Engineering Mathematics - I
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Lecture – 14
Differentiability Functions of Two Variables (Contd.)

So, welcome back to the lectures on Engineering Mathematics-I, and this is lecture number 14. And, again we will continue here the Differentiability of Functions of Two or Several Variables and there will be many worked problems today.

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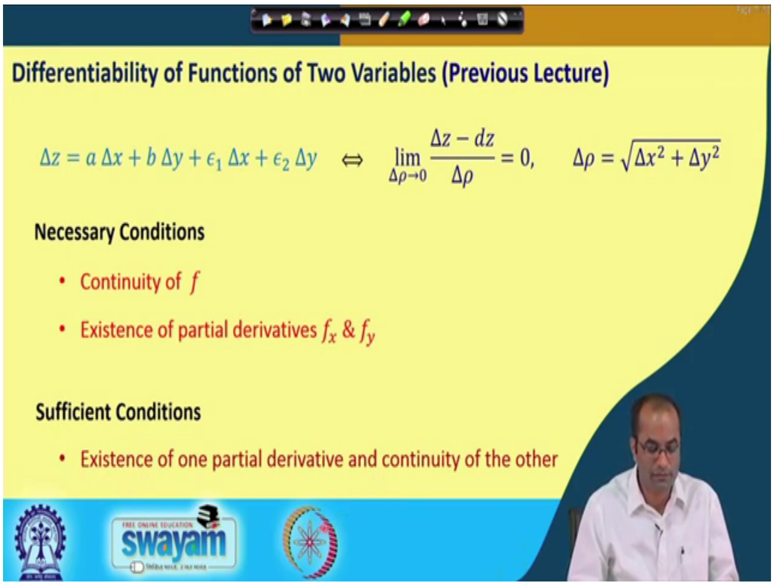

So, what we have observed in the previous lecture that the equivalent definition of the differentiability here instead of expressing this delta z in terms of a delta x be delta y and epsilon delta x plus epsilon delta 2 y we can also find out this limit. And, if this limit equal to 0 then the function is differentiable and we have observed that this limit definition was quite useful for proving the differentiability of various functions.

We have also observed that the necessary conditions are also important because if we check these necessary conditions first and we observe that for example, the function is not continuous or the partial derivatives do not exist. Or one partial derivative does not exist in that case we can immediately conclude there that the function is not differentiable, because these are the necessary conditions for differentiability. And, we have also observed the sufficient the importance of sufficient conditions because if we show that the partial the continuity of one of the partial derivatives. And, then we can conclude that the function is differentiable without proving the differentiability using the definitions here given above.

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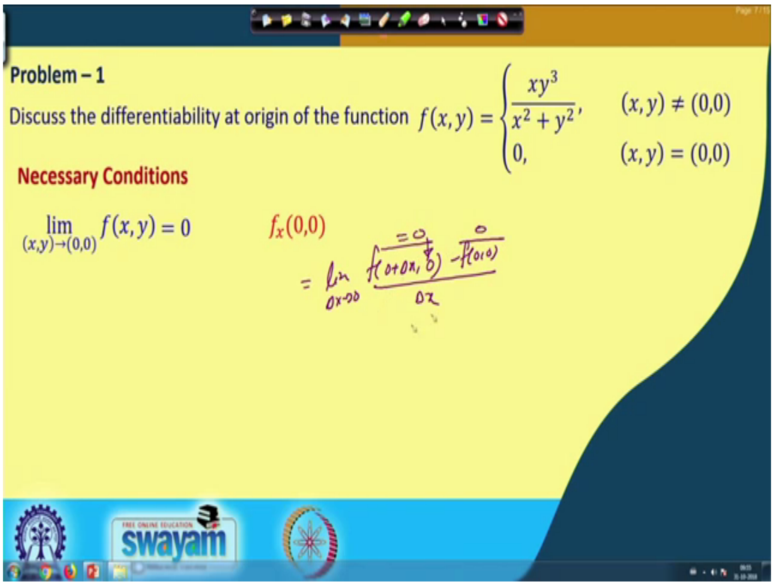

So, let us discuss the problem here the differentiability at the origin of this function f xy is equal to xy cube over x square plus y square and this is 0 when x y equal to 0 so, at origin this is 0. First we check the necessary conditions and we should do basically in all the problems because, these are the necessary these are the conditions which the function must satisfy before going to the differentiability. So, here we will check the continuity of the functions. So, when f x y when x y goes to 0 and in this case without doing much so, we can either change again to polar coordinate then r square term will be coming in the denominator and here in numerator there will be a 4 terms.

And so, we will get r square still in the numerator and with this cos and sin together so, that will make this limit to 0. So, we have the limit of this function f x y as x y goes to 0 is equal to 0 also the partial derivative f x at 0 0, we can easily show the others is like limit delta x goes to 0 and f 0 plus delta x into the y is unchanged here. So, f 0 0 and divided by this delta x so, since this 0 is here and you have we have the product there. So, this will be 0 and here also this will be 0 so, 0 minus 0 we will get this partial derivative as 0. So, this is 0 and similarly the f y will be also 0.

