

So, based on these order for example, here it was n or here minus half we can have some formula for the derivative term in terms of this n.

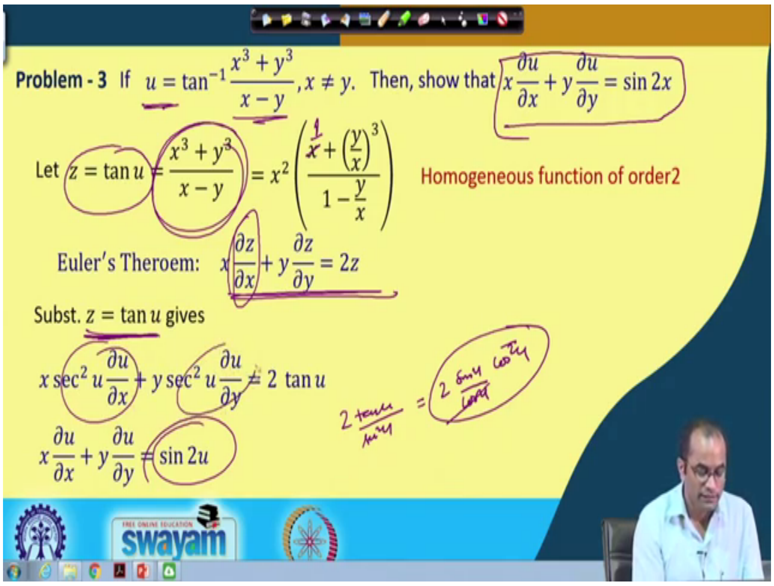
(Refer Slide Time: 20:04)



So, precisely what it is called the Euler's theorem on homogeneous function and it says if z is a homogeneous function of x and y of order n and these partial derivatives exist and so on. So, then we have this expression here x partial derivative of z with respect to x y the partial derivative of z with respect to y will be equal to nz.

So, we do not have to compute this separately, if we know that it is a homogeneous function then x z x y z y will be simply n z and for all x y point in the domain of the order domain of the function here. So, given that z is equal to f x y is a homogeneous function.

(Refer Slide Time: 20:49)



So, if it is a homogeneous function in that case we can write down wait a minute. So, here if it is a homogeneous function of x and y of order n; so, we can write down that this x power n and g y over x and then we know already we have to differentiate with respect to x. So, we can do that.

So, here when we take the partial derivative of this with respect to x so, what we will get here we have to differentiate. So, n x power n minus 1 and this is remain as it is here product rule. So, and then x power n as it is and we differentiate this g of y over x which is g prime y over x. So, the derivative of g with respect to its argument g prime y over x and then here the y over x will be differentiated with respect to x that will give us here minus y over x square.

So, in this case now if we simplify this a bit. So, here we have n x power n minus 1 and g y over x and then here we can have this minus sign there. So, y and n x power n minus 2 and the g prime y over x and the partial derivative of z with respect to y we can get again we have to differentiate now with respect to y. So, x power n will be treated as constant and we have to just differentiate this term.

So, we have g prime y over x and then this y over x the derivative with respect to y will be just 1 over x the partial derivative of y over x with respect to y will be 1 over x. And, then if we add these 2 terms with the product of x here and y there what we will get. So, we have multiply it here by x so, we will get here x power n and here x power n minus 1 and here we have multiply it by y.

So, this is exactly here n x power n and minus y x power n minus 1 and in this case we have x power n and then we have multiply it here by y term. So, this is x power n minus 1 and we have multiply it by y. And, then these 2 terms will get cancelled and we have here n x power n g y over x and what was x power n g y over x this was z or the function f x y. So, here these terms cancel out and then we get simply n into z term.

So, n and this is z here and these 2 gets cancelled. So, the problem number 3; so, here we have a u which is given as tan inverse x cube plus y cube over x minus y x is not equal to y because this is now defined as x is equal to y. And, then we show that x del u over del x plus y del u over del y is equal to sin 2 x. And in this case so, again, but we should note that this given function tan inverse y cube plus x cube over x minus y is not a homogeneous function.

