Engineering Mathematics - I
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Lecture – 13
Differentiability Functions of Two Variables

Welcome back to the lectures on Engineering Mathematics I, and this is lecture number 13.

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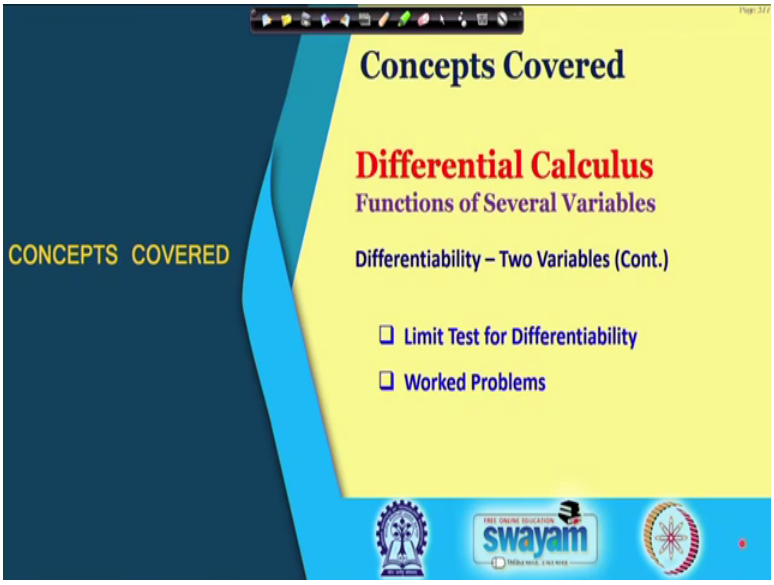

And today we will talk about again Differentiability of Functions of Two Variables. And basically we will go through the limit test for differentiability, and that is very useful to test differentiability of a function of more than one variable. And then some worked problems will be done in this lecture.

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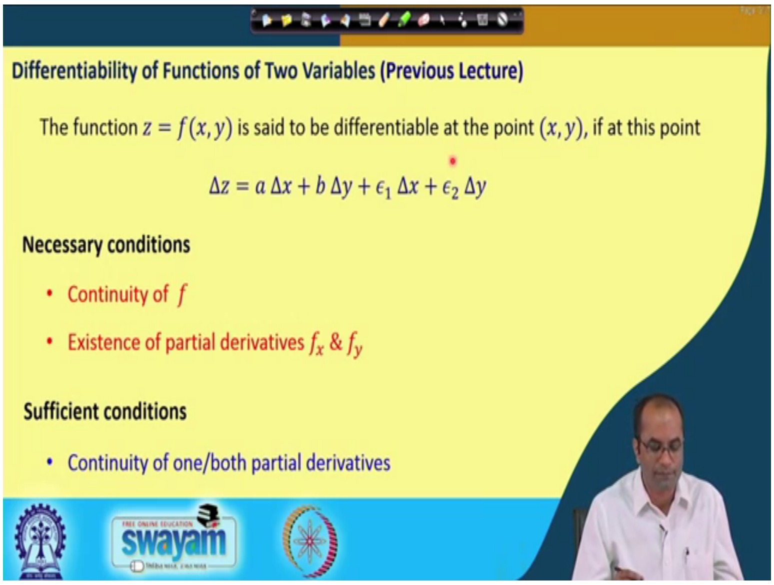

So, just to recall from the previous lecture we have discussed already the differentiability of functions of two variables. And what we have learnt that a function z is said to be differentiable at the point x y, at any point x y; if at this point we can express this delta z which is the variation in z when we when we vary x by delta x and y by delta y. So, if we can express this delta z in terms of delta x and delta y; a times delta x plus b times delta y, that is a linear term a and b are independent of delta x and delta y. Plus, this epsilon times delta x plus epsilon 2 times delta y, and here epsilon and epsilon 2 must go to 0 as delta x and delta y go to 0.

So, the necessary conditions we have learned that the continuity of f is necessary for differentiability and also the existence of partial derivatives f x and f y is necessary for the existence of or for the differentiability. And we have also seen the sufficient conditions where we observe that the continuity of one derivative or continuity of both partial derivatives is sufficient for defining differentiability of functions of two variables. So, what we have seen that to prove the differentiability either we can use the sufficient condition. So, if you observe that the function is or the partial derivative is continuous, then we can claim that the function is differentiable. Or we can test directly this definition, so we have to express this delta z in terms of this delta and delta y, in this form and then we can claim that the function is differentiable.

Today we will learn another way now which is equivalent to this expression here that can be used to prove differentiability easily and that is a limit test.

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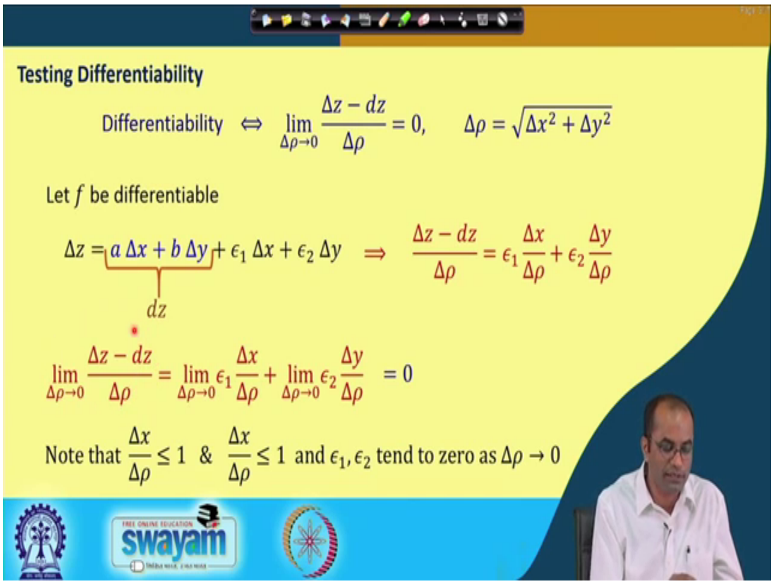

So, here we will now show that this differentiability is equivalent to saying that this limit here delta rho which is square root delta x square plus delta y square. If we take this limit here of this expression delta z minus dz over delta rho is equal to 0, then we can prove that the function is differentiable. Or if we have differentiability it will imply that this limit is 0, and this limit 0 will imply differentiability. So, the both are equivalent definition or testing for differentiability.