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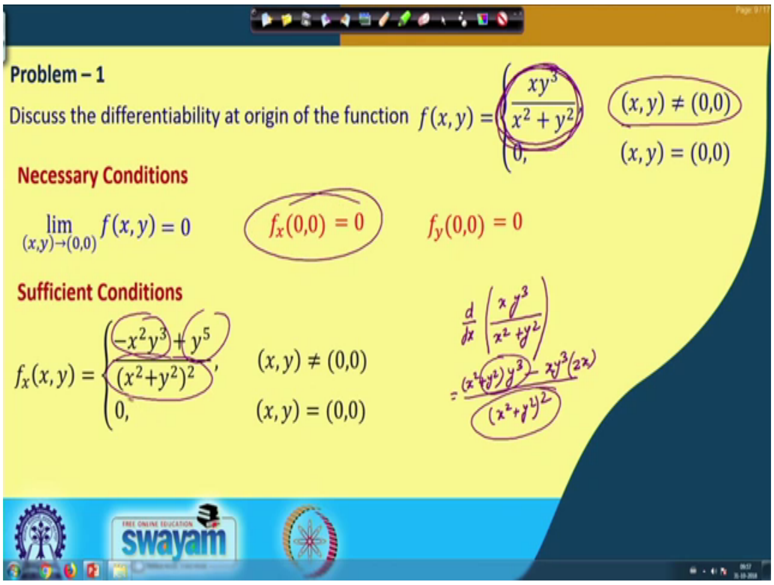

So, we have the existence of the partial derivatives and we have the continuity of the function at 0 0 point, now for the sufficient conditions. So, we need to check the continuity of these partial derivatives at 0 0. So, we need to get these f x the partial derivative of x at all the points here at least in the neighborhood of this 0 0 point. So, to get the partial derivative at x y with not equal to 0 0 we have to differentiate this function with respect to x. So, treating this y constant so, we will just differentiate this with respect to x keeping this y as constant because of these partial derivatives.

So, this will be equal to this is square here whole square and this will be x square plus y square. So, the derivative of this with respect to x will be y cube then minus this x y cube and the partial derivative here with respect to x this will be 2 x. So, this x square y cube here also we have 2 times x square y cube. So, we will get minus this x square y cube and then here y for 5, this is y power 5 here and x square plus y square whole square this is here. So, at this non-zero point x y not equal to 0 0 we can get the derivative partial derivative with respect to x or partial derivatives with respect to y directly from directly taking derivative of this function with respect to that variable and keeping the other variable as constant.

And to get this at 0 0 we have to use this fundamental definition of the differentiability which we have just got here this f x at 0 0 was 0 so, which is 0 here. So, we have now evaluated f x at x y point in the neighborhood and also at 0 0. And, now for continuity we have to take the limit as x y goes to 0 0 of this function whether it is equal to 0 or not. If it is equal to 0 then we can claim that the function is differentiable because of the sufficient condition otherwise we have to go further to test the differentiability.

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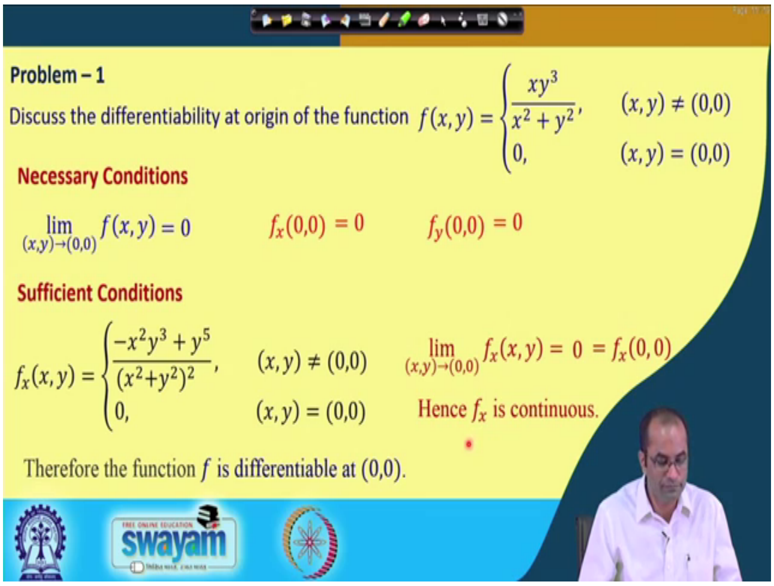

So, here if we take this limit as x y goes to 0 0 f x x y. So, in this case we will observe that like if we substitute here if we change to the polar coordinate now. So, we will get here r square so r 4, but there we will have r power 5 and so, we will get 1 r there in the numerator and since r goes to 0 to go to xy to 0 0 this value will be 0. So, here the f x is continuous because this is equal to 0 and this is the value of the f x at 0 0 point.

So, we have the continuity of the partial derivative and that we can use now to prove the differentiability of this function because the partial derivatives exist, function is continuous. So, we have the necessary conditions for the sufficient condition we have to show that one of the partial derivatives is continuous. So, here we have shown that this f x is continuous and that is enough to prove that the function f is differentiable at 0 0 point.