Because of this tan inverse we cannot bring x power something and the rest we cannot write in terms of y over x. But what we notice that this argument of this tan inverse; that means, x cube plus y cube over x minus y that is a homogeneous function. And, we will make use of this because we can define here z as tan u because, if we take this tan to the left hand side we will get tan u is equal to x cube plus y cube over x minus y. And, now this z here is a function of is a homogeneous function of order 2 because here we can write down as x square.

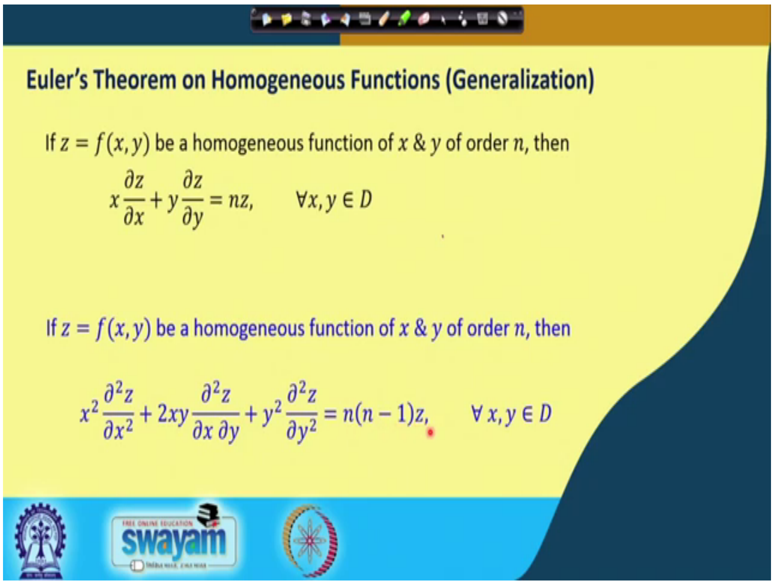
So, x cube we have taken common in the numerator and x here from the denominator. So, we will have x square and this will become 1 plus y over x cube. So, this is 1 here. So, it is 1 plus y over x whole cube and then we have 1 minus y over x. So, this is a homogeneous function of order 2 and then we can apply the Euler's theorem on this function z.

So, applying the Euler's theorem on z is equal to tan u. So, this will give us 2 times the z and z was tan u. So, we get here the x x and then del z over del u. So, we need to see what is this 1 here del z over del u. So, we have z is equal to tan u. So, del z over del x here will be the sec square u and del u over del x, the partial derivative of z with respect to x here the we will take exactly the partial derivative of z with respect to x, but it is given in terms of u. So, we will get the sec square u and then del u over del x.

So, here then y and similarly we have del z over del y. So, we have your tan will become a sec square u and del u over del y is equal to 2 times z. So, z is tan u and then we have here x the sec u it is common here, we can bring to the right hand side. So, we get x and del u over del x plus this y del u over del y and this sec u goes there as a cos square u. So, this is 2 tan u and then here sec square u will be in the denominator which will become cos square.

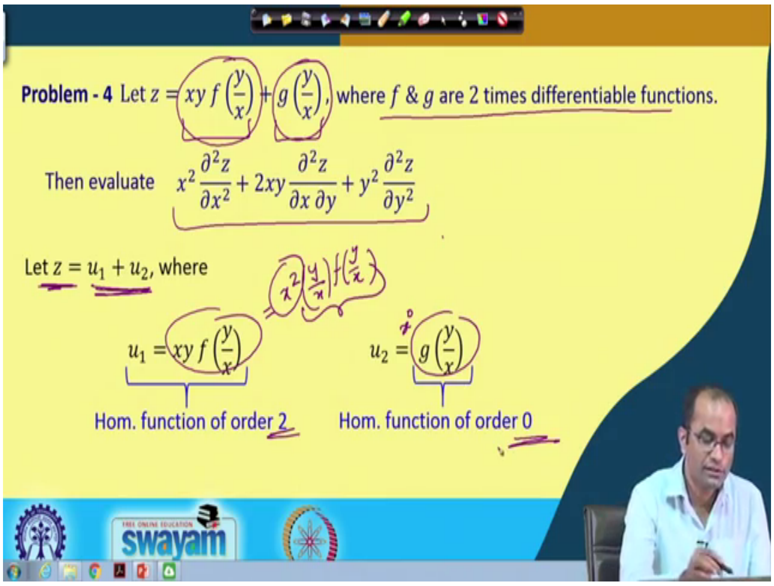
So, here a sin u over cos u for tan and this is cos is square u. So, this gets cancelled we have 2 sin u cos u which is a sin 2 u. So, we get this the z quantity as x del u over del x plus y del u over del y is equal to sin u. So, here the important point is that this u was not a homogeneous function, but by defining this as the z is equal to tan u which is x cube plus y cube over x minus y it becomes homogeneous. And, then we have applied the Euler's result on z and noting this z is equal to tan u, we have computed here partial derivatives of z with respect to x and y and then we get the desired result.