So we now, we will show that if a function is differentiable, then this limit must be 0. So, if f is differentiable; that means, we can express this delta z as a delta x plus b delta y and plus epsilon 1 delta x plus epsilon 2 delta y. And now this is the dz term which we call differential. And if we take this differential term to the left hand side, and divide by this delta rho, delta rho is given as square root of delta x square plus delta y square. So, if we divide this, then we will get the right hand side as epsilon delta x over delta rho a plus epsilon delta y over delta rho term.

And now we will take the limit here as delta x and delta y goes to 0 and observe what is the value right hand side when we take the limit delta x, and delta y goes to 0. We already know that this epsilon 1 and epsilon 2 they go to 0 as delta x goes and delta y go to 0. But we have to make sure that this term here sitting with epsilon 1 that is delta x over delta rho and delta y over delta rho, they both are bounded. But this we can easily see because this delta rho is the square root of delta x square plus delta y square. So, this term here because this is bigger than delta x. Here we have delta y square as well and then the square root.

So, this term is certainly bigger than this delta x. So, the modulus of this term so, absolute value of this delta x over delta rho is bounded by 1. And similarly the absolute value of this delta y over delta rho is also bounded by 1. And then when we take the limit as delta rho goes to 0. So, this side will go to 0, because epsilon 1 will go to 0 and epsilon 2 will go to 0. So, when taking the limit, so we observe because of the boundedness of these 2 terms and we know the properties of these epsilon 1 and epsilon 2 that they go to 0 as delta rho goes to 0 means delta x go to 0 delta y go to 0.

So, in this case we can prove that this limit is equal to 0 because of the boundedness of these terms there. So, this is this limit which we want to show that if f is differentiable this must be equal to 0.

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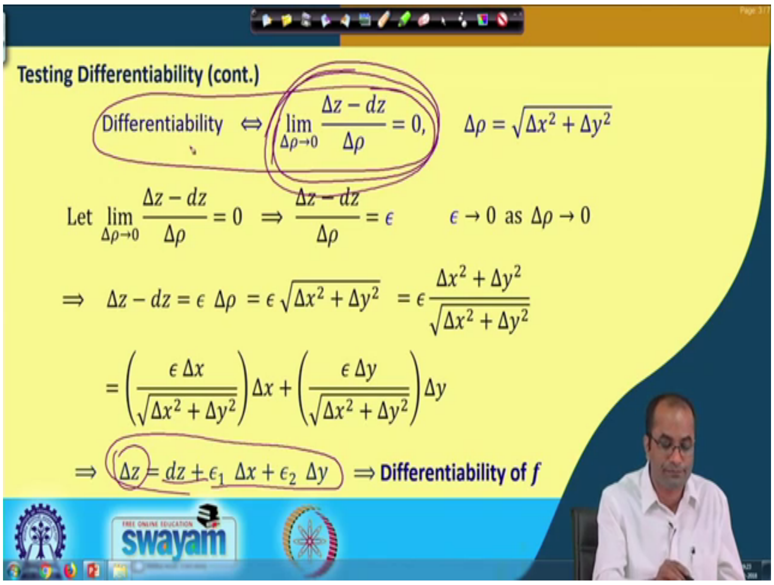

So now we will go the other way round; that means, if this limit is equal to 0 we will show that the function is differentiable. So, in this case we now let that this limit equal to 0. And now we will use the fact which we have also used in the previous lecture, that this limit equal to 0 then we can introduce one epsilon here. So, this term minus 0 which is 0 here so this term is equal to epsilon. And this epsilon will have property that when we take the limit delta rho go to 0 goes to 0 or delta x, and delta y go to 0 then this epsilon must go to 0, because the limit of this is precisely 0. So, the epsilon must go to 0 and we take the limit here as delta rho go to 0 goes to 0.

So, in this case we have this property of the epsilon that this must go to 0 when delta rho goes to 0. And if we take now this term to the right hand side. So, we have delta z minus this dz is equal to epsilon times this delta rho which implies, so the epsilon and this delta rho we have substituted a square root delta x square plus delta y square. Now we can divide by the square root delta x square plus delta y square and multiply it by the same term.

To get this following expression here, so epsilon delta x square plus delta y square divided by this square root here. And now we can break into 2 parts, so here the epsilon and one delta x divided by this term, and the other delta x here we have written down in the product plus, again the same concept here this epsilon together with 1 delta y, and divided by this term square root delta x square plus delta y square and delta y.

So, what we observe now that this delta z variation in z we can write down as dz plus epsilon 1. So, this is our epsilon 1 delta x plus this epsilon 2 delta y. And which implies, but this was the dz that goes to the right hand side, and then this is like epsilon 1 term, and this is epsilon 2 term, and they have the property that they will go to 0 again which we have just learned before.

So, this we will go to 0, this also go to 0, and we have written this delta z in terms of this dz plus epsilon delta x plus epsilon delta y. And that is precisely the definition of the differentiability. So, we have observed now that for the differentiability we have the equivalent definition here. That the differentiability imply this that this limit must be equal to 0, or we have also seen that if this limit equal to 0 then the function must be differentiable.