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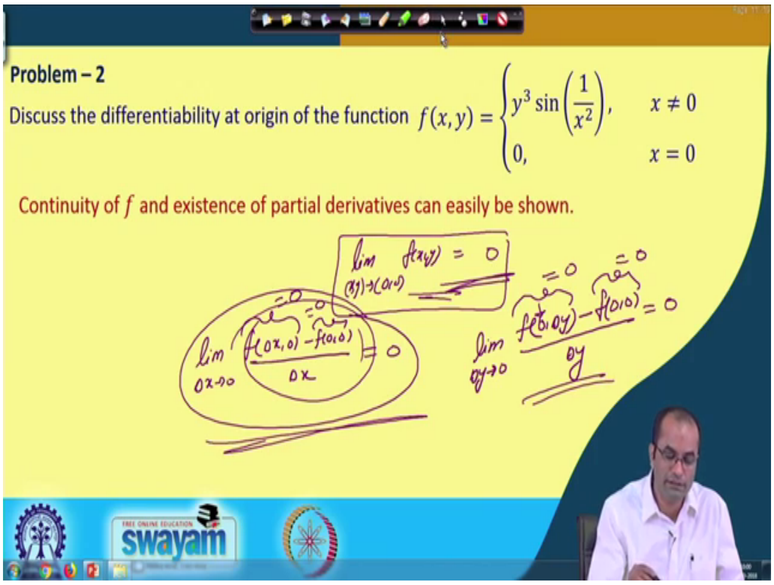

The next problem here we will discuss the differentiability again at the origin of this function y cube sin 1 over y square when x is not equal to 0 and x is equal to 0 this is 0. So, we will test the continuity of f and the existence of the partial derivative which is easy to see again. So, for the continuity we have to take this limit as x y goes to 0 0 of this function x y; and which is trivial here because this is a bounded function sitting and then x y go to 0. So, this y goes to 0 and this will be 0 and this is the function value also at 0 0 point.

So, we have the continuity of this function and also the existence of the partial derivative for that we have to consider for partial derivative with respect to x we need to show that this limit 0 plus delta x. So, 0 and minus this f 0 0 divided by delta x. So, here this y is 0. So, this will be 0 and this is again here 0. So, we have the 0 minus 0 so, this is also 0. So, we have the existence of the partial derivative with respect to x. Similarly we can show the existence of partial derivative with respect to y for that we have to consider the delta y goes to 0 and f 0 delta y minus this f 0 comma 0 and then here delta y.

So, since x is 0 the value of this function is again 0 and here also the value of this function is 0 so, we have again 0. So, the continuity of the function and the existence of the partial derivative we have seen it is a easy.

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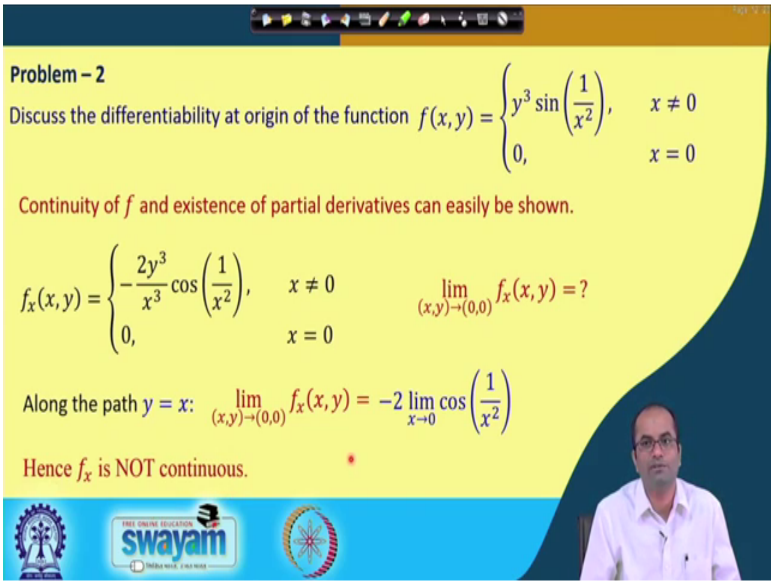

Now, and next we will show the continuity of this f x we will try to get the continuity of the partial derivative with respect to x. So, for that again we need to compute the partial derivative when x is not equal to 0 and we have to also get this we have already seen that this value is 0 and x is equal to 0. So, at least at 0 0 point we have seen it as 0. Now, here to get this when x is not equal to 0 we have to directly differentiate this with respect to x. So, the y cube will be as taken as constant here.

For sin we will get the cos term 1 over x square and the derivative of 1 over x square will be minus 2 over x cube. So, this is the derivative when x is not equal to 0 and we can get this derivative as 0 when x is equal to 0 by the fundamental definition of the derivative. So, in this case and what we will show now that what happens when we take the limit here as xy or basically this x goes to 0, we are approaching to the to the to the origin here what will happen to this limit. So, when we take this limit here f x at x y goes to 0 along this path y is equal to x this is easy to see. So, if we take this path y is equal to x here then what will happen to this limit, this limit will be minus 2 and this cos 1 over x square.

And the limit x goes to 0. So, in this case this limit does not exist if clearly one can see this cos 1 over x square this is not a definite value. So, here this limit does not exist along this particular path itself. So, we can say that the limit of f x as x y goes to 0 0 does not exist and hence this f x is not continuous function in this case. So, it does not help us to prove the differentiability of this function at this point of time.