(Refer Slide Time: 28:04)



So, generalization of this Euler results says that so, this was in case of the first order partial derivatives, but we do have results in case of the second order partial derivatives as well. And, which is the extension of this which says that x square the second order partial derivative 2 x y the mixed from y square again second order derivative with respect to y is equal to n n minus 1 z for all xy in the domain of the function.

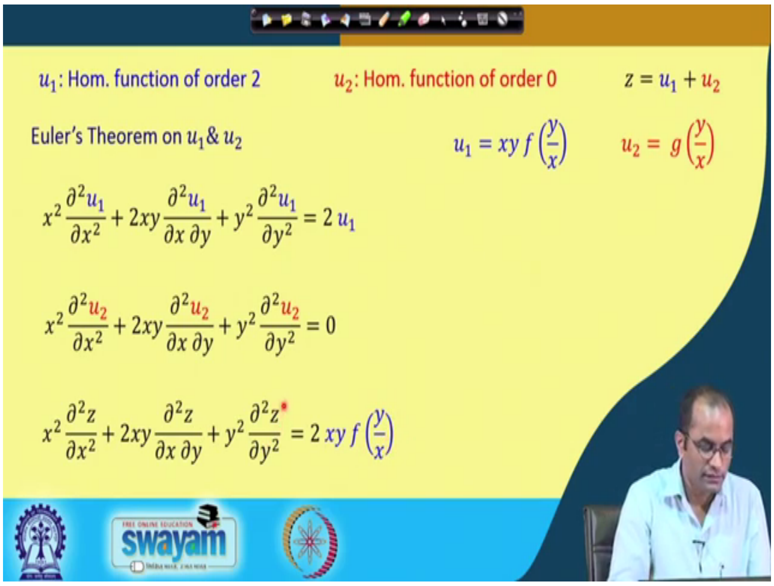
(Refer Slide Time: 28:38)



So, this is the generalization of this result and now we will have this one problem on this. So, suppose we have z is equal to x y a function of y over x and plus g y over x and where this f and g are 2 times differentiable function and then we will evaluate this expression here for z. So, again the same problem the z is not a homogeneous function of x and y, but here this is a homogeneous function and the second g is also a homogeneous function because we can write down as it is a directly given in terms of y over x.

So, what we take here we take z is equal to u 1 plus u 2 the first term here we take as u 1 and here we will take as u 2 ; that means, this u 1 is x y and f y over x and u 2 is g y over x. So, this is a function of homogeneous function of order 2 because this we can again write down as x square and y over x and this f y over x. So, this is a function of y over x and then we have x square there. So therefore, the order is 2 and in this case this g is a function of y over x and here there is no term of x so, x power 0. So, this is a homogeneous function of order 0.

(Refer Slide Time: 30:07)