So, we can use this limit definition here for testing the differentiability, because getting this limit is easier than expressing this delta z in terms of this dz and epsilon delta x delta y term. So, we most of the time you will use this definition to prove the differentiability of the function, because this is easier then that the other definition of the differentiability.

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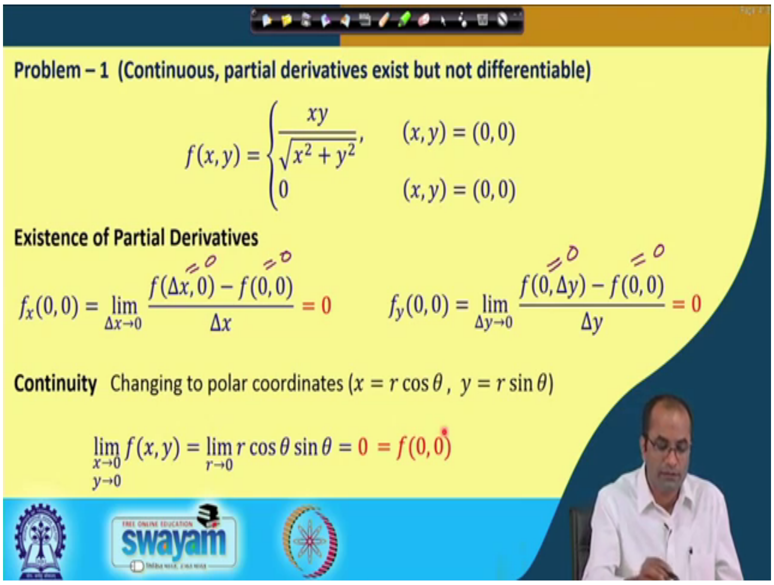

So, here we take this problem number 1, and we will show that in this case the function is continuous and the partial derivatives exists, but the function is not differentiable. And this is because of the reason that the continuity and the partial derivatives the existence of partial derivatives. These two are necessary conditions for the differentiability. So, we cannot claim based on these two conditions that the function is differentiable.

So, this is one example, we will show that this function is continuous and the partial derivatives exist both f x and f y, but the function is not differentiable. So, first the existence of partial derivatives; so, we know the definition of f x at 0 0. And naturally we will show this existence continuity at 0 0 and also that the function is not differentiable at the origin. So, here the f x at 0 0 is as per the definition the limit delta x goes to 0 and f delta x 0 minus f 0 0 over this delta x term.

So, this will be 0, because here we can see this product of x y. So, when we have this argument here in f as 0. So, this will make the function 0 and f 0 0 is defined at 0. So, we have this 0 minus 0. So, this is here 0 and this is also 0. So, 0 minus 0 by delta x and we get this 0. So, similarly for f y; f y 0 0 we have f 0 delta y minus f 0 0, and again this because of this x argument here it is 0. So, this product will make this again 0, and this is 0, so 0 minus 0 and we will get again this value as 0.

So, we have the existence of the partial derivatives at 0 0, and the value of the partial derivative with respect to f x 0 and also the value of the partial derivative at 0 0 with respect to y is also 0. Now for the continuity again it is a simple function we have already tested before. So, we can change this to polar coordinate that is easier. So, x is equal to r cos theta and y is equal to r sin theta we can substitute there, and take the limit as r goes to 0. So, when we substitute here x is equal to r cos theta y is equal to r sin theta, and here we will get simply r the square root of r square, and then one r will get canceled, and we have here r and cos theta sin theta; as r goes to 0 this cos theta sin theta they are bounded. So, here this limit will go to 0.

So, we have and the function is also 0 at 0 0. So, the function is continuous at the point 0 0. So, we have the existence of partial derivatives, we have the continuity of the function and now we will test for the differentiability of this function.

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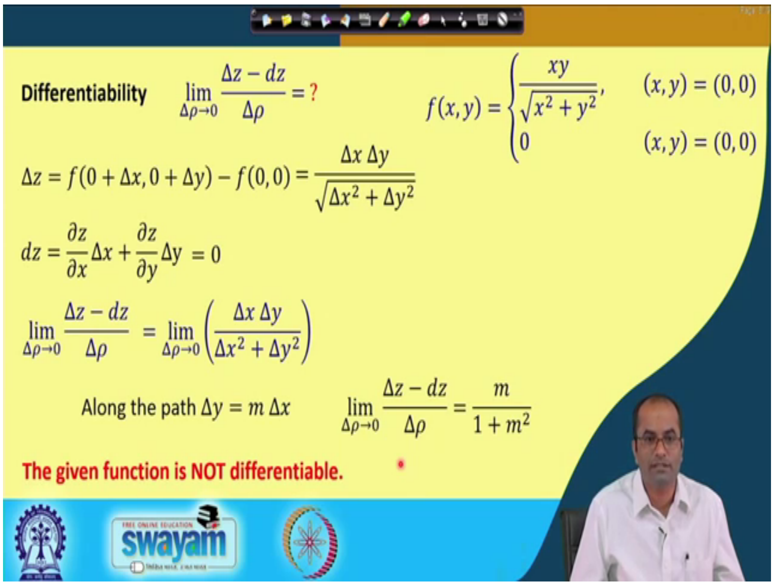

So, for the differentiability of this function we need to now go to this limit definition. Because as I said this is much easier so, we will find out that what is limit here delta z at minus dz at over delta rho.

So, the delta z is this increment in x and y so, at point 0 0; so 0 plus delta x and 0 plus delta y and minus the f 0 0. So, this f 0 0 is 0; so here we have f delta x delta y. So, here this x y will be replaced by delta x delta y, we have delta x delta y and the square root delta x square plus delta y square. And, now this dz the differential of z is partial derivative with respect to x delta x plus the partial derivative of z with respect to y and then we have delta y.