What we can do we can further test the continuity of the other partial derivative f y with respect to. So, the partial derivative of f with respect to y and if that comes to be continuous again we can claim based on the sufficient condition of differentiability, because we need to have the continuity of one of the partial derivatives to claim that the function is differentiable. So, in this case this f x is not continuous. So, we will check now the continuity of the other.

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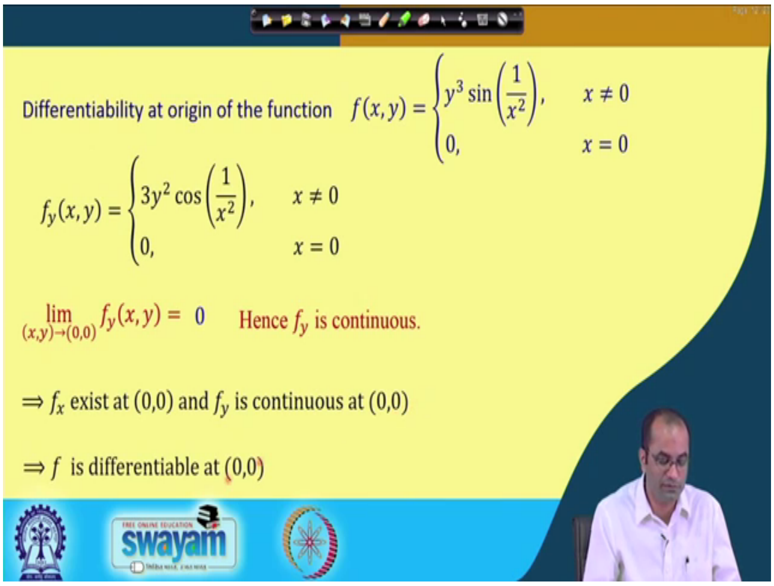

So, in this case again f y we can compute here by treating this x constant. So, this will be just 3 y square when x is not equal to 0 and 0 when x is equal to 0. So, we will take again this limit as xy goes to 0 0 of this f y at the point xy. So, here one now the situation is different. So, here when x y goes to 0 0 so, this term will go to 0 and something finite is sitting here. So, in this case there is no problem to get this limit as 0. So, the limit of this function 3 y square cos 1 over x square when we take xy to 0 0 from any direction. So, this is come this is to be 0.

And this is the value at the origin as well of this function f y. So, hence this f y is continuous, this is a special example where f x was not continuous and now f y is continuous. So, in this case the f x and f y obviously, both exist there and we have on the top this continuity of f y; continuity of one of the partial first order partial derivatives.

And, then we can claim that the function is basically differentiable or we can prove that this function is differentiable based on these sufficient conditions of differentiability.

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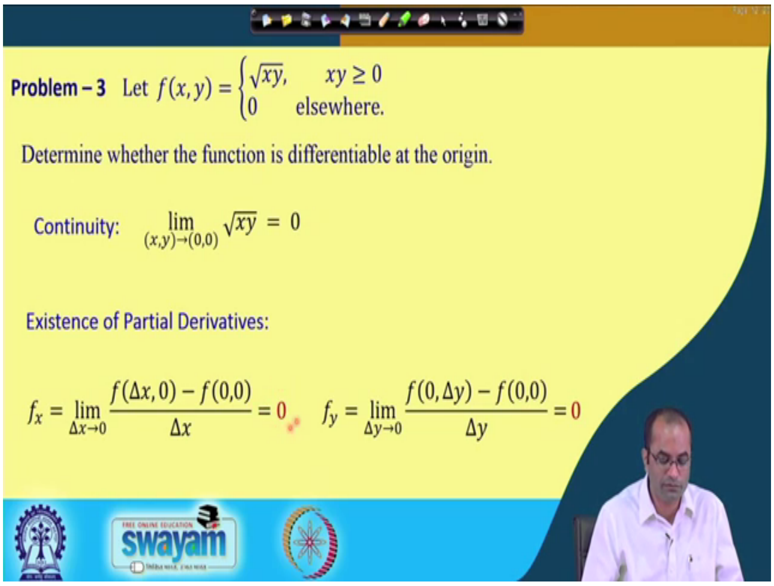

Another problem here we take this f xy is equal to square root x y when we have this xy, this product positive greater than equal to 0 otherwise there will be problem here and the definition and 0 elsewhere. So, we will determine whether the function is differentiable at the origin or not. So, the continuity of this function is again simple because when x y go goes to 0 0 from any direction this will go to 0 and the function is also defined as 0 at the origin.

So, we have the continuity of the function and the existence of partial derivatives also we can use this definition again. So, here f delta x and y is 0 so, this product is 0. So, we will get this again here 0 and this f 0 0 is 0 so, 0 minus 0. So, this f x at 0 0 point is 0 and similarly f y is also 0 at 0 0. So, this function is continuous and the partial order first order derivatives exist.