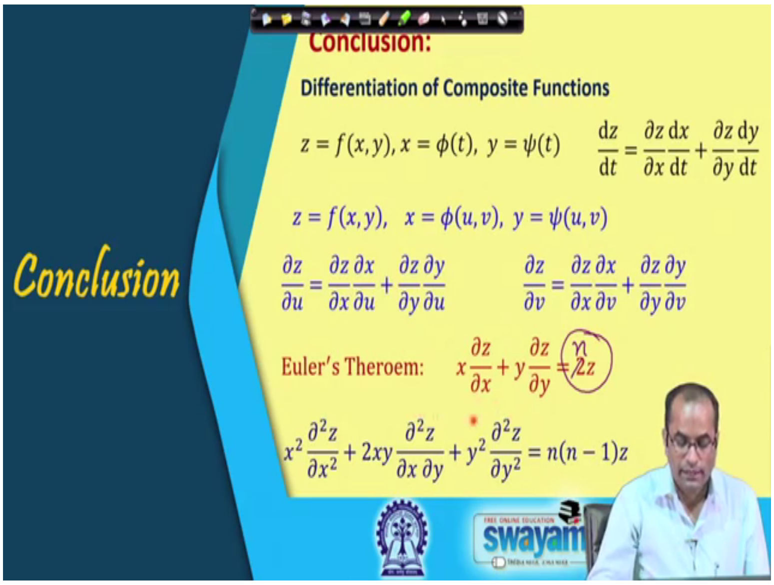


So, well we have u 1 homogeneous function of order 2 u 2 a homogeneous function of order 0 and z was as u 1 plus u 2 u 1 is given by this, u 2 is given by this. So, the Euler's theorem we can apply on u 1 and u 2 because they are the homogeneous functions of order 2 and 0. So, on u 1 for second derivative theorem it says 2 times so, n is 2; so 2 times and n minus 1 and then the u 1.

So, n n minus 1 u 1 so this order was 2; so, we get 2 into 1 into u 1 and this u 1 now we know already is x y f y over x, we will substitute later. And, then for u 2 because that is also a homogeneous function of order 0 and because of that order 0 here right hand side we will get 0 term because here its n n minus 1 and because n is 0 the order is 0. So, this will become 0 there and now we have these 2 expressions directly without computing these derivatives here, we have used this Euler's theorem and we can add them because we want to get this result in terms of z there is u 1 plus u 2.

So, if we add these two so, we have x square and this will become as the partial derivative with respect to 2, the second order u 1 plus u 2 here u 1 plus u 2 here also u 1 plus u 2 and that is z. And, now equal to here 2 u 1 plus 0 so, 2 and u 1 so, plus 0. So, we get only 2 u 1 and u 1 is x y f y u over x. So, we get this result 2 times xy f function of y over x of this expression here.

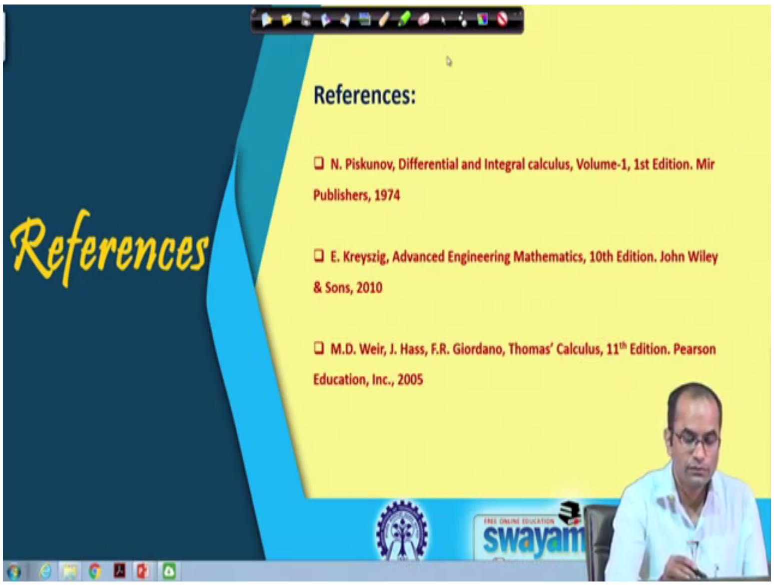
(Refer Slide Time: 31:51)



So, coming to the conclusion we have learn today the differentiation of composite functions which was very useful when z is a given function of x y and x is and y they are the function of t. So, in that case we can directly compute the derivative of z with respect to t; by this formula and then this is generalization of this one that x and y. They may be functions of u v. And in that case also we can compute the partial derivatives now of z with respect to u and the partial derivative of z with respect to v by these formulas.

And, then the Euler's theorem for homogeneous functions we have learnt that if z is a homogeneous function, then we can have this for homogeneous function of order n then it will be here the n z. So, x partial derivative with respect to x plus y partial derivative with respect to y will be equal to n z or there was a generalization that we can also care these second order derivatives as n n minus 1 into z when this z is a homogeneous function of x and y of order n.

(Refer Slide Time: 33:11)



So, these are the references used for preparing these lectures.

Thank you very much.