So, these two here we have just seen before that they are 0. So, we will get this dz term as 0, and then this limit delta z over dz over delta rho. So, this delta z is delta x delta y over delta x square plus delta y square and d z is 0. So, we will get now this delta x delta y and this delta rho was also the square root delta x square plus delta y square and we have one square root here. So, we will get this term without square root.

So, delta x delta y over delta x square plus delta y square and now we will see this limit. So, if we take this path here delta y is equal to m into delta x. One can clearly see because we have this quadratic term there and each of them is also quadratic. So, we will easily realize that, when we take this special part delta y is equal to m delta x the linear path to go to delta x 0 delta y 0. So, we will get here simply when we put delta y is equal to m delta x here also m square delta x square delta x square will get cancel and we will get this m over 1 plus m square.

So, this function is not differentiable, because this limit does not exist the limit depend on this path, so the function is not differentiable. For differentiability this limit must be equal to 0, but what we have seen that this limit does not exist. And hence the given function is not differentiable. So, what we have observed in this example, that the function is continuous and it is partial derivatives exist, but the function is not differentiable.

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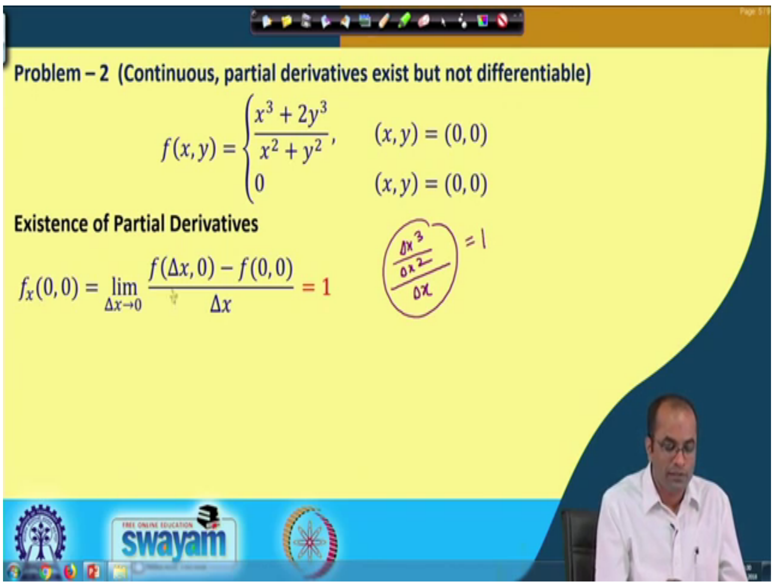

Another example of similar kind here also we will see that the function is continuous partial derivatives exist, but it is not differentiable. So, the existence of partial derivatives again it is easier. So, we have the definition of the partial derivative with respect to x so, f delta x 0, so here if you put y 0. So, we will get delta x cube over this delta x square, and this is f 0 0. So, and this delta x square is also there, so we will get basically this limit as 1. Because f delta x 0 will be delta x cube over delta x square, and then 1 delta x from this definition. So, this will be as 1.

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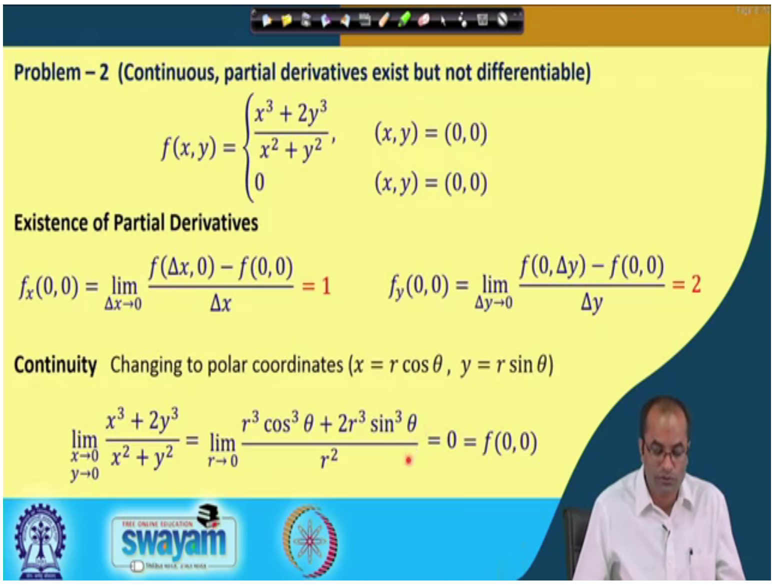

So, this limit is 1, and similarly the f y at 0 0 when we compute. So, in this case we will get this x 0 here 0, and it will get 2 delta y cube and then here also y delta y square and this delta y will make delta y cube. So, this delta y cube will get cancel and we will get this limit as 2. So, the partial derivative with respect to x is 1 and the partial derivative with respect to y is 2.

Now, coming to the continuity of this function again it is simple, and we can show by changing it to polar coordinate as x is equal to r cos theta y is equal to r sin theta and it is easy to see that this will be r square term here. And we will get r cube from there also r cube from there. So, we will get one r in the numerator and together with some bounded function of this cos cube theta plus 2 sin cube theta. So, when r goes to 0 this limit will be 0.

So, we have this limit is equal to 0 the function value is 0. So, hence this function is continuous the partial derivatives exist and the function is continuous. So, we will now move to show that the function is not differentiable at this point.