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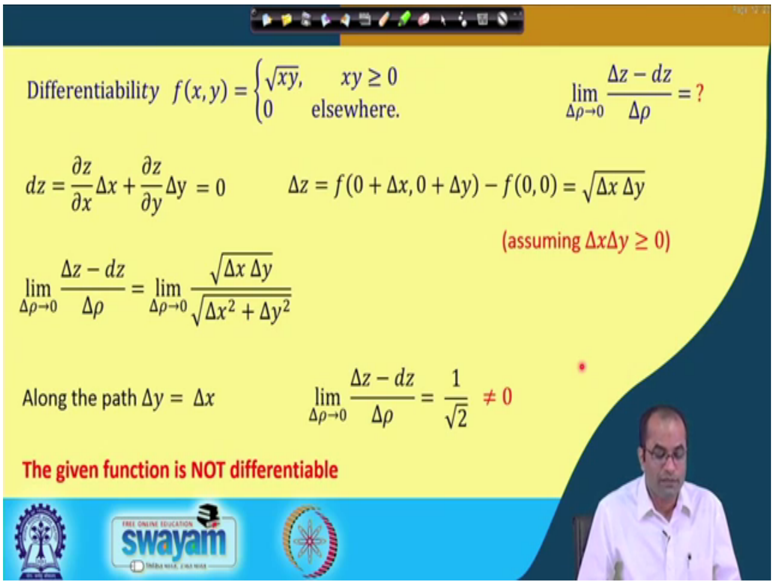

Now, to check the differentiability we will get this limit at what is the value of this limit, if this limit comes to be 0 then the function is differentiable. If we do not get this limit as 0 then the function is not differentiable. So, for that we need to get this dz which is the partial derivative of z with respect to x partial derivative of z with respect to y at the origin which was and both of them was 0. So, we have this dz is equal to 0 and this delta z which is f the 0 plus delta x 0 plus delta y minus this f 0 0. So, this will become as the square root of delta x delta y and this is minus 0 here assuming that this product is 0. So, we get this delta set as square root delta x and delta y.

Now, this constant here delta z minus d z over delta rho will be so, this delta z was square root x and square root x delta x delta y and this data rho was the square root delta x square plus delta y square here. And, in this case when we take this limit as delta rho goes to 0 we will also that this limit does not exist because if we take this path delta y is equal to delta x for example, here. So, what will happen you have delta x there and delta x square here. So, this delta x square terms will vanish there so, we will have 1 over square root 2.

So, along this path delta y is equal to delta x what we observe that delta rho goes to 0 this delta z minus dz over delta rho is 1 over square root 2. Because we will get here 1 when delta x square get cancelled here 1 plus here 1 against the square root 2 will come. So, along this path linear part delta y is equal to delta x this limit is equal to 1 by 2. So, certainly it is the limit cannot be equal to equal to 0. So, here itself by taking one path and showing the limit is different from 0 though the limit may exist or limit does not exist, it does not matter to prove that the function is not differentiable. Because, for differentiability this limit must be equal to 0 and what we have observed that along this particular path this limit is equal to 1 over square root 2.

And hence, we can now say that the function is not differentiable because this limit cannot be equal to 0. So, the function is not differentiable in this case.

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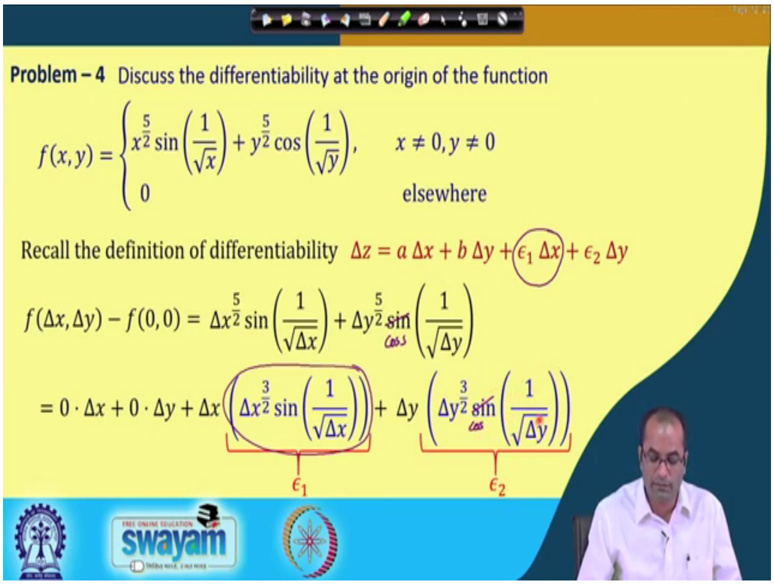

Another example which discuss the differentiability again at the origin of this function: so, x power 5 by 2 and the sin 1 over square root x plus y 5 by 2 and then we have cos 1 over square root y and this is defined for x not equal to 0 and y not equal to 0 and this is 0 elsewhere. So, again this we can directly use this definition of the differentiability which is much easier in this case because of this special structure. We can easily express this function into this form because if you observe that this delta z is f delta x delta y minus this f 0 0 at 0 0 point.

This is equal to just the x and y will be replaced by delta x delta y, this is 0. So, we have here delta x power 5 by 2 then the sin term 1 over square root delta x plus here we have delta y power 5 by 2 and then we have 1 sin and then here we have the cos term this is cos here; so, the cos 1 over square root y. So, in this case we can now write down the 0 into delta x plus 0 into delta y that linear term because there is non-linear term. So, we have just introduced here 0 times delta x and 0 times delta y. Here we have taken this 1 delta x outside because, we need that format here epsilon 1 delta x.