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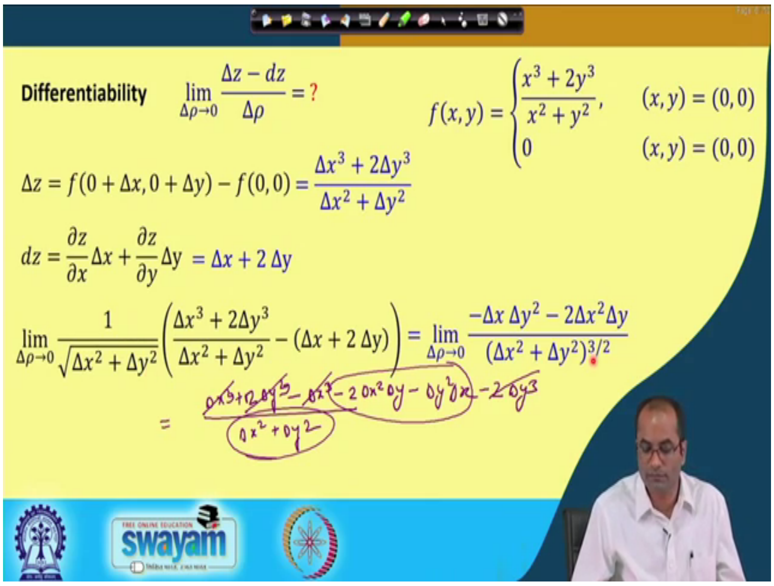

For that we need to show this limit again, where this delta z is this difference, which is again this is 0 and delta x delta y. So, this x y will be replaced simply by delta x and delta y terms. The dz term, so del z over del x at 0 0 is 1 and this was 2 there. So, we have delta x plus 2 delta y, and if we take this limit again here 1 over this is rho and then this delta z which is delta x square plus 2 delta y square, and this delta y square plus delta x square.

Minus this dz term delta x plus 2 delta y, and this one can simply simplify that delta x square delta y square in this denominator. So, we have then delta x square and 2 times delta y square minus this product; which will give us delta x cube as one term, and minus 2 times delta x square and delta y term. Then we will also get this delta y square, delta x term and then minus 2 times delta y cube term.

So, and this is 2 times delta y cube. So, here also this cube, so this delta x cube will get cancelled 2 times delta y cube will get cancel. And you will get only these 2 terms there with this delta y square and x square. So, we get precisely this minus 2 times, this delta x square and this delta y there, and minus delta x and delta y square, and this delta x square plus delta y square together with this you will get this power 3 by 2.

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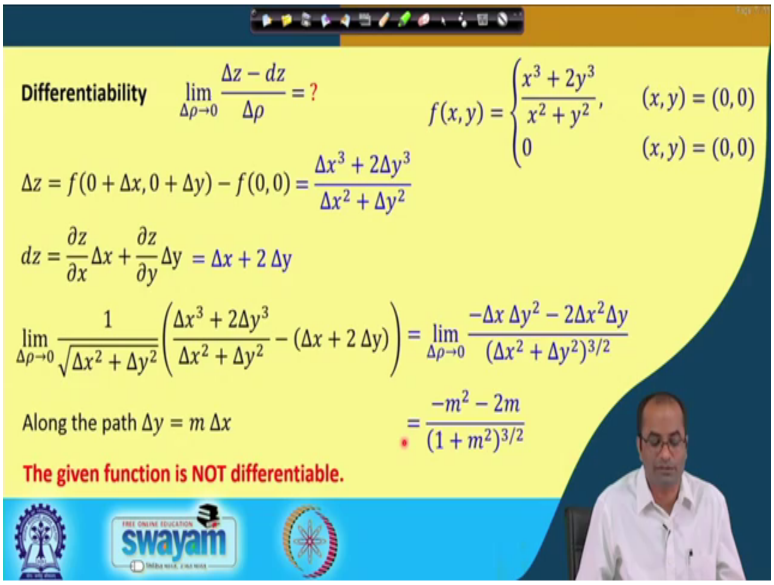

And now if we take this path delta y is equal to m delta x. So, again the same situation, so here we have then delta x cube common here also we will get delta x cube common and this delta y square will become m square delta x square.

So, this is power 3 by 2. So, here also we can take common delta x cube. So, delta x cube will get cancelled everywhere, and we will get the limit minus this is m square minus here m and here 1 plus m square 3 by 2. So, this will be the path dependent limit. Depending on m we have a different number for the limit, and hence this limit does not exist, and again this function is not differentiable.

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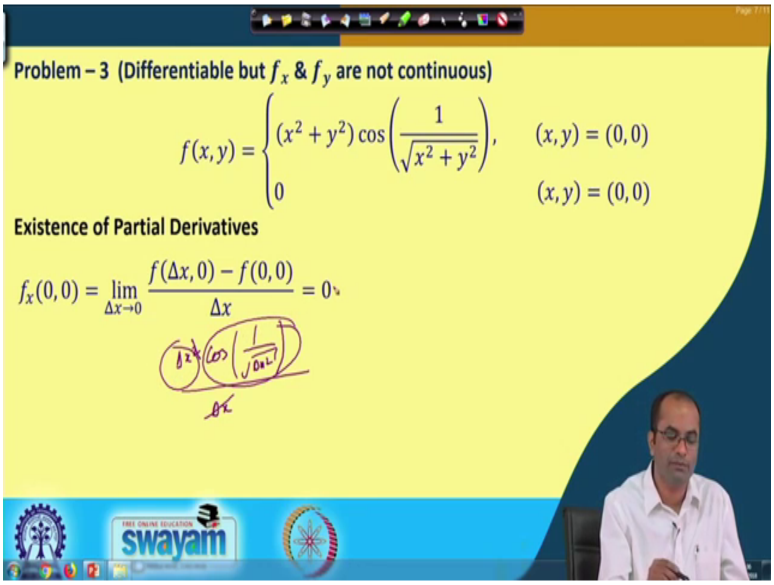

Next problem, here we will show that the function is differentiable, but f x and f y the partial derivatives are not continuous. Remember that the continuity of partial derivatives is sufficient for differentiability. So, the function may be differentiable, but the f x and f y may not be continuous. If we can prove that f x and f y are continuous, that will simply imply that the function is differentiable, but if we if we cannot prove the continuity of f x and f y, we cannot conclude anything about the differentiability of f.

So, this example precisely shows that the function is differentiable, but f x and f y are not continuous in this case. So, the existence of partial derivative now because we have to show that the necessary conditions are satisfied for differentiability, if one of the necessary conditions violated then we can immediately claim that the function is not differentiable.

So, here the existence of partial derivative again we have the definition f delta x 0 minus f 0 0 over delta x. And in this case we can show that this is 0, because here f delta x; so this delta y 0 and we have the delta x. So, we will get this delta x square, and then we will get here the cost term with 1 over square root delta x square, and divided by this delta x term here, and this is 0. So, this will get cancelled. So, here is something bounded and then delta x goes to 0, so this limit will be a 0.

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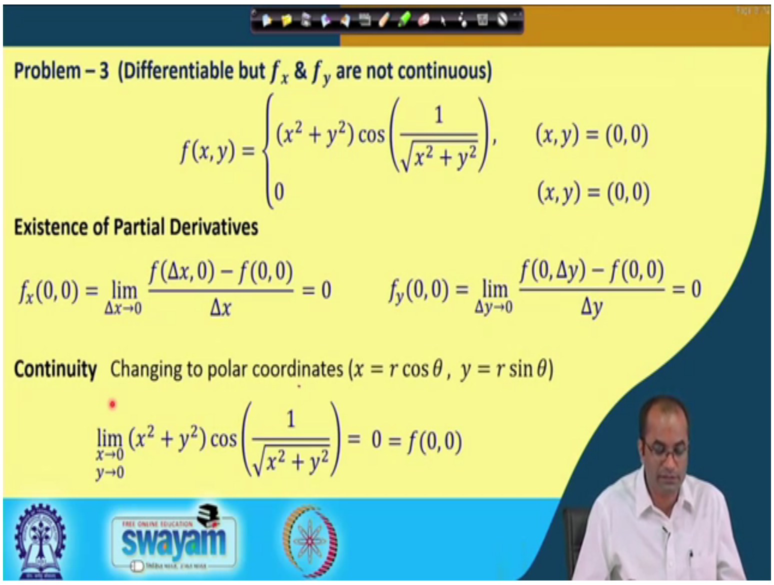

And similarly we can show that f y at 0 0 is also 0, because this function is symmetric anyway. So, we will get this again as 0. Now checking the continuity of the function is again e 0, because when x y goes to 0 0 this is something bounded sitting here and x y goes to 0. So, naturally this f x y will go to 0 as x y goes to 0 0. So, again we can see by changing to the polar coordinate also. So, we have here we can get this like r square, there and then rest everything will be bounded and as r goes to 0. So, this will be 0, whether showing by changing to polar coordinate or directly here when x y goes to 0.

So this term go to 0 and this is something bounded something finite here. So, we will get this limit as 0 directly; which is the function value at 0 0 point. So, we have seen the other function is continuous.

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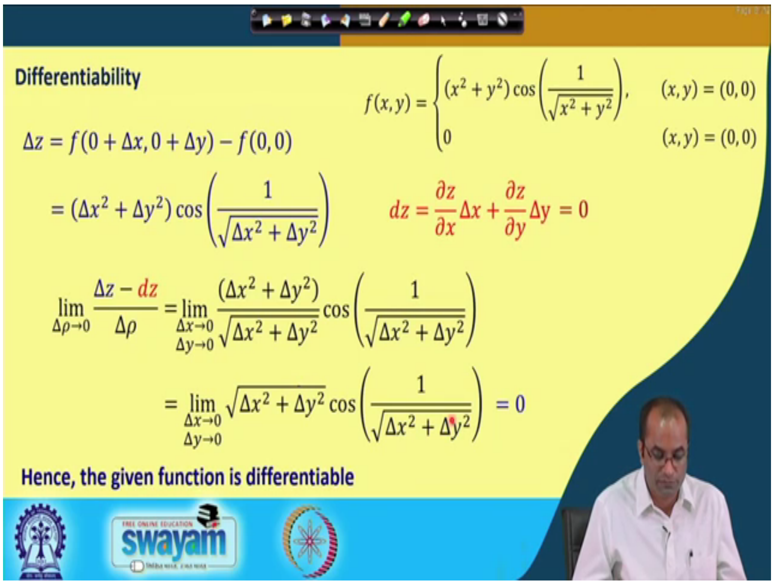

And now coming to the differentiability, so of this function, so we will take this delta z direct definition here, I mean for the delta z, the variance in z. So, f 0 plus delta x 0 plus delta y minus f 0 0 so, this will be because f 0 0 is 0 we have delta x delta y. So, x y will be replaced by delta x delta y term is there. And this dz the partial derivative with respect to x at 0 partial derivative of z with respect to y at 0, and we have seen those values were 0. So, this dz is 0 and now this quotient here, and then we will take the limit.

So, the delta z which is given here delta x square and delta y square cos term and minus this 0; so, and this delta rho is a square root delta x square plus delta y square. So, we will see that what is this limit here, and this can get cancelled. So, we will get in the numerator this is square root term with this cos 1 over this term. And now this term we can easily see that here the limit as delta x goes to 0 delta y goes to 0. Because this term will go to 0 and something bound it is sitting here. So, directly we can show that this value is 0.