This here epsilon 1 delta x so, this will become as epsilon 1 here and then plus from here also we have taken this delta y term away, then we have delta y 5 by 2 and this is again this cos term; so, the cos 1 over square root delta y. So, in this case we have expressed this delta z in this form where this is your epsilon and which has this property that this goes to 0, when f delta x delta y go to 0. Because, of this term here delta x and the positive power 3 by 2 and this is bounded function. So, this will go to 0 and here again the same argument. So, this with cos function also this will go to 0.

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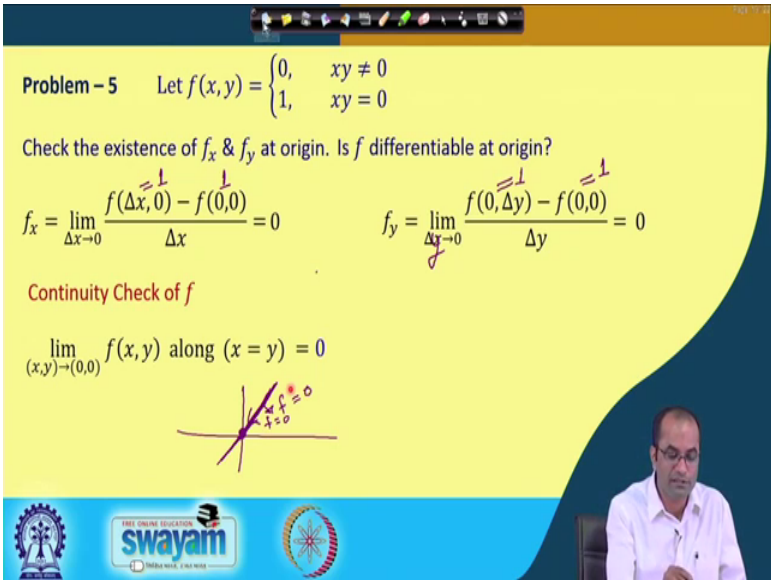

So, in that case this function is differentiable because we can easily express this function in this form a times. So, this is going to be the partial derivative of that function at 0 0 with respect to x and this is going to be the partial derivative with respect to y at 0 0 point and delta y plus this delta x this is epsilon 1 term delta by epsilon 2 term. So, here it was very easy directly because of this function to express in this form as delta z is equal to a delta x plus b delta y plus epsilon delta x plus epsilon delta y. And, then we have proved that this function is differentiable. Another example here that f x y is defined at 0 when x y not equal to 0 and when x y the product is 0 then this function is 1.

And we want to check the existence of f x and f y at the origin and we also want to know whether the function is differentiable at origin or not. So, to get the existence of f x and f y at origin we use this definition, fundamental definition of the derivative. So, we have limit delta x to 0 f delta x 0 minus f 0 0 over delta x and now you note that this f delta x 0. So, this product here of the argument is 0. So, the value will be coming from this 1 so, this is 1 and f 0 0 so, again this is 1 here. So, 1 minus 1 this is 0 that limit is 0, similarly for f y when we have delta y to 0; now this is delta y to 0 and f 0 delta y minus f 0 0 over delta y. So, in this case again this is 1 and this is also 1. So, we have 1 minus 1 again 0 as the partial derivative of f with respect to y at a origin.

So, now we will check the continuity of f at origin. So, for that if we observe that what is the value of this f xy of this limit when x y goes to 0 0 along x is equal to y. So, along this x is equal to y, what we have along x is equal to y line if we approach to this origin point here what will happen. So, here at all these points the x y the product is not equal to 0 at any other point then origin the product is not equal to 0. So, the function will take the value 0. So, when we are approaching along this x is equal to y line towards this 0 0 we have the function value 0 0 0 0 whatever neighborhood we take close to 0 the value of this function is 0.

Because the function is defined as the value is 0 when x y not equal to 0. So, when we are approaching to origin along this line here, the function is taking value as 0 whatever close we are to the origin the function will take the value 0 only at the origin it is 1. So, there is a point of discontinuity here. So, this limit along this x is equal to y is0 whereas, the value of this function at origin is 1. So, we can clearly see then the function is not differentiable or is not continuous.

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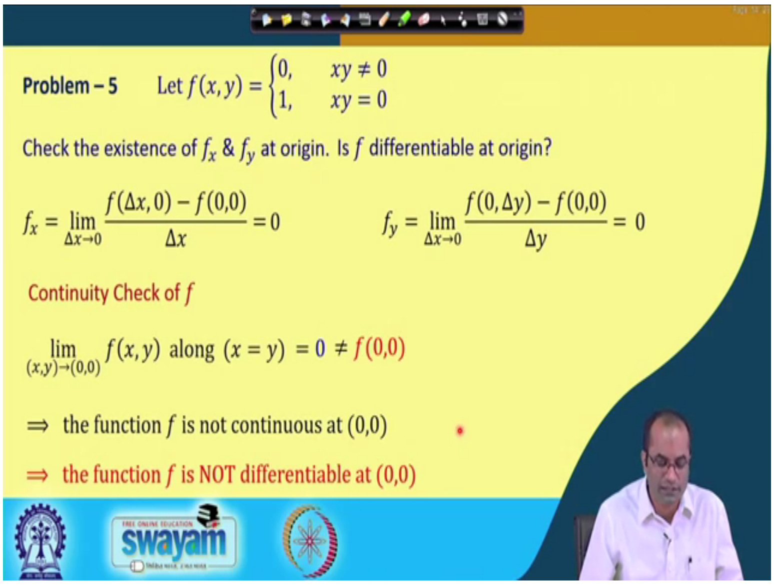

So, this is not equal to the function value at 0 0. Hence, this function is not continuous at origin and since the function is not continuous we can decide about the differentiability. Because the continuity is the necessary condition for differentiability, if the function is not continuous definitely the function is not differentiable. So, this function f is not differentiable at origin. We do not have to go for any other test for differentiability because the function is not continuous though the partial derivatives exist in this case.