That means, if this limit is 0 then the function is differentiable. So, in this case we have observed that this function is differentiable. So, what is now to show, that we will go to the continuity of the partial derivatives and we will observe that the partial derivatives are not continuous in this case.

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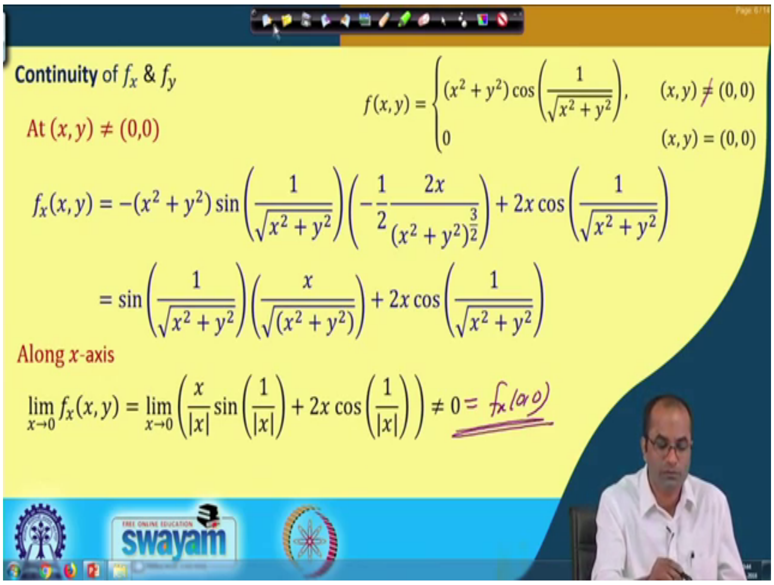

To prove the continuity of f x and f y, we need to get the f x and f y at non origin point. So, here it is non equal to 0, and when equal to 0 we have the 0 0 may be the same here also the same mistake. So, we can hey this was like at 0 0 the function was defined as 0 and then we have 0 when x y equal to 0. So, this is also not equal to 0 and it is 0 when x y is 0.

Well, so now, we will show the continuity of the partial derivatives at 0 0 point. So, we need to compute the continuity of the partial derivative at non 0-point. We have to compute the partial derivatives at non origin here and then the f x and f y at 0 0 which we have already computed the value was 0. So, to get the partial derivative of this function at this non origin point, then we have to we can just directly get the derivative of this function with respect to x treating this y as constant.

So, at x y not equal to 0 0 point, we can get this derivative by the direct differentiation of this treating this y constant. So, this is a product tool. So, this x square y square term, this cos will be with minus sign and 1 over the square root x square plus y square, and the derivative of this term which will be minus 1 by 2 and this 2 x over x square plus y square 3 by 2, and then this will remain unchanged here the cause and the derivative of this term will be 2 x.

So, this is the partial derivative of the function at with respect to x at the point x y which is not equal to 0 0. We can simplify little bit this term here. So, we will get sign of this 1 over square root x square plus y square, and this term will become x over the x square root x square plus y square plus this 2 x cos 1 over a square root x square plus y square. And now if we take path here along the x axis; that means, the delta y this y will be set to 0. So, we are approaching to 0 0, we want to see whether f x as x y goes to 0 is equal to the partial derivative of f at 0 0 point which was 0 there. So, along x axis if we move towards the origin, then what will happen?

The y is 0, now, so we have here x and divided by the square root of x square which is absolute value of x, and then we have this sign here, 1 over again absolute value of x plus; this 2 x and the cos again this here 1 over square more absolute value of x. So, this is along x axis; that means, the y is 0, so we have kept here y 0. So, we are taking a particular path along this x axis. And now if we realize here for example, this one when we go x to 0 from the right side this is plus 1, when x goes to 0 from the negative side this will become as minus 1. And in any case this is also not definite what will be the value here a same situation. At this point here anyway this x goes to 0. So, this is something bounded, so this will vanish this will go to 0.

But at this point here this is like plus 1, and this is undetermined in that case and this is can be minus 1 also and here we do not know what is the value when x goes to 0. So, this limit does not exist or certainly this is not equal to 0 which we were looking for that this limit. If this limit is equal to 0, we can claim that the function is the derivative f x is continuous, because this was the derivative value at 0 0 point, this was f x at 0 0 point. So, in any case this is not equal to this one. In fact, the limit does not exist. So, there is no question about the continuity of this partial derivative hence this f x is not continuous.

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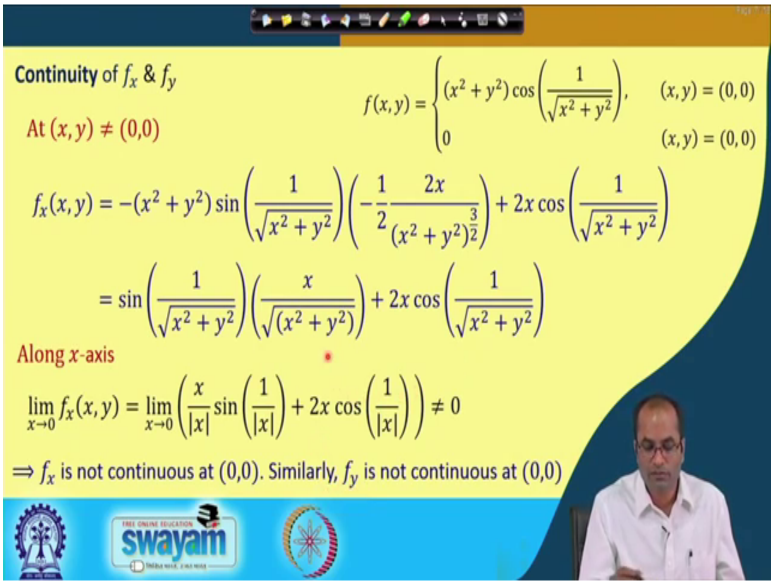

And similarly we can show because this function is just now symmetric. So, we can show also that f y is not continuous. So, in this example we have seen that the partial derivatives are not continuous though the function is differentiable.

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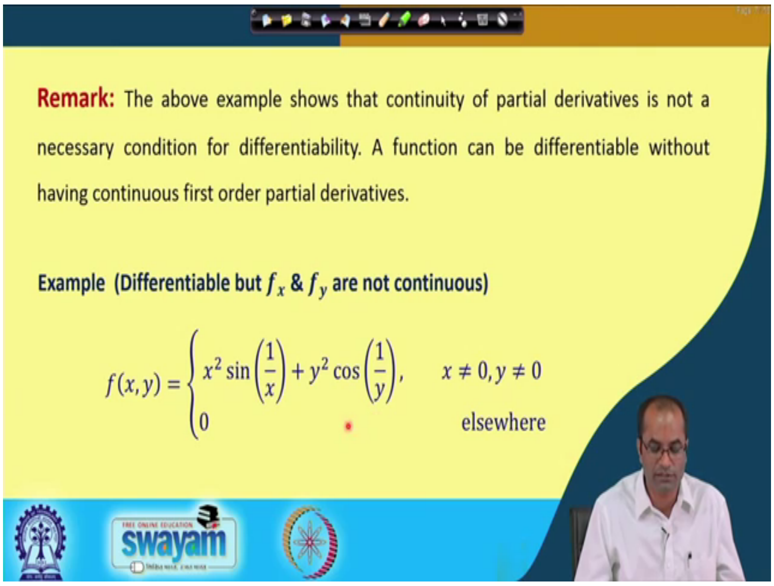

So, this remark the above example shows that the continuity of partial derivatives is not in necessary condition for differentiability. A function can be differentiable without having continuous first order partial derivatives. Another example of similar kind one can show again here that the function is differentiable and f x and f y they are not continuous. So, this is left to the participants we are not going to show that this function is differentiable, but f x and f y are not continuous, but the working steps are similar to the earlier problem. And one can easily show that f x and f y are not continuous for this problem as well.

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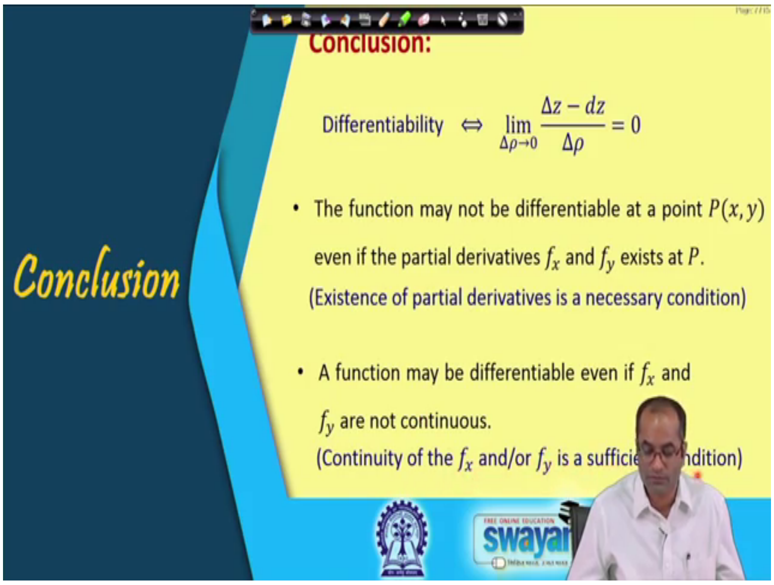

So, the conclusion here, we have what we have seen, we have seen the differentiability an equivalent definition which is the limit here, showing to 0 is equivalent to saying that the function is differentiable. So, this is useful in testing the differentiability of the function. And here the function may not be differentiable at a point even if partial derivative is exist, because the existence of partial derivatives is a sufficient condition for differentiability it is not necessary condition. And, we have also seen that function may be differentiable even if f x and f y are not continuous this is what we have also; so because this is sufficient condition, ok.

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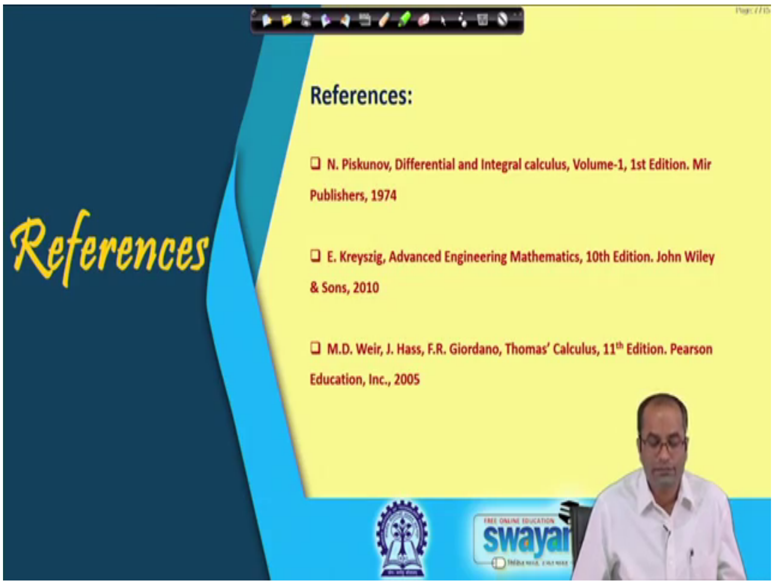

These are the references we have used to prepare these lectures.

Thank you very much.