So, one of the necessary conditions here is fulfilled, but the continuity is also the necessary condition for differentiability which is not fulfilled here. So, the function is not continuous and hence the function is not differentiable without any further test.

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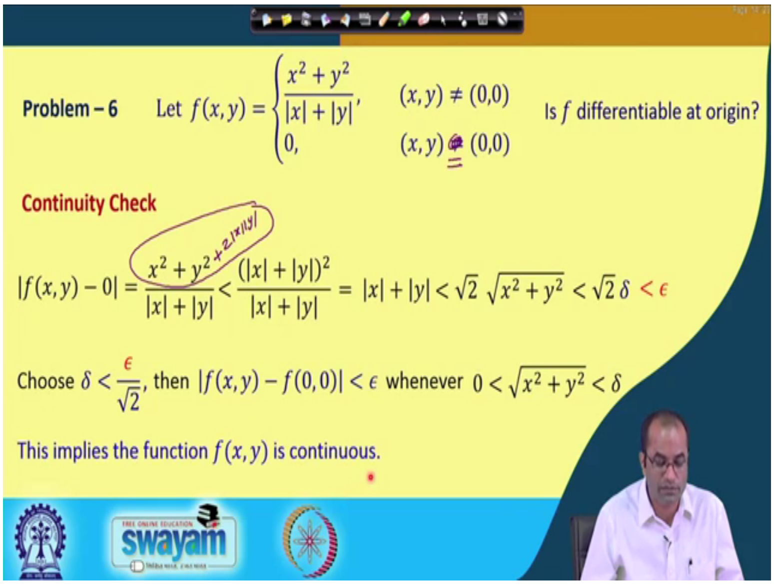

The last example here we take this fx y is equal to x square plus y square divided by absolute value of x plus absolute value of y when x y not equal to 0 0. And the value is 0 when x y equal to 0 0. So, this is a mistake here we have we have this equal to here not this is equal to 0 0. So, we will check whether this function is differentiable at origin. So, the continuity check for this one is can be done by various ways we can change to the polar coordinate or we can also use the epsilon delta approach to show that this function is continuous.

So, here let us take the epsilon delta approach. So, we take the difference of the function f xy xy not equal to 0 0 and minus this f 0 0 which is given as 0. So, we will show that this limit of this f xy is equal to 0 or not by taking this difference and expressing this in terms of the delta and epsilon. So, here this is x square plus y square over modulus x plus modulus y.

And then we can add here plus 2 times absolute value of x plus absolute value of y to make this square here. So, when we are adding in the numerator the quantity will be bigger. So, we have this bigger quantity and that is whole square because we have added this term here plus 2 times absolute value x absolute value of y. And, then this term here becomes this whole square and now this is the same term so, one will get cancelled. So, we have absolute value of x plus absolute value of y. And, in one of the earlier lectures so, we have seen that this here is less than equal to square root 2 and square root x square plus y square which is again we have converted into the delta neighborhood of this 0 0 point. So, this is less than square root 2 and in terms of delta and this everything we want to make less than the arbitrary number epsilon.

So, by this relation here we have found that for given epsilon we can set this delta so, that this f xy minus this 0 0 term is less than the epsilon. So, by choosing this delta here less than epsilon by square root 2 for given epsilon we have this delta now. So, that this difference is less than epsilon and that is the epsilon delta approach for showing the continuity. So, we have shown that if we choose this delta here then this difference will be less than epsilon. And, whenever we take any point xy from this delta neighborhood of this 0 0 point and this implies that the function is continuous. So, moving next now to show the differentiability of this function.

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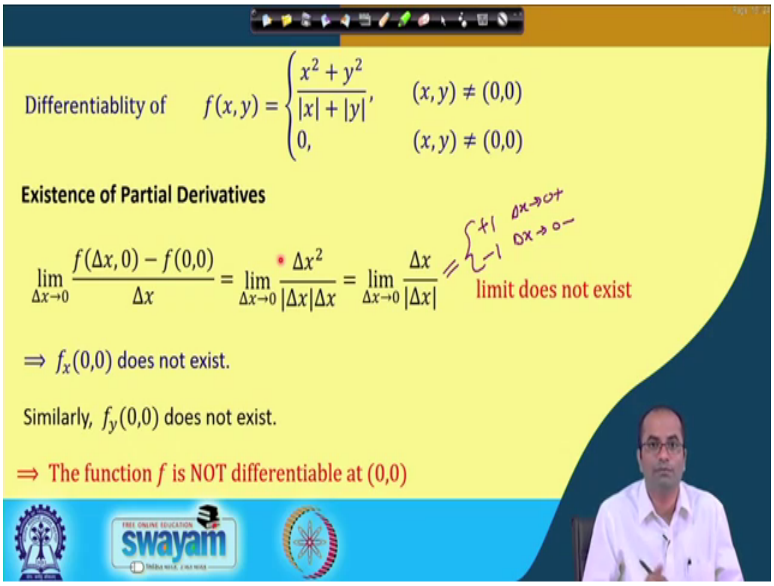

So, now we will show the existence of the partial derivatives again as per the definition here. We have f delta x 0 minus f 0 0 delta x. So, what is this term here the y will be set to 0. So, you have this delta x square and divided by this absolute value of delta x and delta x. So, this will be this term and now so, this delta x will get cancelled we have delta x over more absolute value of delta x. So, if this delta x goes to 0 from the right side then this will be 1 and when it goes to 0 from the left side then this value will become minus 1. So, hence this limit does not exist because these we have here either plus 1 or minus 1 as the limit, when this delta x approaches to 0 from the positive side, when delta x approaches to 0 from the negative side.

So, hence this limit does not exist and in this case we can now we have shown just for the one derivative over the f x here, but since the nature of this function it is a symmetric here x square plus y square over this one; we can also show that f y at 0 0 does not exists. And, in this case since in fact, showing the existence of one of the derivatives is enough to tell that the function is not differentiable because the necessary condition for differentiability is the existence of both the partial order derivatives. So, in this case if this f x does not exist there itself we can tell that a function is not differentiable though in this case both the derivatives do not exist. And hence, the function is not differentiable.

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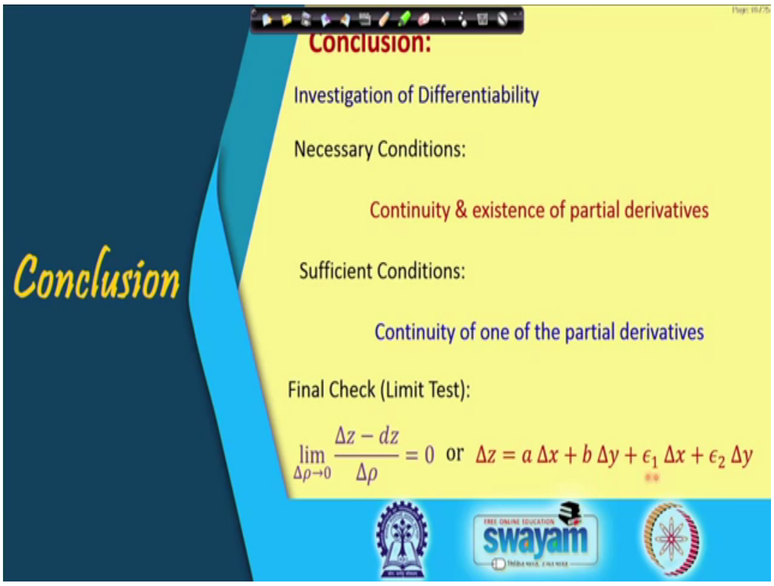

So, coming to the conclusion now; so, what we have observed that the investigation of this differentiability we can use the necessary conditions because the continuity and the existence of partial derivative; if one of them is violated then we can prove that the function is not differentiable. Or using the sufficient conditions also we can discuss the differentiability of a function. So, the continuity of one of the partial order derivatives is sufficient for the differentiability of the function. And the final check if we have observed that the function the necessary conditions are fulfilled. And we are not meeting with the sufficient conditions for example, that the partial derivatives are not continuous In that case we have to go for the final check that one can use the limit test which is often easier to prove that this limit is 0 or it is not equal to 0 or we can directly use the definition of the differentiability in this case.

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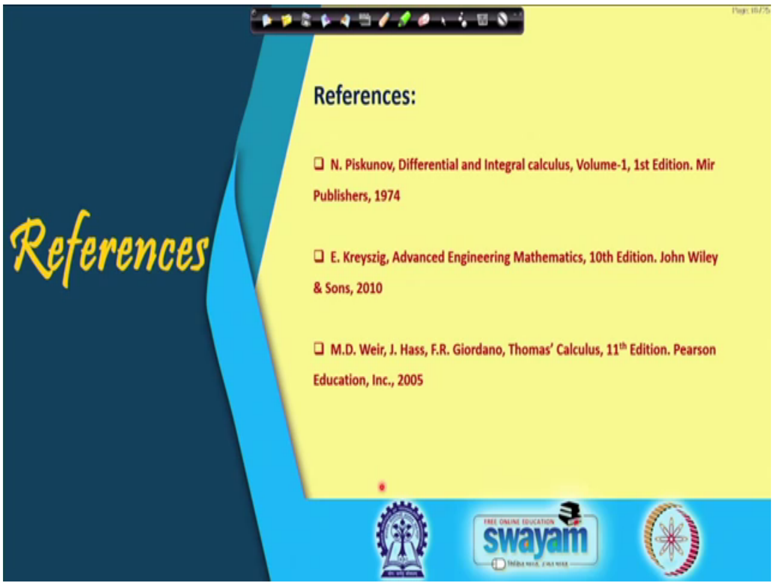

So, these are the references used for preparing the lectures.

Thank you very much.